Pacific Journal of Mathematics

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Vol. 71, No. 2 December 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol. 71, No. 2, 1977

FIXED POINT THEOREMS FOR MAPPINGS WITH A CONTRACTIVE ITERATE

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Several fixed point theorems are proved for metric-space mappings which satisfy a contractive condition involving an iterate of the mapping, where the iterate depends on the point in the space.

Let (X, d) denote a complete metric space. In [3] the second author has established fixed point theorems for mappings which satisfy a variety of contractive conditions. The common property of the mappings discussed in [3] is that the fixed point is unique, and can be found by using repeated iteration, beginning with some initial choice $x_0 \in X$.

The first result in this direction is that of V. M. Sehgal [5] who proved the following.

THEOREM S1. *Let (X, d) be a complete metric space, T a continuous self-mapping of X which satisfies the condition that there exists a real number k, $0 < k < 1$ such that, for each $x \in X$ there exists a positive integer $n(x)$ such that, for each $y \in X$,*

$$(1) \qquad d(T^{n(x)}(x)) \,, \quad T^{n(x)}(y) \leqq kd(x, y) \,.$$

Then T has a unique fixed point in X.

L. F. Guseman, Jr. [1], extended Sehgal's result by removing the condition of continuity of T and weakening (1) to hold on some subset B of X such that $T(B) \subset B$, where, for some $x_0 \in B$, B contains the closure of the iterates of x_0. Further extensions for a single mapping appear in [2] and in [4].

We shall be concerned with a pair of mappings which satisfy the following contractive condition.

Let T_1, T_2 be self-mappings of X such that there exists a constant k, $0 < k < 1$ such that there exist positive integers $n(x)$, $m(y)$ such that for each $x, y \in X$,

$$(2) \qquad \begin{aligned} d(T_1^{n(x)}(x) \,, \quad T_2^{m(y)}(y)) \leqq k \max \{&d(x, y) \,, \quad d(x, T_1^{n(x)}(x)) \,, \\ &d(y, T_2^{m(y)}(y)) \,, \quad [d(x, T_2^{m(y)}(y)) + d(y, T_1^{n(x)}(x))]/2\} \,. \end{aligned}$$

THEOREM 1. *Let T_1, T_2 be self-mappings of a complete metric space (X, d) which satisfy (2). Then T_1 and T_2 have a unique common fixed point.*

Proof. Let $x_0 \in X$, and define the sequence $\{x_n\}$ by $x_1 = T_1^{n(x_0)}(x_0)$, $x_2 = T_2^{m(x_1)}(x_1), \cdots, x_{2n+1} = T_1^{n(x_{2n})}(x_{2n})$, $x_{2n+2} = T_2^{m(x_{2n+1})}(x_{2n+1}), \cdots$. Using (2) and assuming $x_m \neq x_n$ for each $m \neq n$,

$$(3) \qquad d(x_{2n+1}, x_{2n+2}) \leqq k \max\{d(x_{2n}, x_{2n+1}), \; d(x_{2n+1}, x_{2n+2}), \; d(x_{2n}, x_{2n+2})/2\}.$$

If the maximum of the right-hand side of (3) is $d(x_{2n}, x_{2n+2})/2$ then we obtain the contradiction $d(x_{2n+1}, x_{2n+2}) \leqq kd(x_{2n+1}, x_{2n+2})$. Therefore, $d(x_{2n+1}, x_{2n+2}) \leqq kd(x_{2n}, x_{2n+1})$. Similarly $d(x_{2n}, x_{2n+1}) \leqq kd(x_{2n-1}, x_{2n})$, so that $d(x_{2n+1}, x_{2n+2}) \leqq k^{2n}d(x_1, x_2)$ and $d(x_{2n}, x_{2n+1}) \leqq k^{2n}d(x_0, x_1)$. With $r(x_0) = \max\{d(x_0, x_1), d(x_1, x_2)\}$, for any $m > n$,

$$d(x_m, x_n) \leqq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leqq k^{2n}r(x_0)/(1 - k^2).$$

Thus $\{x_n\}$ is Cauchy and hence convergent. Call the limit p.

From (2),

$$(4) \qquad d(x_{2n+1}, T_2^{m(p)}(p)) \leqq k \max\{d(x_{2n}, p), \; d(x_{2n}, x_{2n+1}), \; d(p, T_2^{m(p)}(p)), \; [d(x_{2n}, T_2^{m(p)}(p)) + d(p, x_{2n+1})]/2\}.$$

Taking the limit of (4) as $n \to \infty$ we obtain $d(p, T_2^{m(p)}(p)) \leqq k \max\{0, 0, d(p, T_2^{m(p)}(p)), d(p, T_2^{m(p)}(p))/2\}$, which implies $p = T_2^{m(p)}(p)$. Similarly $p = T_1^{n(p)}(p)$.

Suppose q is also a periodic point of T_1 and T_2; i.e., $q = T_1^{n(q)}(q) = T_2^{m(q)}(q)$. From (2),

$$d(p, q) = d(T_1^{n(p)}(p), \; T_2^{m(q)}(q) \leqq k \max\{d(p, q), 0, d(q, p)\},$$

which implies $p = q$. The condition $p = T_1^{n(p)}(p)$ implies $T_1(p) = T_1^{n(p)}(T_1(p))$, so that $T_1(p)$ is also a periodic point of T_1. From the uniqueness of p, $p = T_1(p)$. Similarly $T_2(p) = p$.

COROLLARY 1. *Let T be a self-mapping of X such that there exists a positive real number k, $0 < k < 1$ such that, for each x, $y \in X$ there exists a positive integer $n(x)$ such that*

$$d(T^{n(x)}(x), \; T^{n(y)}(y)) \leqq k \max\{d(x, y), \; d(x, T^{n(x)}(x)), \; d(y, T^{n(y)}(y)), \; [d(x, T^{n(y)}(y)) + d(y, T^{n(x)}(x)]/2\}.$$

Then T has a unique fixed point in X.

Proof. In Theorem 1 set $T_1 = T_2$, $m(y) = n(y)$.

COROLLARY 2. *Let $\{f_k\}$ be a sequence of self-mappings of X*

satisfying

$$d(f_i^{n(x)}(x)\ ,\quad f_j^{n(y)}(y)) \leqq k \max\{d(x, y)\ ,\quad d(x, f_i^{n(x)}(x))\ ,\\ d(y, f_j^{n(y)}(y))\ ,\quad [d(x, f_j^{n(y)}(y)) + d(y, f_j^{n(x)}(x))]\}$$

for each x, $y \in X$, *each* i, $j = 1, 2, \cdots$. *Then there exists a unique common fixed point.*

THEOREM 2. *Let* $\{f_k\}$ *be a sequence of continuous functions satisfying: there exists a positive constant* k, $0 < k < 1$ *such that for each* x, $y \in X$ *there exists a positive integer* $n(x)$ *such that*

$$d(f_k^{n(x)}(x)\ ,\quad f_k^{n(x)}(y)) \leqq k \max\{d(x, y)\ ,\quad d(x, f_k^{n(x)}(x))\ ,\\ d(y, f_k^{n(x)}(y))\ ,\quad [d(x, f_k^{n(x)}(y)) + d(y, f_k^{n(x)}(x))]/2\}\ . \tag{5}$$

Suppose $\{f_k\}$ *tends pointwise to a continuous function* f. *Then* f *has a unique fixed point* p *and* $p_k \rightarrow p$, *where the* p_k *are the unique fixed points of* f_k.

Proof. In (5) take the limit as $k \rightarrow \infty$ and use the continuity of f, f_k, and d to obtain the result that f satisfies (5). From Corollary 1 f has a unique fixed point p. $d(p_k, p) = d(f_k^{n(pk)}(p_k), f^{n(pk)}(p)) \leqq d(f_k^{n(pk)}(p_k), f_k^{n(pk)}(p)) + d(f_k^{n(pk)}(p), f^{n(pk)}(p))$. From (5),

$$d(f_k^{n(pk)}(p_k)\ ,\quad f_k^{n(pk)}(p)) \leqq h \max\{d(p_k, p)\ ,\quad d(p, f_k^{n(pk)}(p))\}\ .$$

Therefore, $d(p_k, p) \leqq (1 - h)^{-1} d(f_k^{n(pk)}(p), p)$, which tends to zero as $k \rightarrow \infty$.

REMARKS. 1. In each of the results of this paper one can obviously weaken the contractive condition by replacing X with a subset B which is invariant under the mappings involved and which contains the closure of all of the iterates of some $x_0 \in B$.

2. Corollary 1 is a generalization of [4], which in turn generalizes [1], [2] and [5].

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Received January 21, 1977.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72 00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 71, No. 2 December, 1977