$T^n$-ACTIONS ON SIMPLY CONNECTED $(n + 2)$-MANIFOLDS

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In this paper we show that, for each \( n \geq 2 \), there is a unique, closed, compact, connected, simply connected \((n + 2)\)-manifold, \( M_{n+2} \), admitting an action of \( T^n \) satisfying the following condition: there are exactly \( n \) \( T^1 \)-stability groups \( T_1, \ldots, T_n \) with each \( F(T_i, M_{n+2}) \) connected. In this case we have \( T^n \cong T_1 \times \cdots \times T_n \). Any other action \((T^n, M_{n+2})\), \( M_{n+2} \) simply connected, can be obtained from an action \((T^n, M_{n+2})\) by equivariantly replacing copies of \( D^4 \times T^{n-2} \) with copies of \( S^2 \times D^2 \times T^{n-3} \). As an application, we classify all actions of \( T^n \) on simply connected \((n + 2)\)-manifolds for \( n = 3, 4 \).

Several results have been obtained about \( T^n \)-actions on \((n + 2)\)-manifolds. Orlik and Raymond have obtained various classification theorems for the cases \( n = 1, 2 \) (see [11], [12] and [14]). Various general results have been obtained in [4] and [5] for \( n > 2 \). This paper is a continuation of the work done in [4]. We also obtain classification theorems similar to those of [12] for \( n = 3, 4 \).

In [4] it was shown that, for each \( n \), there exist actions of \( T^n \) on simply connected \((n + 2)\)-manifolds. Here we prove the following.

**Theorem.** For each \( n \), there is a unique closed, compact, connected, simply connected \((n + 2)\)-manifold \( M_{n+2} \) admitting an action of \( T^n \) satisfying the following conditions:

(i) There are exactly \( n \) \( T^1 \)-stability groups \( T_1, \ldots, T_n \).

(ii) Each \( F(T_i, M_{n+2}) \) is connected.

Furthermore, \( T^n \cong T_1 \times \cdots \times T_n \).

We then show that any action \((T^n, M_{n+2}), M_{n+2} \) a closed, compact, connected, simply connected \((n + 2)\)-manifold, can be obtained from an action \((T^n, M_{n+2})\) by equivariantly replacing copies of \( D^4 \times T^{n-2} \) with copies of \( S^2 \times D^2 \times T^{n-3} \).

The above results are applied to two specific cases. We show that if \( T^3 \) acts on a simply connected 5-manifold, \( M \), then \( M \) is \( M_5 = S^5 \) or a connected sum of copies of \( S^2 \times S^3 \). For \( T^1 \)-actions on simply connected 6-manifolds, \( M \), we show that \( M \) is \( M_6 = S^5 \times S^3 \) or \( M \) is a connected sum of copies of \( S^2 \times S^4 \) and \( S^3 \times S^3 \).

1. **Preliminaries.** We shall use standard terminology and notation throughout (e.g. see [2]). Unless otherwise stated, all mani-
folds are closed, connected and compact. All actions are assumed to be locally smooth and effective.

Let \((G, M)\) and \((G, N)\) be two \(G\)-actions. We shall use \((G, M) \equiv_{eq} (G, N)\) or \(M \equiv_{eq} N\) to mean that \(M\) and \(N\) are equivariantly homeomorphic. Given actions \((G, M)\) and \((H, N)\), \((G \times H, M \times N)\) will indicate the obvious product action.

The \(n\)-dimensional torus \(T^n = S^1 \times \cdots \times S^1\) (\(n\) factors) can parameterized as:

\[
T^n = \{(e^{i\phi_1}, \ldots, e^{i\phi_n}) \mid 0 \leq \phi_i \leq 2\pi\}
\]

We simplify this as \(T^n = \{ (\phi_1, \ldots, \phi_n) \mid 0 \leq \phi_i \leq 2\pi \}\). Similarly, we write:

\[
D^n = \{(r_1, \theta_1, \ldots, r_{[n+1/2]}, \theta_{[n+1/2]}) \mid \sum r_i^2 \leq 1, 0 \leq \theta \leq 2\pi, \theta_{[n+1/2]} = 0 \text{ if } n \text{ odd}\}
\]

Of course for \(S^n\), we have \(\sum r_i^2 = 1\).

**Example 1.1.** We have an action of \(T^n\) on \(D^4 \times T^{n-2}\) defined as follows. If \(t = (\phi_1, \ldots, \phi_n) \in T^n\) and \(z = ((r_1, \theta_1, r_2, \theta_2), (\theta_3, \ldots, \theta_n)) \in D^4 \times T^{n-2}\), let

\[
tz = ((r_1, \theta_1 + a_{11}\phi_1 + \cdots + a_{1n}\phi_n, r_2, \theta_2 + a_{21}\phi_1 + \cdots + a_{2n}\phi_n),
(\theta_3 + a_{31}\phi_1 + \cdots + a_{3n}\phi_n, \ldots, \theta_n + a_{n1}\phi_1 + \cdots + a_{nn}\phi_n)).
\]

This action defines a matrix \(A = (a_{ij})\). For the action to be effective, we must have \(\det(A) \neq 0\). We shall frequently define such an action by giving the matrix \(A\).

We shall often use the following (see [8]). Suppose \(M\) is an \(m\)-manifold with boundary and \(G \cong T^n\) acts on \(M\) with \(m > n\). If \(M^*\) is a closed cone with vertex \(x_0^*\) and \(G_{x_0} \cong T^k\), \(0 \leq k \leq n\) (\(T^0 = \text{id}\)), then \((T^n, M) \equiv_{eq} (T^{n-k} \times T^k, T^{n-k} \times D^{m-n+k})\).

Suppose \(T^n\) acts on a simply connected \((n + 2)\)-manifold \(M\). It was shown in [4] that the orbit space, \(M^*\), will be \(D^2\), with points on the boundary corresponding to singular orbits and interior points corresponding to principal orbits. Isolated points on the boundary correspond to orbits of type \(T^{n-2}\) and the remaining boundary points correspond to orbits of type \(T^{n-1}\). The result mentioned above shows that an invariant tubular neighborhood of an orbit of type \(T^{n-2}\) will be \(D^4 \times T^{n-2}\).

It was also shown in [4] that, for all \(n\), actions of \(T^n\) on simply connected \((n + 2)\)-manifolds exist. The following picture shows how such actions can be constructed.
Each sector of the disk \( D^2 \cong M^* \) represents an invariant tubular neighborhood of an orbit of type \( T^{n-2} \) which, as mentioned above, must be \( D^4 \times T^{n-2} \). These are attached to one another along subspaces of the boundary homeomorphic to \( D^2 \times T^{n-1} \). Another result of [4] is that the circle stability groups of the action must span \( T^n \). Hence, we must have at least \( n \) copies of \( D^4 \times T^{n-2} \).

We shall say that \((T_1, T_2)\) is an adjacent pair of \( T^1\)-stability groups for an action \((T^n, M^{n+2})\), if there is an invariant \( D^4 \times T^{n-2} \) so that the induced action \((T^n, D^4 \times T^{n-2})\) has stability groups \( T_1, T_2 \) and \( T_1 \times T_2 \). \((T_1, T_2, T_3)\) will be called an adjacent triple of \( T^1\)-stability groups if \((T_1, T_2)\) and \((T_2, T_3)\) are adjacent pairs, with invariant copies of \( D^4 \times T^{n-2} \), \( (D^4 \times T^{n-2})_1 \) and \( (D^4 \times T^{n-2})_2 \), respectively, such that \( (D^4 \times T^{n-2})_1 \cap (D^4 \times T^{n-2})_2 \cong D^2 \times T^{n-1} \) and \( 0 \times T^{n-1} \subseteq F(T^2, M^{n+2}) \). In this case \( (D^4 \times T^{n-2})_1 \) and \( (D^4 \times T^{n-2})_2 \) are said to be adjacent.

2. Orbit structure. Suppose \( T^n \) acts on a simply connected \((n + 2)\)-manifold \( M \). As mentioned above, we know that the \( T^1\)-stability groups span \( T^n \). In this section we show that, in certain cases, \( T^n \) is the direct product of the \( T^1\)-stability groups. If \( G \) is a group and \( S \subseteq G \) is a subset let \( \langle S \rangle \) denote the subgroup spanned by \( S \).

**Lemma 2.1.** If \( T^n \) acts on a simply connected \((n + 2)\)-manifold \( M \), there exists an adjacent triple \((T_1, T_2, T_3)\) such that \( \langle T_1 \cup T_2 \cup T_3 \rangle \cong T_1 \times T_2 \times T_3 \).

**Proof.** Let \((T_1, T_2)\) be an adjacent pair so that we have an invariant \((D^4 \times T^{n-2})_1 \) with stability groups \( T_1, T_2 \) and \( T_1 \times T_2 \). Write \( T^n = T_1 \times T_2 \times T^{n-2} \) and parameterize so that the action \((T^n, (D^4 \times T^{n-2})_1)\) is defined by the matrix \( I \) (see 1.1).

Consider an adjacent invariant \((D^4 \times T^{n-2})_2 \) with \( T^1\)-stability
groups $T_i$ and $C$ so that $(T_2, T_1, C)$ is an adjacent triple. The action $(T^n, (D^4 \times T^{n-2})_2)$ will be determined by a matrix of the form

$$A = \begin{pmatrix}
1 & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{n2} & \cdots & a_{nn}
\end{pmatrix}.$$ 

We may consider $N = (D^4 \times T^{n-2})_1 \cup f(D^4 \times T^{n-2})_2$, where $f: (D^4 \times T^{n-1})_1 \rightarrow (D^2 \times T^{n-1})_2$ is an equivariant attaching homeomorphism, as an invariant subspace of $M$. $f$ is determined by $A$ in the following manner. If

$$z = ((r_1, \theta_1, r_2, \theta_2), (\theta_3, \cdots, \theta_n)) \in (D^2 \times T^{n-1})_1 \subseteq (D^4 \times T^{n-2})_1$$

then $f(z) = ((r_1, \theta'_1, r_2, \theta'_2), (\theta'_3, \cdots, \theta'_n))$ where

$$\bar{\theta}' = (\theta'_1 \cdots \theta'_n)^t = A(\theta_1 \cdots \theta_n)^t = A\bar{\theta}.$$ 

To show that $f$ is equivariant, we ignore the $r$'s for convenience. Let $\alpha = (\varphi_1 \cdots \varphi_n)^t = \bar{\varphi} \in T^n$. Then

$$\alpha(f(z)) = \alpha A\bar{\theta}$$

$$= A\bar{\theta} + A\bar{\varphi}$$

$$= A(\bar{\theta} + \bar{\varphi}) = f(\alpha z).$$

Now note that if, for each $j$, there exists an $i > 2$ with $a_{ij} \neq 0$, then $f_1: \pi_1((D^2 \times T^{n-1})_1) \rightarrow \pi_1((D^2 \times T^{n-1})_2)$ is injective. In this case, it follows that $\pi_1(N) \cong \mathbb{Z}^{n-1}$. Since $M$ is obtained, as described above, by attaching successive copies of $D^4 \times T^{n-2}$, some attaching map must kill an element of some $\pi_1(D^4 \times T^{n-1})$. Hence, let us assume that $a_{in} = 0$ for all $i > 2$.

It now follows that the stability group $C$ is defined by the following system of equations:

$$\varphi_1 + a_{12}\varphi_2 + \cdots + a_{1n}\varphi_n \equiv 0(2\pi)$$

$$a_{22}\varphi_2 + \cdots + a_{2,n-1}\varphi_{n-1} \equiv 0(2\pi)$$

$$\vdots$$

$$a_{n2}\varphi_2 + \cdots + a_{n,n-1}\varphi_{n-1} \equiv 0(2\pi).$$

Since the action is effective, it follows that $\varphi_2 = \cdots = \varphi_{n-1} = 0$. Hence, $C = \{(-a_{1n}\varphi_n, 0, \cdots, 0, \varphi_n) | 0 \leq \varphi_n \leq 2\pi\}$. It is easy to see that $\langle T_1 \cup T_2 \cup C \rangle \cong T_1 \times T_2 \times C$.

**Corollary 2.2.** If $T^n$ acts on a simply connected $(n + 2)$-manifold, $M$, there exists an invariant $D^2 \times S^3 \times T^{n-3}$ with the
standard product action \((T^1 \times T^2 \times T^{n-3}, D^2 \times S^3 \times T^{n-3})\).

Proof. By the lemma, one can find an adjacent triple of \(T^1\)-stability groups \((T_2, T_1, T_3)\) so that \(T^n = T_1 \times T_2 \times T_3 \times T^{n-3}\). Let \((D^4 \times T^{n-2})_1\) and \((D^4 \times T^{n-2})_2\) be the adjacent copies of \(D^4 \times T^{n-2}\) corresponding to the adjacent pairs \((T_2, T_1)\) and \((T_1, T_3)\), respectively. Let \(N = (D^4 \times T^{n-2})_1 \cup (D^4 \times T^{n-2})_2\) as in the proof of 2.1 so that we have the action \((T^n, N)\). We have the standard action

\[
(T^n, D^2 \times S^3 \times T^{n-3}) = (T_1 \times (T_2 \times T_3) \times T^{n-3}, D^2 \times S^3 \times T^{n-3})
\]

with weighted orbit space equivalent to \(N^*\). It follows from standard techniques that \(N \cong e_2D^2 \times S^3 \times T^{n-3}\).

In case there are only \(n\) \(T^1\)-stability groups we have the following much stronger result.

**Theorem 2.3.** Suppose \(T^n\) acts on a simply connected \((n+2)\)-manifold, \(M\), so that there are exactly \(n\) \(T^1\)-stability groups \(T_1, \cdots, T_n\) with each \(F(T_i, M)\) connected. Then \(T^n \cong T_1 \times \cdots \times T_n\).

Proof. First remove nonintersecting neighborhoods \(D^4 \times T^{n-2}\) of each orbit of type \(T^{n-2}\). We obtain a \(T^n\)-manifold with boundary, \(N\), with \(N^*\) as shown below.

Using the Seifert-Van Kampen theorem, it is easy to see that \(\pi_1(N) \cong 0\). Let \(U_{n+1} \cong T^n \times D^2\) be as shown. For each \(i, 1 \leq i \leq n\), choose \(U_i \cong (D^2 \times T^{n-1})\), so that \(F(T_i, N) \subseteq U_i\) and \(U_i \cap U_j = U_{n+1}\), \(1 \leq i < j \leq n\).

Each inclusion \(\pi_1(U_{n+1}) \to \pi_1(U_i)\) has kernel isomorphic to \(Z\), generated by an element \(z_i \in \pi_1(U_{n+1})\) corresponding to \(T_i \subseteq T^n\). Let \(K = \langle z_1, \cdots, z_n \rangle\). For each \(i\), we have the following commutative
The vertical map is the inclusion, $\rho_{n+1}$ is the natural projection and $\rho_t$ is defined to make the diagram commute. By the Seifert-Van Kampen theorem, we then have the commutative diagram.

\[
\begin{array}{ccc}
\pi_1(U_{n+1}) & \xrightarrow{\rho_{n+1}} & \pi_1(U_{n+1})/K \\
\downarrow & & \downarrow \\
\pi_1(U_t) & & \\
\end{array}
\]

Therefore $\rho_{n+1} = 0$ and $K = \pi_1(U_{n+1})$.

Label the $T_i$'s so that $T_i \times \cdots \times T_k$ is a direct product and $k$ is a maximum. Suppose $k < n$. For each $i > k$ we have $\langle T_i \cup \cdots \cup T_i \rangle \cong T^i$. For $1 \leq i \leq k$ let $C_i = T_i$ and for $i > k$ let $C_i \cong T^i$ be such that $\langle T_i \cup \cdots \cup T_i \rangle \cong C_i \times \cdots \times C_i$ and $T_i \not\subseteq C_i \times \cdots \times C_{i-1}$. Parameterize $T^n \cong C_1 \times \cdots \times C_n$ in the obvious manner. For $1 \leq i \leq k$, $T_i = \{0, \cdots, 0, \varphi_i, 0, \cdots, 0\}$ if $0 \leq \varphi_i \leq 2\pi$. For $i > k$, we have $T_i = \{(c_{1i}, \varphi_i), \cdots, c_{ii}, \varphi_i, 0, \cdots, 0\}$ if $0 \leq \varphi_i \leq 2\pi$. Let $\delta_{ij}$ be the Kronecker delta. If we write $\pi_1(U_{n+1}) \cong \pi_1(C_i) \times \cdots \times \pi_1(C_n)$, then for $1 \leq i \leq k$, $z_i = (\delta_{1i}, \cdots, \delta_{ni})$ and for $i > k$, $z_i = (c_{1i}, \cdots, c_{ii}, 0, \cdots, 0)$. Since $T_i \cap (C_1 \times \cdots \times C_k) \neq \text{id}$ and $T_i \not\subseteq C_1 \times \cdots \times C_{i-1}$ for $i > k$, we have $c_{ii} > 1$. Therefore,

\[
\det \begin{pmatrix}
1 \\
\vdots \\
1 \\
0 \\
\vdots \\
c_{n+1,k+1} \\
0 \\
\vdots \\
c_{n,n}
\end{pmatrix} \neq 1.
\]

This would imply that $K \neq \pi_1(U_{n+1})$, a contradiction. Therefore $k = n$ and $T^n \cong T_1 \times \cdots \times T_n$.

3. The manifolds $M_{n+2}$ and the construction of actions $(T^n, M^{n+2})$. In this section we show the existence of basic simply connected $(n + 2)$-manifolds admitting actions of $T^n$. We then show
how any action of $T^n$ on a simply connected $(n + 2)$-manifold can be obtained from some action $(T^n, M_{n+2})$.

**Theorem 3.1.** For each $n = 2$ there exists a unique manifold $M_{n+2}$ admitting an action of $T^n$ satisfying the following condition: there are exactly $n$ $T^1$-stability groups with each $F(T_i, M_{n+2})$ connected.

**Proof.** By the construction in [4] there exists a simply connected $(n + 2)$-manifold $M$ and an action $\theta: T^n \times M \to M$ with $T^1$-stability groups $T_1, \cdots, T_n$ satisfying the stated conditions. Let $\varphi: T^n \times N \to N$ be another such action with $T^1$-stability groups $C_1, \cdots, C_n$. We assume the $T_i$'s and $C_i$'s are labeled in a clockwise direction going around the orbit spaces. We must show that $M \cong N$.

By 2.3, $T^n \cong T_1 \times \cdots \times T_n = C_1 \times \cdots \times C_n$. We have the obvious isomorphism $f: T^n \to T^n$ with $f(C_i) = T_i$. Define an action $\theta': T^n \times M \to M$ by $\theta'(t, m) = \theta(f(t), m)$. It is easy to see that the weighted orbit space of this action is equivalent to that for $\varphi$. By the equivariant classification theorem of [4], $M \cong N$.

While it is not true that all actions of $T^n$ on $M_{n+2}$ are equivalent, the above proof shows the following

**Corollary 3.2.** Any two actions of $T^n$ on $M_{n+2}$ are weakly equivalent.

The standard actions $(T^2, S^4)$, $(T^3, S^5)$ and $(T^5, S^3 \times S^3)$ show that $M_4 = S^4$, $M_5 = S^5$ and $M_6 = S^3 \times S^3$. The manifolds $M_{n+2}$, $n > 4$, have not been identified at this time.

The manifolds $M_{n+2}$ provide a starting point for the construction of $T^n$-actions on simply connected $(n + 2)$-manifolds.

**Theorem 3.3.** Suppose $T^n$ acts on a simply connected $(n + 2)$-manifold $M$. Then the action $(T^n, M)$ can be obtained from an action $(T^n, M_{n+2})$ by equivariantly replacing copies of $D^4 \times T^{n-2}$ with copies of $S^3 \times D^2 \times T^{n-3}$.

**Proof.** Consider the action $(T^n, M)$. By 2.2, $M$ contains an invariant $S^3 \times D^2 \times T^{n-3}$. When this is replaced equivariantly with a $D^4 \times T^{n-2}$, the number of $T^{n-2}$-orbits is decreased by one. If this process is continued, $M_{n+2}$ will be obtained. Reversing the process proves the theorem.

From [4] we know that if $T^n$ acts on a simply connected $(n + 2)$-
COROLLARY 3.4. Suppose $T^n$ acts on a simply connected $(n+2)$-manifold $M$. Then there are $T^1$-stability groups $T_1, \ldots, T_n$ such that $T^n \cong T_1 \times \cdots \times T_n$.

Proof. Obtain $M_{n+2}$ from $M$ as in the proof of 3.3. Then $T^n \cong T_1 \times \cdots \times T_n$ where $T_1, \ldots, T_n$ are the $T^1$-stability groups of the resulting action $(T^n, M_{n+2})$. However these will also be $T^1$-stability groups for the original action $(T^n, M)$.

4. The cases $n = 3, 4$. It was noted that $M_4 = S^4$, $M_5 = S^5$ and $M_6 = S^3 \times S^3$. These are the only $M_{n+2}$'s identified. In fact no explicit actions of $T^n$ on simply connected $(n + 2)$-manifolds have been identified for $n > 4$.

In [12], Orlik and Raymond classify actions of $T^2$ on simply connected 4-manifolds. In this section we use results of Wall, [16], and Barden, [1], to classify actions of $T^3$ and $T^4$ on simply connected 5- and 6-manifolds, respectively.

Recall that the orbit space, $D^2$, of an action $(T^n, M_{n+2})$ has isolated points on the boundary, each corresponding to an orbit of type $T^{n-2}$.

THEOREM 4.1. Suppose $T^n$ acts on a simply connected 5-manifold $M$ so that there are $k$ distinct orbits of type $T^1$. If $k = 3$, $M \cong S^5$. If $k > 3$, $M$ is a connected sum of $k - 3$ copies of $S^2 \times S^3$.

Proof. If $k = 3$, then $M \cong M_5 = S^5$. Suppose the theorem is true for some $k \geq 3$. Let $T^3$ act on $M$ with $k + 1$ orbits of type $T^1$. $M$ is obtained from a manifold $N$ by equivariantly replacing an $S^1 \times D^4$ with a $D^2 \times S^3$. Since $N$ has $k$ orbits of type $T^1$, $N$ is a connected sum of $k - 3$ copies of $S^2 \times S^3$ or $S^5$ if $k - 3 = 0$. By the Mayer-Vietoris sequence

$$H^p(M) \cong \begin{cases} \mathbb{Z} & p = 0, 5 \\ \mathbb{Z}^{k-2} & p = 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

By results in [1], the above construction can be done in $\mathbb{R}^7$ so $M$ embeds in $\mathbb{R}^7$. It follows that $\omega_k(\nu^2) = 0$ for all $k \geq 1$, where $\nu^2$ is the normal bundle of $M$ and $\omega_k$ is the $k$th Stiefel-Whitney class. By Whitney Duality, $\omega_k(M) = 0$. Therefore, by [1], $M$ is a connected sum of $k - 2$ copies of $S^2 \times S^3$. 

manifold, $M$, then the $T^1$-stability groups span $T^n$. We now have the following.
It is worthwhile to note that $M$ will not be an equivariant connected sum. In fact, equivariant connected sums of codimension two actions cannot exist for $n \geq 3$ since $T^n$ cannot act on $S^{n+1}$ for $n \geq 3$.

It was noted that all $T^n$-actions on $M_{n+2}$ are weakly equivalent. The following example shows that this is not true for $T^n$-actions on other simply connected $(n+2)$-manifolds.

**Example 4.2.** Let $T^3$ act on $S^5$ with $T^1$-stability groups $T_1$, $T_2$ and $T_3$ so that $T^3 = T_1 \times T_2 \times T_3$. Define an action $(T^3, S^3 \times D^2)$ as follows:

$$tz = ((r_1, \theta_1 + \varphi_1 - \varphi_2, r_2, \theta_2 + \varphi_1 - \varphi_3), (r_3, \theta_3 + \varphi_1)).$$

This action has $T^1$-stability groups $T_2$, $T_3$ and

$$T_4 = \{(\varphi_1, \varphi_2, \varphi_3) | \varphi_1 = \varphi_2 = \varphi_3\}.$$

Replace the $D^4 \times S^1 \subseteq S^5$ containing $F(T_2 \times T_3)$ with $S^3 \times D^2$ to obtain an action $(T^3, S^2 \times S^3)$ with $T^1$-stability groups $T_1$, $T_2$, $T_3$ and $T_4$. However, we have another action $(T^3, S^2 \times S^3) = (T_1 \times (T_2 \times T_3), S^2 \times S^3)$ with $T^1$-stability groups $T_1$, $T_3$ and $T_4$ where $F(T, S^2 \times S^3)$ has two components. It is obvious that these actions are not weakly equivalent.

We now consider $T^4$-actions on 6-manifolds.

**Theorem 4.2.** Suppose $T^4$ acts on a simply connected 6-manifold, $M$, with $k$ orbits of type $T^2$. Then $M$ is a connected sum of $k - 4$ copies of $S^2 \times S^4$ and $k - 3$ copies of $S^3 \times S^3$.

*Proof.* For $k = 4$, $M \cong M_6 = S^3 \times S^3$. Assume the theorem is true for some $k \geq 4$, and let $T^4$ act on $M$ with $k + 1$ orbits of type $T^2$. $M$ is obtained from a $T^4$-manifold $N$ by equivariantly replacing $V_1 \cong D^4 \times T^2$ with $V_2 \cong S^3 \times D^2 \times T^1$. Since $N$ has $k$ orbits of type $T^2$, $N$ is a connected sum of $k - 4$ copies of $S^2 \times S^4$ and $k - 3$ copies of $S^3 \times S^3$. We may assume $N = U \cup V_1$, $M = U \cup V_2$ and $V_i \cap U = S^3 \times T^2$. Applying the Mayer-Vietoris sequence, it is easy to see that $H^4(U) \cong \mathbb{Z}^{k-4}$. Also, by examining the pair $(U, \partial U)$, one can show that $H^2(\partial U) \rightarrow H^2(U, \partial U)$ is injective so that $H^2(U) \rightarrow H^2(\partial U) \cong H^2(T^2 \times S^3)$ is trivial. Therefore, the following sequence is exact.

$$0 \rightarrow H^4(V_2) \rightarrow H^4(U \cup V_2) \rightarrow H^4(M) \rightarrow H^4(U) \rightarrow 0.$$ 

It follows that $H^2(M) \cong \mathbb{Z}^{k-3} \cong H^4(M)$. Since there are no fixed
points, $\chi(M) = 0$, so $H^s(M) \cong \mathbb{Z}^{2k-4}$.

Since $\omega_2(N) = 0$, $\omega_2(U) = 0$, so if $\omega_2(M) \neq 0$, it must be the generator of $K = \ker (H^2(M) \rightarrow H^2(U))$ (with $\mathbb{Z}$-coefficients). One can choose as a generator of $K$ a two cochain vanishing off $L$ where $L^*$ is as shown below.

Now $L$ is a closed, compact 4-manifold admitting an action of $T^3$ so, by [13], $L \cong L(p, q) \times S^1$. Since each factor of $L$ is parallelizable, $L$ is and $\omega_2(L) = 0$. Therefore $\omega_2(M) = 0$. The result follows from [16].

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