

# Pacific Journal of Mathematics

**OPERATORS INVERTIBLE MODULO THE WEAKLY  
COMPACT OPERATORS**

KUNG-WEI YANG

## OPERATORS INVERTIBLE MODULO THE WEAKLY COMPACT OPERATORS

KUNG-WEI YANG

**A continuous linear operator is a Fredholm operator if and only if it is invertible modulo the compact operators. In this note, we will generalize several theorems on Fredholm operators to theorems concerning operators invertible modulo the weakly compact operators.**

1. Preliminaries. We fix the following notation.

$C$  = the complex field

$B$  = the category of complex Banach spaces and continuous linear operators

$B(X, Y)$  = the Banach space of continuous linear operators from  $X$  to  $Y$  (with the sup norm  $|\cdot|$ )

$WK(X, Y)$  = the closed subspace of all weakly compact operators in  $B(X, Y)$

$X^* = B(X, C)$ , the conjugate space

$F^* = B(F, C)$ , the adjoint of  $F: X \rightarrow Y$

$I_X$  = the identity operator on  $X$

$\bar{X} = X^{**}/n_X(X)$ , where  $n_X: X \rightarrow X^{**}$  is the natural injection.

If  $F \in B(X, Y)$ , then the commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \xrightarrow{n_X} & X^{**} & \longrightarrow & \bar{X} \longrightarrow 0 \\
 & & \downarrow F & & \downarrow F^{**} & & \downarrow \bar{F} \\
 0 & \longrightarrow & Y & \xrightarrow{n_Y} & Y^{**} & \longrightarrow & \bar{Y} \longrightarrow 0
 \end{array}$$

uniquely defines an operator  $\bar{F} \in B(\bar{X}, \bar{Y})$ . (Here  $n_X, n_Y$  are the natural injections.)

We will need the following results (1.1)-(1.7) from [9].

1.1.  $X$  is reflexive if and only if  $\bar{X} = 0$ . [9, (3.1)]

1.2.  $F \in WK(X, Y)$  if and only if  $\bar{F} = 0$ . [9, (4.1)]

1.3.  $\bar{I}_X = I_{\bar{X}}$ . [9, (2.3)]

1.4. If  $E \in B(X, Y)$  and  $F \in B(Y, Z)$ , then  $\overline{FE} = \overline{FE}$ . [9, (2.3)]

1.5.  $|\bar{F}| \leq |F|$ . [9, (2.3)]

1.6. For any  $a, b \in C$  and  $E, F \in B(X, Y)$ ,  $\overline{aE + bF} = a\bar{E} + b\bar{F}$ . [9, (2.4)]

1.7. There exists a natural topological isomorphism  $N_X: (\bar{X})^* \rightarrow (\bar{X}^*)$ . i.e., given any  $F \in B(X, Y)$ , the diagram

$$\begin{array}{ccc}
 (\bar{X})^* & \xrightarrow{N_X} & \bar{X}^* \\
 (\bar{F})^* \uparrow & & (\bar{F}^*) \uparrow \\
 (\bar{Y})^* & \xrightarrow{N_Y} & \bar{Y}^*
 \end{array}$$

is commutative. [9, (2.8)] (The naturality is not explicitly stated in [9].)

**THEOREM 1.8.** *If  $E \in B(X, X_1)$  and  $F \in B(Y, Y_1)$ , then  $\overline{E \oplus F} = \bar{E} \oplus \bar{F}$ , where  $\oplus$  denotes direct sum.*

*Proof.* Use  $(E \oplus F)^{**} = E^{**} \oplus F^{**}$  and the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X \oplus Y & \longrightarrow & X^{**} \oplus Y^{**} & \longrightarrow & \bar{X} \oplus \bar{Y} \longrightarrow 0 \\
 & & E \oplus F \downarrow & & E^{**} \oplus F^{**} \downarrow & & \bar{E} \oplus \bar{F} \downarrow \\
 0 & \longrightarrow & X_1 \oplus Y_1 & \longrightarrow & X_1^{**} \oplus Y_1^{**} & \longrightarrow & \bar{X}_1 \oplus \bar{Y}_1 \longrightarrow 0.
 \end{array}$$

**2. The operators invertible modulo the weakly compact operators.** An operator  $F \in B(X, Y)$  is *left (right) invertible modulo the weakly compact operators* if there exists an operator  $E \in B(Y, X)$  such that  $\overline{EF} = I_{\bar{X}}$  ( $\overline{FE} = I_{\bar{Y}}$ ). An operator is *invertible modulo the weakly compact operators* if it is left and right invertible modulo the weakly compact operators. Notice that this condition is quite different from merely requiring  $\bar{F}$  to be invertible. We let  $\Psi_l(X, Y)$ ,  $\Psi_r(X, Y)$ , denote the set of all operators left, respectively right, invertible modulo the weakly compact operators and let  $\Psi(X, Y)$  denote the set of all operators invertible modulo the weakly compact operators.

**THEOREM 2.1.** *If  $E \in \Psi(X, Y)$  and  $F \in \Psi(Y, Z)$ . Then  $FE \in \Psi(X, Z)$ .*

*Proof.* By assumption there exist  $E_1 \in B(Y, X)$ ,  $F_1 \in B(Z, Y)$  such that  $\overline{E_1 E} = I_{\bar{X}}$ ,  $\overline{E E_1} = I_{\bar{Y}}$ ,  $\overline{F_1 F} = I_{\bar{X}}$ ,  $\overline{F F_1} = I_{\bar{Z}}$ . Clearly,

$$\overline{(E_1 F_1)(FE)} = I_{\bar{X}}$$

and  $\overline{(FE)(E_1 F_1)} = I_{\bar{Z}}$ .

**THEOREM 2.2.** *Let  $E \in B(X, Y)$ ,  $F \in B(Y, Z)$ . Assume  $FE \in \Psi(X, Z)$ . Then,*

- (1)  $E \in \Psi(X, Y)$  if and only if  $F \in \Psi(Y, Z)$ ;
- (2) If  $F \in \Psi_l(Y, Z)$ , then  $E \in \Psi(X, Y)$  and  $F \in \Psi(Y, Z)$ ;
- (3) If  $E \in \Psi_r(X, Y)$ , then  $E \in \Psi(X, Y)$  and  $F \in \Psi(Y, Z)$ .

*Proof.* By assumption, there exist  $G \in B(Z, X)$  such that  $\overline{GFE} = I_{\bar{X}}$  and  $\overline{FEG} = I_{\bar{Z}}$ .

(1) If  $E \in \Psi(X, Y)$ , then there exists  $E_1 \in B(Y, X)$  such that  $\overline{E_1E} = I_{\bar{X}}$  and  $\overline{EE_1} = I_{\bar{Y}}$ . Hence  $\overline{EGF} = I_{\bar{Y}}$  and  $\overline{F(EG)} = I_{\bar{Z}}$ . This means  $F \in \Psi(Y, Z)$ . The implication in the other direction is proved similarly.

(2) If  $F \in \Psi_i(Y, Z)$ , then there exists  $F_1 \in B(Z, Y)$  such that  $\overline{F_1F} = I_{\bar{Y}}$ . This clearly implies  $\overline{(GF)E} = I_{\bar{X}}$ ,  $\overline{E(GF)} = I_{\bar{Y}}$  and  $\overline{F(EG)} = I_{\bar{Z}}$ ,  $\overline{(EG)F} = I_{\bar{Y}}$ . Hence,  $E \in \Psi(X, Y)$  and  $F \in \Psi(Y, Z)$ .

(3) is proved similarly.

**THEOREM 2.3.** *Let  $F \in B(X, Y)$ . If there exist  $E_1, E_2 \in B(Y, X)$  such that  $E_1F$  and  $FE_2$  are invertible modulo the weakly compact operators, then  $F \in \Psi(X, Y)$ .*

*Proof.* Since  $E_1F$  and  $FE_2$  are invertible modulo the weakly compact operators, there exist  $G_1 \in B(X, X)$ ,  $G_2 \in B(Y, Y)$  such that  $\overline{G_1(E_1F)} = I_{\bar{X}}$  and  $\overline{(FE_2)G_2} = I_{\bar{Y}}$ . Hence  $F \in \Psi(X, Y)$ .

**THEOREM 2.4.** *If  $F \in \Psi(X, Y)$ , then  $F^* \in \Psi(Y^*, X^*)$ .*

*Proof.* Let  $F \in \Psi(X, Y)$ . Then there exists  $E \in B(Y, X)$  such that  $\overline{EF} = I_{\bar{X}}$  and  $\overline{FE} = I_{\bar{Y}}$ . By 1.7,  $\overline{E^*F^*} = (N_Y(\bar{E})^*N_{\bar{X}}^{-1})(N_X(\bar{F})^*N_{\bar{Y}}^{-1}) = I_{\bar{Y}^*}$  and  $\overline{F^*E^*} = (N_X(\bar{F})^*N_{\bar{Y}}^{-1})(N_Y(\bar{E})^*N_{\bar{X}}^{-1}) = I_{\bar{X}^*}$ . Hence  $F^* \in \Psi(Y^*, X^*)$ .

**THEOREM 2.5.** *If  $F \in \Psi(X, Y)$  and  $K \in WK(X, Y)$ , then  $F + K \in \Psi(X, Y)$ .*

*Proof.*  $\overline{F + K} = \bar{F} + \bar{K} = \bar{F}$ .

As is shown in [9, Theorem 5.10], if the Banach spaces  $X$  and  $Y$  enjoy the property that every closed reflexive subspace of  $X$  is complemented and every closed subspace of  $Y$  with reflexive quotient is complemented, then every generalized Fredholm operator is invertible modulo the weakly compact operators.

There are, however, other kinds of operators invertible modulo the weakly compact operators. Let  $X$  be any Banach space. Let  $U: X \rightarrow X$  be an invertible operator, and  $K \in WK(X, X)$ . Then, clearly,  $U + K \in \Psi(X, X)$ .

To construct a nontrivial operator invertible modulo the weakly compact operators which is not a generalized Fredholm operator, we start an operator  $F = U + K \in \Psi(X, X)$  such as the one constructed above. We choose a reflexive Banach space  $Y$  and an operator  $G \in B(Y, Y)$  which does not have a closed range. If we form the direct

sum  $F \oplus G: X \oplus Y \rightarrow X \oplus Y$ , we see, by Theorem 1.8,  $\overline{F \oplus G} = \overline{F} \oplus \overline{G} = \overline{F} \oplus 0$ . Hence  $F \oplus G$  is invertible modulo the weakly compact operators but it is definitely not a generalized Fredholm operator because it does not have a closed range. (Also see [3, V. 2.6].)

**3. The operators left (right) invertible modulo the weakly compact operators.**

**THEOREM 3.1.** (1) *If  $F \in \Psi_i(X, Y)$ ,  $K \in WK(X, Y)$ , then  $F + K \in \Psi_i(X, Y)$ .*

(2) *If  $E \in \Psi_i(X, Y)$ ,  $F \in \Psi_i(Y, Z)$ , then  $FE \in \Psi_i(X, Z)$ .*

(3) *If  $E \in B(X, Y)$ ,  $F \in B(Y, Z)$  and  $FE \in \Psi_i(X, Z)$ , then  $E \in \Psi_i(X, Y)$ .*

(4) *If  $F \in \Psi_r(X, Y)$ ,  $K \in WK(X, Y)$ , then  $F + K \in \Psi_r(X, Y)$ .*

(5) *If  $E \in \Psi_r(X, Y)$ ,  $F \in \Psi_r(Y, Z)$ , then  $FE \in \Psi_r(X, Z)$ .*

(6) *If  $E \in B(X, Y)$ ,  $F \in B(Y, Z)$  and  $FE \in \Psi_r(X, Z)$ , then  $F \in \Psi_r(Y, Z)$ .*

*Proof.* (1)  $\overline{F + K} = \overline{F} + \overline{K} = \overline{F}$ .

(2) If  $E \in \Psi_i(X, Y)$  and  $F \in \Psi_i(Y, Z)$ , then there exist  $E_1 \in B(Y, X)$ ,  $F_1 \in B(Z, Y)$  such that  $\overline{E_1 E} = I_{\overline{X}}$  and  $\overline{F_1 F} = I_{\overline{Z}}$ . Clearly,  $\overline{E_1 F_1 F E} = I_{\overline{X}}$ . Hence  $FE \in \Psi_i(X, Z)$ .

(3) If  $FE \in \Psi_i(X, Z)$ , then there exists  $G \in B(Z, X)$  such that  $\overline{G(FE)} = I_{\overline{X}}$ . Hence  $E \in \Psi_i(X, Y)$ .

(4), (5), (6) are similarly proved.

**THEOREM 3.2.** (1) *If  $F \in \Psi_i(X, Y)$ , then  $F^* \in \Psi_r(Y^*, X^*)$ .*

(2) *If  $F \in \Psi_r(X, Y)$ , then  $F^* \in \Psi_i(Y^*, X^*)$ .*

*Proof.* (1) If  $F \in \Psi_i(X, Y)$ , then there exists  $F_1 \in B(Y, X)$  such that  $\overline{F_1 F} = I_{\overline{X}}$ . Hence  $(\overline{F})^*(\overline{F_1})^* = I_{(\overline{X})^*}$ . By 1.7,

$$(N_X(\overline{F})^* N_{\overline{Y}^*}^{-1})(N_Y(\overline{F_1})^* N_{\overline{X}^*}^{-1}) = I_{\overline{X}^*},$$

whence  $\overline{F^* F_1^*} = I_{\overline{X}^*}$ . This shows  $F^* \in \Psi_r(Y^*, X^*)$ .

(2) is proved similarly.

**4. Perturbation.** Let  $A$  be a Banach algebra with identity (1),  $A^\circ$  be the group of invertible elements in  $A$ ,  $A_i^\circ(A_r^\circ)$  be the set of all left(right) invertible elements of  $A$ . Let

$R(A)$  = the radical of  $A$

$$= \{r \in A \mid 1 + ar \in A^\circ \text{ for every } a \in A\} \text{ [6, p. 163]}$$

$$= \{r \in A \mid 1 + ar \in A^\circ \text{ for every } a \in A^\circ\} \text{ [5, p. 4]}$$

$Q(A)$  = the set of all quasi-nilpotent (topologically nilpotent [7, p. 12]) elements of  $A$

$$= \{q \in A \mid 1 + kq \in A^\circ \text{ for every } k \in C\} \text{ [4, p. 699]}$$

$$= \{q \in A \mid |q^n|^{1/n} \rightarrow 0 \text{ as } n \rightarrow \infty\}. \text{ [1, p. 23]}$$

For a semigroup  $S$  in  $A$ , let

$$P(S) = \{a \in A \mid a + S \subset S\}.$$

The following theorem is proved in [5].

**THEOREM 4.1.**  $P(A^\circ) = P(A_i^\circ) = P(A_r^\circ) = R(A)$ .

**THEOREM 4.2.**

$Q(A) = \{q \in A \mid I + aq \in A^\circ \text{ for all } a \in A^\circ \text{ such that } aq = qa\}$ .

*Proof.* The set on the right is obviously contained in  $Q(A)$ . Now take an element  $q \in Q(A)$ , and let  $a \in A^\circ$  be such that  $aq = qa$ . Clearly,  $|(aq)^n|^{1/n} \rightarrow 0$  as  $n \rightarrow \infty$ . Hence  $aq \in Q(A)$ . So  $1 + aq \in A^\circ$ . This shows that  $q$  is in the set on the right hand side.

**THEOREM 4.3.** (1) *Let  $q \in A$ . Then,  $q \in Q(A)$  if and only if for all  $a \in A^\circ$  such that  $aq = qa, a + q \in A^\circ$ ;*

(2) *If  $q_1, q_2 \in Q(A)$  and  $q_1q_2 = q_2q_1$ , then  $q_1 + q_2 \in Q(A)$ .*

*Proof.* (1) Clearly,  $aq = qa$  is equivalent to  $a^{-1}q = qa^{-1}$ . Hence,  $q \in Q(A) \Leftrightarrow 1 + aq \in A^\circ$  for all  $a \in A^\circ$  such that  $aq = qa \Leftrightarrow 1 + a^{-1}q \in A^\circ$  for all  $a \in A^\circ$  such that  $aq = qa \Leftrightarrow a + q = a(1 + a^{-1}q) \in A^\circ$  for all  $a$  such that  $aq = qa$ .

(2) Let  $q_1, q_2 \in Q(A)$  be such that  $q_1q_2 = q_2q_1$ . Let  $k$  be an arbitrary complex number. Clearly,  $(1 + kq_1) \in A^\circ$  and  $kq_2 \in Q(A)$ . Since  $(1 + kq_1)(kq_2) = (kq_2)(1 + kq_1)$ , by (1),  $1 + k(q_1 + q_2) \in A^\circ$ . Since  $k$  is arbitrary,  $q_1 + q_2 \in Q(A)$ .

We remark that (2) also follows from [7, Th. 1.4.1(v), p. 10].

Now we shall apply these theorems to the specific problem of perturbation of operators (left, right) invertible modulo the compact operators.

Let  $X$  be a Banach space. Let  $B(X) = B(X, X)$ ,  $WK(X) = WK(X, X)$ ,  $\Psi(X) = \Psi(X, X)$ ,  $\Psi_l(X) = \Psi_l(X, X)$ ,  $\Psi_r(X) = \Psi_r(X, X)$ ,  $\bar{B}(X) = B(X)/WK(X)$  and  $\tau: B(X) \rightarrow \bar{B}(X)$  be the natural projection. Notice that  $\bar{B}(X)$  can be considered as a subalgebra of  $B(\bar{X})$  [9, (5.11)] and we have  $\Psi(X) = \tau^{-1}[(\bar{B}(X))^\circ]$ ,  $\Psi_l(X) = \tau^{-1}[(\bar{B}(X))_l^\circ]$ ,  $\Psi_r(X) = \tau^{-1}[(\bar{B}(X))_r^\circ]$ .

**THEOREM 4.4.**  $P(\Psi(X)) = P(\Psi_l(X)) = P(\Psi_r(X)) = \tau^{-1}[R(\bar{B}(X))]$ .

*Proof.* Use Theorem 4.1 and the above remark.

**COROLLARY 4.5.** *If  $R(\bar{B}(X)) = 0$ , then  $P(\Psi(X)) = P(\Psi_i(X)) = P(\Psi_r(X)) = WK(X)$ .*

Since  $\bar{B}(X)$  can be considered as a subalgebra of  $B(\bar{X})$  and since  $R(B(\bar{X})) = 0$  [4, p. 702],  $R(\bar{B}(X)) = 0$  if  $\bar{B}(X) = B(\bar{X})$ .

Now, let  $\Omega(X) = \tau^{-1}[Q(\bar{B}(X))]$ . The classical counterpart of  $\Omega(X)$  is the set of all Riesz operators [2, p. 323] [8].

**THEOREM 4.6.** *Let  $E \in B(X)$ . Then,  $E \in \Omega(X)$  if and only if  $E + F \in \Psi(X)$  for all  $F \in \Psi(X)$  such that  $\overline{EF} = \overline{FE}$ .*

*Proof.* Apply Theorem 4.3.1.

**THEOREM 4.7.** *If  $E_1, E_2 \in \Omega(X)$  and  $\overline{E_1E_2} = \overline{E_2E_1}$ , then  $E_1 + E_2 \in \Omega(X)$ .*

*Proof.* Apply Theorem 4.3.2.

Note that if we apply Theorem 4.3 to the case  $A = B(X)/K(X)$ , where  $K(X)$  is the closed 2-sided ideal of all compact operators, then we obtain Theorems 9, 12, 13, of [8]. They are the classical "relatives" of Theorems 4.6 and 4.7.

It would be an interesting problem to characterize the operators in  $\Psi(X, Y)$  ( $\Psi_i(X, Y)$ ,  $\Psi_r(X, Y)$ ) intrinsically.

We would like to thank Professor Martin Schechter for his help.

## REFERENCES

1. F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York, 1973.
2. J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York, 1967.
3. S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill, New York, 1966.
4. E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ., 1957.
5. A. Lebow and M. Schechter, *Semigroups of operators and measures of noncompactness*, J. Functional Analysis, **7** (1971), 1-26.
6. M. A. Naimark, *Normed Rings*, P. Noordhoff, N. V. Groningen, 1959.
7. C. E. Rickart, *General Theory of Banach Algebras*, V. Nostrand, Princeton, N. J., 1960.
8. M. Schechter, *Riesz operators and Fredholm perturbations*, Bull. Amer. Math. Soc., **74** (1968), 1139-1144.
9. K.-W. Yang, *The generalized Fredholm operators*, Trans. Amer. Math. Soc., (to appear).

Received February 11, 1976 and in revised form February 4, 1977.

WESTERN MICHIGAN UNIVERSITY  
KALAMAZOO, MI 49008

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)

University of California  
Los Angeles, CA 90024

CHARLES W. CURTIS

University of Oregon  
Eugene, OR 97403

C. C. MOORE

University of California  
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University  
Stanford, CA 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA, RENO  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF HAWAII  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).

8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan



Krishnaswami Alladi and Paul Erdős, <i>On an additive arithmetic function</i> .....	275
James Bailey and Dale Rolfsen, <i>An unexpected surgery construction of a lens space</i> .....	295
Lawrence James Brenton, <i>On the Riemann-Roch equation for singular complex surfaces</i> .....	299
James Glenn Brookshear, <i>Projective ideals in rings of continuous functions</i> .....	313
Lawrence Gerald Brown, <i>Stable isomorphism of hereditary subalgebras of <math>C^*</math>-algebras</i> .....	335
Lawrence Gerald Brown, Philip Palmer Green and Marc Aristide Rieffel, <i>Stable isomorphism and strong Morita equivalence of <math>C^*</math>-algebras</i> ....	349
N. Burgoyne, Robert L. Griess, Jr. and Richard Lyons, <i>Maximal subgroups and automorphisms of Chevalley groups</i> .....	365
Yuen-Kwok Chan, <i>Constructive foundations of potential theory</i> .....	405
Peter Fletcher and William Lindgren, <i>On <math>w\Delta</math>-spaces, <math>w\sigma</math>-spaces and <math>\Sigma^\sharp</math>-spaces</i> .....	419
Louis M. Friedler and Dix Hayes Pettey, <i>Inverse limits and mappings of minimal topological spaces</i> .....	429
Robert E. Hartwig and Jiang Luh, <i>A note on the group structure of unit regular ring elements</i> .....	449
I. Martin (Irving) Isaacs, <i>Real representations of groups with a single involution</i> .....	463
Nicolas P. Jewell, <i>The existence of discontinuous module derivations</i> .....	465
Antonio M. Lopez, <i>The maximal right quotient semigroup of a strong semilattice of semigroups</i> .....	477
Dennis McGavran, <i><math>T^n</math>-actions on simply connected <math>(n + 2)</math>-manifolds</i> .....	487
Charles Anthony Micchelli and Allan Pinkus, <i>Total positivity and the exact <math>n</math>-width of certain sets in <math>L^1</math></i> .....	499
Barada K. Ray and Billy E. Rhoades, <i>Fixed point-theorems for mappings with a contractive iterate</i> .....	517
Fred Richman and Elbert A. Walker, <i>Ext in pre-Abelian categories</i> .....	521
Raymond Craig Roan, <i>Weak* generators of <math>H^\infty</math> and <math>l^1</math></i> .....	537
Saburou Saitoh, <i>The exact Bergman kernel and the kernels of Szegő type</i> ...	545
Kung-Wei Yang, <i>Operators invertible modulo the weakly compact operators</i> .....	559