Pacific Journal of Mathematics

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Vol. 72, No. 1 January 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol. 72, No. 1, 1977

WHEN IS A REPRESENTATION OF A BANACH *-ALGEBRA NAIMARK-RELATED TO A *-REPRESENTATION?

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Conditions are given which imply that a continuous Banach representation of a Banach *-algebra is Naimark-related to a *-representation of the algebra.

1. **Introduction.** The representation theory of a Banach algebra necessarily includes the notion of comparing representations to determine when they are essentially the same or related in important ways. Thus, if the algebra is a Banach *-algebra, then two *-representations are considered essentially the same if they are unitarily equivalent. When π is a representation of a Banach algebra on a Banach space X, we denote this Banach representation by the pair (π, X). A strong notion used to compare Banach representations is that of similarity.

DEFINITION. The Banach representations (π, X) and (φ, Y) of a Banach algebra A are similar if there exists a bicontinuous linear isomorphism V defined on X and mapping onto Y such that

$$\varphi(f)V = V\pi(f) \quad (f \in A) .$$

If (π, X) and (φ, Y) are similar, then the representation spaces X and Y are bicontinuously isomorphic. Thus the concept of similarity is limited to comparing representations that act on essentially the same Banach space. A notion that has proved useful in comparing representations that act on perhaps different representation spaces is that of Naimark-relatedness.

DEFINITION. Let (π, X) and (φ, Y) be Banach representations of a Banach algebra A. Then π and φ are Naimark-related if there exists a closed densely-defined one-to-one linear operator V defined on X with dense range in Y such that

(i) the domain of V is π-invariant, and
(ii) $\varphi(f)V\xi = V\pi(f)\xi$ for all $f \in A$ and all ξ in the domain of V.

The relation of being Naimark-related is in some ways a rather weak way of comparing representations. For this relation is not in general transitive [15, p. 242], and an irreducible representation can be Naimark-related to a reducible one [15, p. 243]. On the positive side, *-representations that are Naimark-related are unitarily equivalent [15, Prop. 4.3.1.4], and the relation is transitive on certain kinds of irreducible representations [15, p. 232]. Also, the concept has proved useful in comparing Banach representations of the algebra $L^1(G)$ for certain locally compact groups G.

In this paper we are concerned with the question: when is a Banach representation of a Banach *-algebra Naimark-related to a *-representation of the algebra? We are mainly interested in the cases where the algebra is either a B^*-algebra ($\equiv C^*$-algebra) or $L^1(G)$, for these cases occur in the theory of weakly continuous group representations of locally compact groups. Some results on this question are known, a few are classical. In the latter category is a theorem of A. Weil that every continuous finite dimensional representation of $L^1(G)$ is similar to a *-representation [8, p. 353]. Another well-known result is that if G is an ammenable locally compact group (in particular if G is abelian or compact), then every continuous representation of $L^1(G)$ on Hilbert space is similar to a *-representation [7, Theorem 3.4.1]. R. Gangoli has recently proved that if G is a locally compact motion group, then every continuous topologically completely irreducible Banach representation of $L^1(G)$ is Naimark-related to a *-representation [6, Cor. 1.3]. In the case of a B^*-algebra, J. Bunce has shown that for a GCR algebra (or more generally, a strongly ammenable algebra), every continuous representation of the algebra on Hilbert space is similar to a *-representation [3, Theorem 1]. The present author proves in [2, Cor. 1] that every continuous irreducible representation of a B^*-algebra on Hilbert space is Naimark-related to a *-representation. Also in [2] conditions are given which imply that such a representation is similar to a *-representation.

In this paper we give conditions on representations of certain Banach *-algebras that imply that the given representation is Naimark-related to a *-representation. The main results are Theorem 3 and its corollaries and Theorem 7. Among the results we prove are: any cyclic representation of a separable B^*-algebra on Hilbert space is Naimark-related to a *-representation [§ 4, Corollary 4]; for unimodular second countable locally compact groups, any weakly continuous bounded irreducible group representation which has a nonzero square integrable coefficient lifts to a representation of $L^1(G)$ which is Naimark-related to a *-representation [§ 4, Corollary 6]; and under very general conditions, a finite dimensionally spanned representation of a Banach *-algebra is Naimark-related to a *-representation [§ 5, Theorem 7].

2. Notation and a basic construction.

Throughout this paper

A is a Banach *-algebra. The Gelfand-Naimark pseudonorm γ on A is defined by

$$\gamma(f) = \sup\{||\varphi(f)||\}$$

where the sup is taken over all *-representations φ of A on Hilbert space. In general $\gamma(f)$ is an algebra pseudonorm with the property that $\gamma(f^*f) = \gamma(f)^2$ for all $f \in A$ [12]. When γ is a norm, then A is called an A^*-algebra. In this case we denote by $\bar{A}$ the completion of A with respect to this norm. Then $\bar{A}$ is a B^*-algebra. We use the standard meanings of state and pure state of A. If α is a state of A, then the left kernel of α is the left ideal

$$K_\alpha = \{f \in A: \alpha(f^*f) = 0\} .$$

We use the notions of modular maximal left ideal, primitive ideal, and Jacobson semisimplicity as in C. Rickart's book [14]. If M is a left ideal of A, then $A - M$ is the usual quotient space of A modulo M. We denote the elements of $A - M$ by $f + M$ where $f \in A$. If M is closed, then $A - M$ is a Banach space in the quotient norm

$$||f + M|| = \inf\{||f + g||: g \in M\} .$$

Let π be a representation of A on a Banach space X. We often designate such a pair by (π, X). The representation (π, X) is irreducible provided that the only *closed* π-invariant subspaces of X are $\{0\}$ and X. It is algebraically irreducible provided that the only π-invariant subspaces of X are $\{0\}$ and X. A representation (π, X) is essential if whenever $\xi \in X$, $\xi \neq 0$, then there exists $f \in A$ such that $\pi(f)\xi \neq 0$.

If V is a linear operator with domain and range in given linear spaces, then we use the notation $\mathscr{D}(V)$, $\mathscr{N}(V)$, and $\mathscr{R}(V)$ for the domain of V, null space of V, and the range of V, respectively.

Now we describe a basic construction which occurs frequently in what follows. In (I) and (II) below, (π, X) is a given Banach representation of A, and under the appropriate hypothesis, a *-representation of A is formed which is closely related to π. Then (III) deals with the case where the intertwining operator which is involved has a closure.

(I). Assume $\xi_0 \in X$. If

$$\{f \in A: \pi(f)\xi_0 = 0\} = K_\alpha$$

for some state α of A, then

$$\langle \pi(f)\xi_0, \pi(g)\xi_0 \rangle = \alpha(g^*f) \qquad (g, f \in A)$$

defines an inner-product on $\pi(A)\xi_0$ with the property that

$$\langle \pi(h)\xi, \eta \rangle = \langle \xi, \pi(h^*)\eta \rangle \qquad (\xi, \eta \in \pi(A)\xi_0, h \in A) .$$

Proof. Assume that $\pi(f_1)\xi_0 = \pi(f_2)\xi_0$ and $\pi(g_1)\xi_0 = \pi(g_2)\xi_0$. Then by hypothesis $f_1 - f_2 \in K_\alpha$ and $g_1 - g_2 \in K_\alpha$. It follows that $\alpha(g_1^* f_1) = \alpha(g_2^* f_2)$, and therefore the form is well-defined. That the form is an inner product is clear.

Now assume that $h, f, g \in A$. Then

$$\begin{aligned} \langle \pi(h)\pi(f)\xi_0, \pi(g)\xi_0 \rangle &= \alpha(g^* h f) \\ &= \alpha((h^* g)^* f) = \langle \pi(f)\xi_0, \pi(h^*)\pi(g)\xi_0 \rangle . \end{aligned}$$

(II). Let X_0 be a π-invariant subspace of X with $\langle \cdot, \cdot \rangle$ an inner product on X_0 such that

$$\langle \pi(f)\xi, \eta \rangle = \langle \xi, \pi(f^*)\eta \rangle \qquad (\xi, \eta \in X_0, f \in A) .$$

Let H_0 denote the inner-product space $(X_0, \langle \cdot, \cdot \rangle)$, and define φ_0 on H_0 by

$$\varphi_0(f)\xi = \pi(f)\xi \qquad (\xi \in H_0, f \in A) .$$

Let H be the Hilbert space completion of H_0. Define a linear operator $U: X \rightarrow H$ with $\mathscr{D}(U) = X_0$ by $U\xi = \xi$ for $\xi \in X_0$. Then

(1) φ_0 has a unique extension to a *-representation φ on H, and

(2) $\mathscr{D}(U)$ is π-invariant and $\varphi(f)U\xi = U\pi(f)\xi$ $(\xi \in \mathscr{D}(U), f \in A)$.

Proof. By definition φ_0 is a *-representation of A on the inner-product space H_0. Then by a result of T. Palmer $\varphi_0(f)$ is a bounded operator on H_0 for each $f \in A$ and $f \rightarrow \varphi_0(f)$ is a continuous map of A into the algebra of bounded linear operators on H_0 [12, Proposition 5]. Thus, (1) holds. Part (2) follows immediately from the definitions given.

(III). Assume that (π, X) and (φ, Y) are continuous Banach representations of A. Assume that $U: X \rightarrow Y$ is a linear operator with $\mathscr{D}(U)$ π-invariant and

$$\varphi(f)U\xi = U\pi(f)\xi \qquad (\xi \in \mathscr{D}(U), f \in A) .$$

Furthermore assume that U has closure $\bar{U}$. Then $\mathscr{D}(\bar{U})$ is π-invariant and

$$\varphi(f)\bar{U}\xi = \bar{U}\pi(f)\xi \qquad (\xi \in \mathscr{D}(\bar{U}), f \in A) .$$

Proof. Assume that $\xi \in \mathscr{D}(\bar{U})$. Then by the definition of $\bar{U}$ there exists $\{\xi_n\} \subset \mathscr{D}(U)$ such that $\xi_n \to \xi$ and $U\xi_n \to \bar{U}\xi$. Then $\pi(f)\xi_n \to \pi(f)\xi$ and $U\pi(f)\xi_n = \varphi(f)U\xi_n \to \varphi(f)\bar{U}\xi$. Again, by the definition of $\bar{U}$ we have

$$\pi(f)\xi \in \mathscr{D}(\bar{U}) \quad \text{and} \quad \bar{U}\pi(f)\xi = \varphi(f)\bar{U}\xi .$$

3. **Symmetry and Naimark-relatedness.** In this paper we are basically concerned with conditions that imply that a given Banach representation of A is Naimark-related to a *-representation. In this regard it is natural to ask what Banach algebras have the property that every continuous irreducible Banach representation is Naimark-related to a *-representation? It is known that every irreducible representation of a B^*-algebra on Hilbert space is Naimark-related to a *-representation [2, Cor. 1]. The next result shows that if a Banach *-algebra A has the property that every algebraically irreducible Banach representation is Naimark-related to a *-representation, then A must be symmetric. In fact, the symmetry of A can be characterized in this fashion. The symmetry of a Banach *-algebra has other implications for the representation theory of the algebra; see Corollaries 5 and 11.

THEOREM 1. *Let A be a Banach *-algebra. The following are equivalent*:

(1) *A is symmetric*;

(2) *every modular maximal left ideal of A is the left kernel of some state of A* (which in this case may be chosen to be a pure state);

(3) *every algebraically irreducible Banach representation of A is Naimark-related to a *-representation of A* (which in this case may be chosen to be irreducible).

Proof. By [13, Theorem] (1) and (2) are equivalent.

Assume that (2) holds. Let (π, X) be an algebraically irreducible representation of A. Fix $\xi_0 \in X$, $\xi_0 \neq 0$. A simple algebraic argument verifies that $M = \{f \in A : \pi(f)\xi_0 = 0\}$ is a modular maximal left ideal of A. Therefore by hypothesis there exists a state α of A such that $M = K_\alpha$ (and α may be chosen to be a pure state). Define an inner-product $\langle \cdot, \cdot \rangle$ on $X = \pi(A)\xi_0$ as in (I), i.e.,

$$\langle \pi(f)\xi_0, \pi(g)\xi_0 \rangle = \alpha(g^*f) \qquad (f, g \in A) .$$

Let (φ, H) be the *-representation of A, and let U be the intertwining operator constructed as in (II).

Consider the map $\psi: A - M \to X$ defined by

$$\psi(f + M) = \pi(f)\xi_0 \qquad (f \in A).$$

Clearly ψ is continuous, and therefore bicontinuous by the Open Mapping Theorem. Hence there exists $B > 0$ such that for all $f \in A$

$$\inf \{\|f + g\|: g \in M\} = \|f + M\| \leqq B\|\pi(f)\xi_0\|_X .$$

If $f \in A$, $g \in M$, then

$$\|U\pi(f)\xi_0\|_H^2 = \alpha((f + g)^*(f + g)) \leqq \gamma(f + g)^2 \leqq \|f + g\|^2 .$$

Taking the infimum over all $g \in M$ we have for all $f \in A$

$$\|U\pi(f)\xi_0\|_X \leqq \|f + M\| \leqq B\|\pi(f)\xi_0\|_X .$$

This proves that $U: X \to H$ is bounded on X and is therefore closed. It follows that π is Naimark-related to φ. This verifies that (2) implies (3).

Conversely, assume that (3) holds. Let M be a modular maximal left ideal of A. Let π be the algebraically irreducible representation of A on $A - M$ given by

$$\pi(f)(g + M) = fg + M \qquad (f, g \in A).$$

By (3) there exists a *-representation (φ, H) of A Naimark-related to π (φ may be chosen to be irreducible). Let U be a closed one-to-one linear operator with π-invariant domain in $A - M$ such that

$$\varphi(f)U\xi = U\pi(f)\xi \qquad (\xi \in \mathscr{D}(U), f \in A).$$

Since π is algebraically irreducible and $\mathscr{D}(U)$ is π-invariant, we have $\mathscr{D}(U) = A - M$. Fix $u_0 \in A$ such that $fu_0 - f \in M$ for all $f \in A$. Define α on A by

$$\alpha(f) = (\varphi(f)U(u_0 + M), U(u_0 + M)) \qquad (f \in A).$$

Clearly, α is a positive linear functional on A. Also,

$$\begin{aligned} f \in M &\Longleftrightarrow f(u_0 + M) = 0 \\ &\Longleftrightarrow U\pi(f)(u_0 + M) = 0 \\ &\Longleftrightarrow \varphi(f)U(u_0 + M) = 0 \\ &\Longleftrightarrow \alpha(f^*f) = 0 . \end{aligned}$$

Thus, $M = K_\alpha$. Finally, some constant multiple of α is a state of A, and if φ is irreducible, then this multiple of α is a pure state.

4. **Representations on a Hilbert space.** In this section we

investigate a variety of conditions on A and on a representation (π, H) of A, H a Hilbert space, that imply that π is Naimark-related to a *-representation of A. In order to construct a *-representation of A by the methods of (I) and (II), some reasonable hypothesis is necessary to insure that certain closed left ideals of A are left kernels of a state of A. The next lemma provides a useful tool in this regard.

LEMMA 2. *Let A be a separable A^*-algebra. Let M be a γ-closed left ideal of A. Then there exists a state α of A such that $M = K_\alpha$.*

Proof. Let $\bar{M}$ be the closure of M in $\bar{A}$. Since $\gamma(f) \leqq ||f||$ for all $f \in A$, $\bar{A}$ is separable. If there exists a state $\bar{\alpha}$ on $\bar{A}$ such that $\bar{M} = K_{\bar{\alpha}}$, then $M = K_\alpha$ where α is the restriction of $\bar{\alpha}$ to A. Thus we may assume that A is a separable B^*-algebra and that M is a closed left ideal of A.

Let Δ be the set of all pure states ω of A such that $M \subset K_\omega$. Define for all $f + M \in A - M$

$$||f + M||_\Delta = \sup \{\omega(f^*f)^{1/2} : \omega \in \Delta\} .$$

Since for every state ω we have

$$\omega((f + g)^*(f + g))^{1/2} \leqq \omega(f^*f)^{1/2} + \omega(g^*g)^{1/2} \qquad (f, g \in A) ,$$

it follows that

$$||(f + g) + M||_\Delta \leqq ||f + M||_\Delta + ||g + M||_\Delta \qquad (f, g \in A) .$$

Now because A is a B^*-algebra we have $M = \bigcap \{K_\omega : \omega \in \Delta\}$ [5, Théorème 2.9.5]. This fact and the inequality above prove that $||\cdot||_\Delta$ is a norm on $A - M$. Also, $||f + M||_\Delta \leqq ||f||$ by [5, Prop. 2.7.1], and therefore $A - M$ is separable in the norm $||\cdot||_\Delta$. Choose $\{f_n + M : n \geqq 1\}$ a countable dense subset of $\{g + M : ||g + M||_\Delta = 1\}$. For each $n \geqq 1$ choose $\omega_n \in \Delta$ such that $\omega_n(f_n^* f_n) > 1/2$. Suppose there exists $g \in \bigcap_{n \geqq 1} K_{\omega_n}$ such that $g \notin M$. We may assume $||g + M||_\Delta = 1$. Take f_n such that

$$||(g - f_n) + M||_\Delta < \frac{1}{2} .$$

Then

$$\frac{1}{4} > ||(g - f_n) + M||_\Delta^2 \geqq \omega_n((g - f_n)^*(g - f_n)) = \omega_n(f_n^* f_n) > \frac{1}{2} .$$

This contradiction proves that $M = \bigcap_{n \geqq 1} K_{\omega_n}$. Finally, set $\alpha = \sum_{n=1}^{\infty} (1/2)^n \omega_n$. Then α is a state of A with $K_\alpha = M$.

Now we state and prove the main result of this section.

THEOREM 3. *Let π be a continuous essential representation of A on a Hilbert space H. Assume that either*

(1) *(π, H) is irreducible, and for some $\xi_0 \in H$, $\xi_0 \neq 0$, $\{g \in A: \pi(g)\xi_0 = 0\}$ is the left kernel of a state of A, or*

(2) *there exists a dense π-invariant subspace H_0 of H which is the algebraic direct sum of subspaces of the form $\pi(A)\xi$ where $\xi \in H$, and every left ideal of the form $\{g \in A: \pi(g)\eta = 0\}$ is the left kernel of some state of A.*

*Then (π, H) is Naimark-related to a *-representation (φ, K) of A where K is a closed subspace of H.*

Proof. Under either of the hypotheses (1) or (2), we can use (I) to construct an inner-product $\langle \cdot, \cdot \rangle$ defined on a dense π-invariant subspace H_0 with the property that

$$\langle \pi(f)\xi, \eta \rangle = \langle \xi, \pi(f^*)\eta \rangle \qquad (\xi, \eta \in H, f \in A) .$$

In the case of (2), the, inner-product $\langle \cdot, \cdot \rangle$ is constructed by forming the sum of inner-products defined on the direct summands of H_0 of the form $\pi(A)\xi$. By [10, Theorem 1.27, p. 318, and Theorem 2.23, p. 331] there exists an operator U with $\mathscr{D}(U) = H_0$ and with closure $\bar{U}$ such that

$$\langle \xi, \eta \rangle = (U\xi, U\eta) \qquad (\xi, \eta \in H_0) .$$

For $f \in A$ define $\varphi_0(f)$ on $K_0 = UH_0$ by

$$\varphi_0(f)U\xi = U\pi(f)U^{-1}(U\xi) \qquad (\xi \in H_0) .$$

Then

$$\varphi_0(f)U\xi = U\pi(f)\xi \qquad (\xi \in H_0, f \in A) .$$

Also, for $\xi = U\xi_0$, $\eta = U\eta_0$ where $\xi_0, \eta_0 \in H_0$, we have

$$\begin{aligned}(\varphi_0(f)\xi, \eta) &= (U\pi(f)\xi_0, U\eta_0) \\ &= \langle \pi(f)\xi_0, \eta_0 \rangle \\ &= \langle \xi_0, \pi(f^*)\eta_0 \rangle \\ &= (U\xi_0, U\pi(f^*)U^{-1}(U\eta_0)) \\ &= (\xi, \varphi_0(f^*)\eta) .\end{aligned}$$

By [12, Prop. 5] there is a unique extension of φ_0 to a *-representation φ of A on K, the closure of K_0 in H. Then by (III) $\mathscr{D}(\bar{U})$ is π-invariant, and

$$\varphi(f)\bar{U}\xi = \bar{U}\pi(f)\xi \qquad (\xi \in \mathscr{D}(\bar{U}), f \in A) .$$

To complete the proof that (π, H) is Naimark-related to (φ, K) it remains to be shown that $\bar{U}$ is one-to-one on $\mathscr{D}(\bar{U})$. Since $\bar{U}$ is closed, $\mathscr{N}(\bar{U})$ is a closed subspace. If $\xi \in \mathscr{N}(\bar{U})$, then $\bar{U}\pi(f)\xi = \varphi(f)\bar{U}\xi = 0$ for all $f \in A$. Therefore $\mathscr{N}(\bar{U})$ is π-invariant. Assume that (1) holds. Then π being irreducible, it follows that $\mathscr{N}(\bar{U}) = \{0\}$.

Now assume that (2) holds. Let $\mathscr{T}$ be the collection of all inner-products $N(\xi, \eta)$ defined on a subspace $\mathscr{D}(N)$ of H such that

(i) $H_0 \subset \mathscr{D}(N)$,

(ii) $\mathscr{D}(N)$ is π-invariant, and

(iii) $N(\pi(f)\xi, \eta) = N(\xi, \pi(f^*)\eta)$ $(\xi, \eta \in \mathscr{D}(N), f \in A)$.

Partially order the nonempty collection $\mathscr{T}$ by $N_1 \leqq N_2$ provided that

$$\mathscr{D}(N_1) \subset \mathscr{D}(N_2) \quad \text{and} \quad N_1(\xi, \eta) = N_2(\xi, \eta) \qquad (\xi, \eta \in \mathscr{D}(N_1)) .$$

A straightforward Zorn's lemma argument establishes the existence of a maximal element N in $\mathscr{T}$. Following the argument in the first paragraph of the proof with N replacing $\langle\cdot, \cdot\rangle$ and $\mathscr{D}(N)$ replacing H_0, we can construct as before an operator U with closure $\bar{U}$ and a *-representation (φ, K) of A such that

$$N(\xi, \eta) = (U\xi, U\eta) \qquad (\xi, \eta \in \mathscr{D}(N)) ,$$

$\mathscr{D}(\bar{U})$ is π-invariant, and

$$\varphi(f)\bar{U}\xi = \bar{U}\pi(f)\xi \qquad (\xi \in \mathscr{D}(\bar{U}), f \in A) .$$

Suppose that $\bar{U}$ is not one-to-one. Choose $\eta_0 \in \mathscr{N}(\bar{U})$, $\eta_0 \neq 0$. By hypothesis exists a state α of A such that

$$K_\alpha = \{g \in A: \pi(g)\eta_0 = 0\} .$$

Now $\|\bar{U}\xi\|^2 = N(\xi, \xi)$ for $\xi \in \mathscr{D}(N)$, and therefore $\bar{U}$ is one-to-one on $\mathscr{D}(N)$. Thus, $\mathscr{D}(N) \cap \pi(A)\eta_0 = \{0\}$. Also note that $\pi(A)\eta_0 \neq \{0\}$ since π is essential. Let

$$\mathscr{D}(M) = \mathscr{D}(N) + \pi(A)\eta_0 .$$

Now by (I)

$$\langle\pi(f)\eta_0, \pi(g)\eta_0\rangle = \alpha(g^*f) \qquad (g, f \in A)$$

defines an inner-product on $\pi(A)\eta_0$ with properties (i), (ii), (iii) above. For $\xi, \eta \in \mathscr{D}(M)$, $\xi = \xi_1 + \xi_2$ and $\eta = \eta_1 + \eta_2$ where $\xi_1, \eta_1 \in \mathscr{D}(N)$, $\xi_2, \eta_2 \in \pi(A)\eta_0$, define

$$M(\xi, \eta) = N(\xi_1, \eta_1) + \langle\xi_2, \eta_2\rangle .$$

Then $M \in \mathscr{T}$, $M \geqq N$, and $M \neq N$. This contradicts the maximality of N. Thus, $\bar{U}$ must be one-to-one.

By Lemma 2 and Theorem 3 we have:

COROLLARY 4. *Let A be a separable B^*-algebra. If π is a continuous essential representation of A on a Hilbert space H, and there exists a π-invariant subspace H_0 having the property described in part* (2) *of Theorem* 3 (in particular, if π is cyclic), *then π is Naimark-related to a *-representation of A.*

COROLLARY 5. *Let A be a symmetric Banach *-algebra. If π is a continuous irreducible representation of A on a Hilbert space H, and π acts algebraically irreducibly on some π-invariant subspace $H_0 \subset H$, then π is Naimark-related to a *-representation of A.*

Proof. Fix $\xi_0 \in H_0$, $\xi_0 \neq 0$. Since π acts algebraically irreducibly on H_0, $\{g \in A: \pi(g)\xi_0 = 0\}$ is a modular maximal left ideal of A. By Theorem 1 this left ideal is the left kernel of a state of A. Thus Theorem 3 applies.

COROLLARY 6. *Let G be a unimodular locally compact group such that $L^1(G)$ is separable. Assume that π is a bounded weakly continuous irreducible representation of G on a Hilbert space H. Assume that there exist $\xi_0 \neq 0$, $\eta_0 \neq 0$ in H such that $x \rightarrow (\pi(x)\xi_0, \eta_0)$ is in $L^2(G)$. Then π is Naimark-related to a unitary representation of G.*

Proof. Let W be the subspace consisting of the vectors $\eta \in H$ such that $x \rightarrow (\pi(x)\xi_0, \eta) \in L^2(G)$. Note that if $\eta \in W$ and $y \in G$, then

$$x \longrightarrow (\pi(x)\xi_0, \pi(y)^*\eta) = (\pi(yx)\xi_0, \eta) \in L^2(G) .$$

Therefore W is invariant under the set of operators $\{\pi(y)^*: y \in G\}$. Thus $W^\perp$ is π-invariant. It follows that $W^\perp = \{0\}$, and hence that W is dense in H.

Now for each $\eta \in W$ let

$$g_\eta(y) = (\pi(y^{-1})\xi_0, \eta) \qquad (y \in G) .$$

Since G is unimodular, $g_\eta \in L^2(G)$ for all $\eta \in W$. Denote again by π the integrated form on $L^1(G)$ of the group representation π, that is, for $\xi, \eta \in H$ and $f \in L^1(G)$,

$$(\pi(f)\xi, \eta) = \int_G f(x)(\pi(x)\xi, \eta)dx .$$

Let $K = \{f \in L^1(G): \pi(f)\xi_0 = 0\}$. The set K is a closed left ideal of $L^1(G)$. We proceed to prove that K is γ-closed. Assume that $\{f_n\} \subset K$ and $\gamma(f_n - f) \to 0$. Since for $h \in L^1(G)$ and $g \in L^2(G)$

$$\gamma(h)\|g\|_2 \geqq \|h * g\|_2 ,$$

we have

$$(\sharp) \qquad (f_n - f) * g \to 0 \text{ in } L^2(G) \text{ whenever } g \in L^2(G) .$$

If h is a function on G and $x \in G$, then we use the notation

$$h_x(y) = h(xy) \qquad (y \in G) .$$

For $\eta \in W$ we have by $(\sharp)$ that

$$\begin{aligned}(f_n - f) * g_\eta(x) &= \int_G \{f_n(xy) - f(xy)\}(\pi(y)\xi_0, \eta)dy \\ &= (\{\pi((f_n)_x) - \pi(f_x)\}\xi_0, \eta) \\ &\longrightarrow 0 \quad \text{in} \quad L^2(G) .\end{aligned}$$

Now K is a closed left ideal of $L^1(G)$ and hence $(f_n)_x \in K$ for all $n \geqq 1$ and all $x \in G$. Thus $x \to (\pi(f_x)\xi_0, \eta)$ is 0 a.e. on G. Since this function is continuous on G, $(\pi(f_x)\xi_0, \eta) = 0$ for all $x \in G$. Then $(\pi(f)\xi_0, \eta) = 0$ for all $\eta \in W$, so that $\pi(f)\xi_0 = 0$. This proves that K is γ-closed. Therefore Lemma 2 and Theorem 3 imply the result.

5. Representations containing operators with finite dimensional range. Let (π, X) be a continuous Banach representation of A, let (φ, H) be a continuous *-representation of A, and assume that π is Naimark-related to φ. Then $\ker(\pi) = \ker(\varphi)$, and since φ is γ-continuous, it follows that $\ker(\pi)$ is γ-closed. In this section we prove a converse of this fact in the case where there are sufficiently many operators with finite dimensional range in the image of π. More precisely we hypothesize that π is finite dimensional spanned (FDS) in the sense of [15, p. 231].

THEOREM 7. *Let A be an A^*-algebra. Let (π, X) be a continuous Banach representation of A such that π is* FDS. *If* $\ker(\pi)$ *is γ-closed, then π is Naimark-related to a direct sum of irreducible *-representations of A.*

We begin the proof of Theorem 7 by proving several preliminary results, and also, since the proof depends heavily on results concerning Banach algebras with minimal left ideals, we briefly review the necessary material from that area.

Let A be a Jacobson semisimple (complex) Banach algebra.

Denote the complex number field by C. An element $e \in A$ is a minimal idempotent (abbreviation: m.i.) of A if $eAe = \{\lambda e: \lambda \in C\}$ [14, Cor. (2.1.6)]. Every minimal left ideal L of A has the form $L = Ae$ where e is a m.i. of A [14, Lemma (2.1.5)]. Furthermore, if A has an involution * which is proper ($f^*f = 0 \Rightarrow f = 0$) then the m.i. e above may be chosen such that $e = e^*$ [14, Lemma (4.10.1)]. The socle of A, denoted $\operatorname{soc}(A)$, is an ideal which is the algebraic sum of all the minimal left ideals of A or $\{0\}$ if A has no minimal left ideals [14, p. 46]. Also, $\operatorname{soc}(A)$ is the direct algebraic sum of minimal ideals of A each of which has the form AeA for some m.i. e of A.

LEMMA 8. *Let A be an A^*-algebra, and let (π, X) be a continuous Banach representation of A. Assume that e is a* m.i. *of A with $e = e^*$. Fix $\xi \in \mathscr{R}(\pi(e))$, $\xi \neq 0$. Then*

(1) *π acts algebraically irreducibly on $\pi(A)\xi$;*

(2) *the form $\langle \cdot, \cdot \rangle$ defined on $\pi(A)\xi$ by the formula*

$$\langle \pi(f)\xi, \pi(g)\xi \rangle e = eg^*fe \qquad (f, g \in A)$$

is an inner-product on $\pi(A)\xi$, and

$$\langle \pi(g)\eta, \delta \rangle = \langle \eta, \pi(g^*)\delta \rangle \qquad (\eta, \delta \in \pi(A)\xi,\ g \in A);$$

(3) *if φ is defined on the Hilbert space completion H of $(\pi(A)\xi, \langle \cdot, \cdot \rangle)$ as in* (II), *then (φ, H) is an irreducible *-representation of A;*

(4) *if $\{\xi_1, \cdots, \xi_n\}$ is a basis for $\mathscr{R}(\pi(e))$, then $\pi(AeA)X$ is the algebraic direct sum of the spaces $\{\pi(A)\xi_k: 1 \leqq k \leqq n\}$.*

Proof. Assume that $\pi(f)\xi \neq 0$ and $\pi(g)\xi$ are given. Since Ae is a minimal left ideal [14, Lemma (2.1.8)], there exists $h \in A$ such that $ge = hfe$. Then $\pi(h)(\pi(f)\xi) = \pi(hfe)\xi = \pi(ge)\xi = \pi(g)\xi$. This proves (1).

Let $J = \{f \in A: \pi(f)\xi = 0\}$. Clearly $A(1-e) \subset J$. Then since $A(1-e)$ is a maximal left ideal, $A(1-e) = J$. If $\pi(f_1)\xi = \pi(f_2)\xi$ and $\pi(g_1)\xi = \pi(g_2)\xi$, then $f_1 - f_2 \in A(1-e)$ and $g_1 - g_2 \in A(1-e)$. Therefore $f_1e = f_2e$ and $g_1e = g_2e$. It follows that $\langle \cdot, \cdot \rangle$ is well-defined. Now the map $fe \rightarrow \pi(f)\xi$ is an isomorphism of Ae onto $\pi(A)\xi$. Given this identification of Ae and $\pi(A)\xi$, the proof of [14, Theorem (4.10.3)] is easily adapted to prove (2).

Let (φ, H) be as in (3). If $\eta \in H$, choose $\{f_n\} \subset A$ such that $\|\pi(f_n)\xi - \eta\|_H \rightarrow 0$. For each n there exists a scalar μ_n such that $ef_ne = \mu_ne$. Then

$$\mu_n\xi = \pi(e)\pi(f_ne)\xi = \varphi(e)\pi(f_n)\xi \longrightarrow \varphi(e)\eta\ .$$

Thus, $\varphi(e)\eta = \mu\xi$ for some $\mu \in C$. This proves that

$$\varphi(e)H = \{\lambda\xi : \lambda \in C\} .$$

Let K be a nonzero closed φ-invariant subspace of H. Then either $\varphi(e)K \neq \{0\}$ or $\varphi(e)K^\perp \neq \{0\}$. In the former case we have $\xi \in \varphi(e)K$, which implies $\pi(A)\xi \subset K$, so that $K = H$. In the latter case, $K^\perp = H$. This proves that φ is irreducible on H.

To prove (4), we first show that the subspaces $\{\pi(A)\xi_k : 1 \leqq k \leqq n\}$ are independent. Assume that $f_k \in A$, $1 \leqq k \leqq n$, and

$$\sum_{k=1}^{n} \pi(f_k)\xi_k = 0 .$$

Then for all $g \in A$,

$$\sum_{k=1}^{n} \pi(egf_ke)\xi_k = 0 .$$

Since egf_ke is just a scalar multiple of e and $\{\xi_1, \cdots, \xi_n\}$ is an independent set of vectors, we have $egf_ke = 0$ for all $g \in A$ and $1 \leqq k \leqq n$. In particular for each k, $ef_k^*f_ke = 0$, so that $f_ke = 0$ since * is proper. Then finally,

$$\pi(f_k)\xi_k = \pi(f_ke)\xi_k = 0 , \qquad 1 \leqq k \leqq n .$$

This proves our first assertion. Now clearly

$$\sum_{k=1}^{n} \pi(A)\xi_k \subset \pi(A)\pi(e)X \subset \pi(AeA)X .$$

Assume $f, g \in A$ and $\xi \in X$. Then $\pi(eg)\xi = \lambda_1\xi_1 + \cdots + \lambda_n\xi_n$ for some scalars $\lambda_1, \cdots, \lambda_n$. Then

$$\pi(feg)\xi = \lambda_1\pi(f)\xi_1 + \cdots + \lambda_n\pi(f)\xi_n \subset \sum_{k=1}^{n} \pi(A)\xi_k .$$

Therefore $\pi(AeA)X = \sum_{k=1}^{n} \pi(A)\xi_k$.

LEMMA 9. *Let A be an A^*-algebra. Assume that I is a γ-closed ideal of A. Then I is a *-ideal of A and the quotient algebra A/I is an A^*-algebra where the involution in A/I is defined as usual by*

$$(f + I)^* = f^* + I \qquad (f \in A) .$$

Proof. Let $\bar{I}$ be the closure of I in $\bar{A}$. Since I is γ-closed, $I = \bar{I} \cap A$. By [14, Theorem (4.9.2)] $\bar{I}$, and therefore I, is a *-ideal. Now $\bar{A}/\bar{I}$ is a B^*-algebra [14, Theorem (4.9.2)], and the map $f + I \rightarrow f + \bar{I}$ is a *-isomorphism of A/I onto a *-subalgebra of $\bar{A}/\bar{I}$. Thus

A/I is an A^*-algebra.

Now assume the notation and hypotheses in the statement of Theorem 7. By Lemma 9 $A/\ker(\pi)$ is an A^*-algebra. Thus, the proof of Theorem 7 reduces to the case where $\ker(\pi) = \{0\}$. From this point until the end of the proof of Theorem 7 *we make the assumption that* $\ker(\pi) = \{0\}$. Let $F = \{g \in A : \pi(g)$ has finite dimensional range$\}$.

LEMMA 10. $F = \operatorname{soc}(A)$.

Proof. First we prove

(1) if $g \in A$, $gF = \{0\}$ or $Fg = \{0\}$, then $g = 0$.

Assume that $gF = \{0\}$. Then $\pi(g)\pi(f) = 0$ for all $f \in F$. Since $\bigcup\{\mathscr{R}(\pi(f)) : f \in F\}$ is dense in X, we have $\pi(g) = 0$. Therefore $g = 0$. Suppose $Fg = \{0\}$. Then $(gF)^2 = \{0\}$, so that gF is a nilpotent right ideal of A. An A^*-algebra is Jacobson semisimple [14, Theorem (4.1.19)], and in particular, has no nonzero nilpotent left or right ideals. Therefore $gF = \{0\}$ which implies $g = 0$. This proves (1).

Let M be a minimal ideal of A in $\operatorname{soc}(A)$. Then either $M \cap F = \{0\}$ or $M \subset F$. But in the former case $MF \subset M \cap F = \{0\}$ which is impossible by (1). Then since $\operatorname{soc}(A)$ is the algebraic sum of minimal ideals of A, $\operatorname{soc}(A) \subset F$.

In order to prove the opposite inclusion we need the technical result:

(2) if $f \in F$, $f \neq 0$, then there exists a nonzero idempotent $e \in \operatorname{soc}(A)$ such that $\mathscr{R}(\pi(e)) \subset \mathscr{R}(\pi(f))$.

Choose $g \in F$ such that $gf \neq 0$. The algebra fAg is isomorphic to $\pi(f)\pi(A)\pi(g)$, and therefore is finite dimensional. If for some n $(fAg)^n = \{0\}$, then $(Agf)^{n+1} = \{0\}$. This contradicts the fact that A has no nilpotent left ideals. By classical Wedderburn theory [9, pp. 38, 53, 54] there exists a nonzero idempotent $e \in fAg$. Then clearly $\mathscr{R}(\pi(e)) \subset \mathscr{R}(\pi(f))$.

Assume $f \in F$. Choose $g \in \operatorname{soc}(A)$ such that $\mathscr{R}(\pi(f - gf))$ has the smallest possible dimension. Suppose $f - gf \neq 0$. Then by (2) there exists a nonzero idempotent $e \in \operatorname{soc}(A)$ such that $\mathscr{R}(\pi(e)) \subset \mathscr{R}(\pi(f - gf))$. Consider

$$h = (f - gf) - e(f - gf) = f - (g + e - eg)f\,.$$

Then $\dim(\mathscr{R}(\pi(h))) < \dim(\mathscr{R}(\pi(f - gf)))$ which contradicts the minimal dimension of $\mathscr{R}(\pi(f - gf))$. Therefore $f = gf \in \operatorname{soc}(A)$

Now we complete the proof of Theorem 7. Let $\{M_\delta: \delta \in \Delta\}$ be the set of all minimal ideals of A in $\mathrm{soc}(A)$. For each $\delta \in \Delta$ choose e_δ a m.i. of A with $e_\delta^* = e_\delta$ such that $M_\delta = Ae_\delta A$. By Lemma 10 each element $e_\delta \in F$. Let $n(\delta)$ be the dimension of the range of $\pi(e_\delta)$. For each $\delta \in \Delta$, choose a basis $\{\xi_{\delta,1}, \cdots, \xi_{\delta,n(\delta)}\}$ for the range of $\pi(e_\delta)$. Form the spaces

$$X_{\delta,k} = \pi(A)\xi_{\delta,k} \qquad (\delta \in \Delta, 1 \leqq k \leqq n(\delta)) .$$

Note that if $\delta, \tau \in \Delta$, $\delta \neq \tau$, then $e_\delta A e_\tau \subset M_\delta \cap M_\tau = \{0\}$. From this fact and part (4) of Lemma 8 it is easy to see that the spaces

$$\{X_{\delta,k}: \delta \in \Delta, 1 \leqq k \leqq n(\delta)\} \quad \text{are independent.}$$

Combining the facts that $\pi(F)X$ is dense in X and $F = \mathrm{soc}(A) = \sum_{\delta \in \Delta} Ae_\delta A$ with Lemma 8 (4), we have

$$\sum \{X_{\delta,k}: \delta \in \Delta, 1 \leqq k \leqq n(\delta)\} \quad \text{is dense in} \quad X.$$

For convenience of notation we index the collection in the sum above by an index set Λ. Set

$$X_0 = \sum \{X_\lambda: \lambda \in \Lambda\} .$$

We have proved that X_0 is the algebraic direct sum of the spaces $\{X_\lambda: \lambda \in \Lambda\}$ and that X_0 is dense in X.

For each λ let $\langle \cdot, \cdot \rangle_\lambda$ be the inner-product defined on $\pi(A)\xi_\lambda$ as in Lemma 8 (2). Define an inner-product on X_0 by

$$\langle \xi, \eta \rangle = \sum_{\lambda \in \Lambda} \langle \xi_\lambda, \eta_\lambda \rangle_\lambda$$

where $\xi = \sum \xi_\lambda$, $\eta = \sum \eta_\lambda$, $\xi_\lambda, \eta_\lambda \in X_\lambda$ for all $\lambda \in \Lambda$. For each $f \in A$ define $\varphi_0(f)$ on X_0 by

$$\varphi_0(f)(\sum_{\lambda \in \Lambda} \pi(g_\lambda)\xi_\lambda) = \sum_{\lambda \in \Lambda} \pi(fg_\lambda)\xi_\lambda .$$

Then φ_0 is a *-representation of A on $(X_0, \langle \cdot, \cdot \rangle)$ as in (II). Let H be the Hilbert space completion of $(X_0, \langle \cdot, \cdot \rangle)$, and extend φ_0 to a *-representation of A on H, again as in (II). For each $\lambda \in \Lambda$, let H_λ be the closure of X_λ in H, and let φ_λ be the restriction of φ to the φ-invariant subspace H_λ. By Lemma 8 (3) each of the representations $(\varphi_\lambda, H_\lambda)$, $\lambda \in \Lambda$ is an irreducible *-representation of A. If $\xi \in X_\lambda$, $\eta \in X_\mu$ where $\lambda \neq \mu$, then by definition $\langle \xi, \eta \rangle = 0$. It follows that $H_\lambda \perp H_\mu$. Since $X_0 \subset \sum \{H_\lambda: \lambda \in \Lambda\}$, H is the orthogonal direct sum of $\{H_\lambda: \lambda \in \Lambda\}$. Then φ is direct sum of the irreducible *-representations $(\varphi_\lambda, H_\lambda)$, $\lambda \in \Lambda$.

It remains to be shown that (π, X) is Naimark-related to (φ, H). To begin we establish the technical fact that

(1) if $\psi \in H$, $\psi \neq 0$, then there exists $f \in F$ such that $\varphi(f)\psi \neq 0$.

For $\psi = \sum_{\lambda \in \Lambda} \psi_\lambda$ where $\psi_\lambda \in H_\lambda$, $\lambda \in \Lambda$. There is some $\mu \in \Lambda$ such that $\psi_\mu \neq 0$. By the construction of H_μ there exists a m.i. e of A such that $\varphi_\mu(e) \neq 0$. Also, since φ_μ is irreducible, $\varphi(A)\psi_\mu$ is dense in H_μ. It follows that there exists $g \in A$ such that $\varphi(eg)\psi_\mu \neq 0$. Then $eg \in F$ by Lemma 10. This proves (1).

Define a linear operator V with $\mathscr{D}(V) = X_0 \subset X$ and with range in H by $V\eta = \eta$, $\eta \in X_0$. Clearly

$$\varphi(f)V\xi = V\pi(f)\xi \qquad (\xi \in X_0, f \in A).$$

By Lemma 8 (4) and by construction we have $\operatorname{soc}(A)X \subset X_0$. Thus, given $f \in F = \operatorname{soc}(A)$, the range of $\pi(f)$ is in X_0. The restriction of V to the finite dimensional subspace $\mathscr{R}(\pi(f))$ is a bounded map from $\mathscr{R}(\pi(f))$ into H. Therefore we have

(2) for every $f \in F$, $V\pi(f)$ is a bounded everywhere defined operator from X to H.

Now we prove that V has a closure $\bar{V}$ and that $\bar{V}$ is one-to-one. Assume that $\{\psi_n\} \subset \mathscr{D}(V) = X_0$, $\psi \in H$, $\|\psi_n\|_X \to 0$, and $\|V\psi_n - \psi\|_H \to 0$. Suppose that $\psi \neq 0$. Then by (1) there exists $f \in F$ such that $\varphi(f)\psi \neq 0$. By (2), $\|V\pi(f)\psi_n\|_H \to 0$. Also, $\|\varphi(f)V\psi_n - \varphi(f)\psi\|_H \to 0$. Since $\varphi(f)V\psi_n = V\pi(f)\psi_n$ for all n, we have $\varphi(f)\psi = 0$. This contradiction proves that $\psi = 0$, and hence, that V has a closure, $\bar{V}$. Assume that $\xi \in \mathscr{D}(\bar{V})$ and $\bar{V}(\xi) = 0$. Then there exists $\{\xi_n\} \subset \mathscr{D}(V) = X_0$ such that $\|\xi_n - \xi\|_X \to 0$ and $\|V\xi_n\|_H \to 0$. For all $f \in F$ we have by (2) $\|V\pi(f)\xi_n - V\pi(f)\xi\|_H \to 0$. Also, $\|\varphi(f)V\xi_n\|_H \to 0$. Therefore $V\pi(f)\xi = 0$ for all $f \in F$. Thus, $\pi(F)\xi = 0$, and since π is FDS, $\xi = 0$. This proves that $\bar{V}$ is one-to-one. Then (π, X) and (φ, H) are Naimark-related by (III).

COROLLARY 11. *Let A be a symmetric A^*-algebra. Then any irreducible Banach representation (π, X) of A that contains a non-zero operator of finite rank in its image is Naimark-related to an irreducible $*$-representation of A.*

Proof. There exists a dense subspace X_0 of X such that π acts algebraically irreducibly on X_0 [15, p. 231]. Thus $\ker(\pi)$ is primitive in this case, and then the symmetry of A implies that $\ker(\pi)$ is γ-closed. Also, π is FDS. Therefore the result follows from Theorem 7.

6. An example. In this section we construct a symmetric

Banach *-algebra A and a continuous irreducible representation π of A on a Hilbert space H with the properties:

(1) (π, H) is not similar to any *-representation of A, and

(2) π is not γ-continuous.

The question of whether any continuous irreducible representation of a B^*-algebra on a Hilbert space is similar to a *-representation is open.

Let $I = (0, 1]$, and set $S = I \times I$. If $J(x, y)$ is a bounded function on S, let

$$||J||_u = \sup\{|J(x, y)|: (x, y) \in S\}\ .$$

Let A be the collection of all complex-valued functions $K(x, y)$ defined on S such that $K(x, y)(xy)^{-1}$ is continuous and bounded on S. Clearly A is a complex linear space with the usual operations. Norm A by

$$||K(x, y)|| = ||K(x, y)(xy)^{-1}||_u \qquad (K \in A)\ .$$

Note that $||K||_u \leqq ||K||$ for all $K \in A$. It is easy to see that the norm $||\cdot||$ is a complete norm on A. Define multiplication in A by

$$(K \cdot J)(x, y) = \int_I K(x, t)J(t, y)dt$$

where $K, J \in A$, $(x, y) \in S$. It is clear that $K \cdot J \in A$ whenever $K, J \in A$, and that A is a complex algebra with respect to this multiplication operation. Furthermore, if $(x, y) \in S$, then

$$|(K \cdot J)(x, y)(xy)^{-1}| \leqq \int_I |(K(x, t)x^{-1}J(t, y)y^{-1}|dt \leqq ||K||\,||J||\ .$$

Therefore $||K \cdot J|| \leqq ||K||\,||J||$, so that A is a Banach algebra. For $K \in A$, let

$$K^*(x, y) = \overline{K(y, x)} \qquad (x, y) \in S\ .$$

Then $K \rightarrow K^*$ is an isometric involution on A.

For $K \in A$, let $\tau(K)$ be the Fredholm integral operator on $L^2(I)$ determined by K, that is,

$$\tau(K)f(x) = \int_I K(x, y)f(y)dy \qquad (x \in I, f \in L^2(I))\ .$$

Then

$$||\tau(K)f||_2 \leqq ||K||_u||f||_2 \leqq ||K||\,||f||_2$$

whenever $f \in L^2(I)$. A standard argument proves that $K \rightarrow \tau(K)$ is a faithful continuous *-representation of A on $L^2(I)$. Let D be the set of all complex-valued functions f on I such that $f(x)x^{-1}$ is continuous and bounded on I. If $f_k, g_k \in D$ for $1 \leqq k \leqq n$, then

$$K(x, y) = \sum_{k=1}^{n} f_k(x) g_k(y) \in A .$$

The set of such kernels is exactly the socle of A, and this set is dense in A. For every kernel K of this form, $\tau(K)$ is an operator with finite dimensional range. Furthermore, $K \to \tau(K)$ acts algebraically irreducibly on the subspace $D \subset L^2(I)$. The fact that a primitive Banach algebra with proper involution and dense socle is symmetric follows from an argument similar to the one used to establish [4, Theorem 3.8]. To summarize:

(IV). A is a primitive symmetric Banach *-algebra with dense socle.

Now we construct a continuous representation of A on $H = L^2(I, y^2 dy)$ with the properties (1) and (2) stated above. We denote the norm of $f \in H$ by

$$|f|_2 = \left(\int_I |f(y)|^2 y^2 \, dy\right)^{1/2} .$$

For $K \in A$ let

$$\pi(K)f(x) = \int_I K(x, y) f(y) dy \qquad (x \in I, f \in H) .$$

Then for all $K \in A, f \in H$, and $x \in I$ we have

$$\begin{aligned} |\pi(K)f(x)| &= \left|\int_I K(x, y) y^{-1}(f(y)y) dy\right| \\ &\leqq \|K(x, y) y^{-1}\|_u \left(\int_I |f(y)|^2 y^2 \, dy\right)^{1/2} \\ &\leqq \|K\| \, |f|_2 . \end{aligned}$$

Therefore

$$\int_I |\pi(K)f(x)|^2 x^2 dx \leqq \int_0^1 \|K\|^2 |f|_2^2 x^2 dx \leqq \|K\|^2 |f|_2^2 .$$

Thus

$$|\pi(K)f|_2 \leqq \|K\| \, |f|_2 \qquad (f \in H, K \in A) .$$

This proves that $K \to \pi(K)$ is a continuous representation of A on H. Using the fact that π acts algebraically irreducibly on $D \subset H$, it is not difficult to verify that (π, H) is irreducible. Suppose that (π, H) is similar to a *-representation of A (which is then necessarily irreducible). It can be shown that an algebra with the properties listed in (IV) has a unique irreducible *-representation up to unitary equivalence. Therefore in this case τ is the unique irreducible *-representation of A. Thus π must be similar to τ. We show that this is impossible. For assume that there is a bicontinuous linear isomorphism W mapping $L^2(I)$ onto H such that

$$\pi(K)W = W\tau(K) \qquad (K \in A) .$$

Assume $h \in D$. Choose $g \in D$, $g \neq 0$. Let $K(x, y) = h(x)\overline{g(y)}$ $(x, y) \in S$. Then $K \in A$. Now $\pi(K)Wg = W(\tau(K)g)$, that is,

$$\int_I h(x)\overline{g(y)}(Wg)(y)dy = W\Big(\int_I h(x)|g(y)|^2 dy\Big) .$$

This equation proves that Wh is a scalar multiple of h. Since D is dense in $L^2(I)$ and W is continuous, Wh is a scalar multiple of h for all $h \in L^2(I)$. But $g(y) = y^{-1} \in H$ and $g \notin L^2(I)$. Thus W can not map onto H. This contradiction proves the assertion (1).

If π is γ-continuous, then π has a continuous extension $\bar{\pi}$ to the B^*-algebra $\bar{A}$. Then by [1, Cor. 2.3], the representation $\bar{\pi}$, and hence π, is similar to a *-representation. This contradiction proves (2).

7. Some open questions.

There are many open questions concerning Naimark-relatedness of representations of Banach *-algebras. In this section we list several interesting questions in the area.

Question 1. Let A be a symmetric Banach *-algebra. Is every continuous essential Banach representation of A with γ-closed kernel Naimark-related to a *-representation?

Question 1 has an affirmative answer if the representation is algebraically irreducible [Theorem 1], if the representation is irreducible and contains in its image an operator with finite dimensional range [Corollary 11], or if the hypotheses of Corollary 5 are satisfied.

Question 2. Is every continuous representation of a B^*-algebra on Hilbert space similar to a *-representation?

J. Bunce has proved that this question has an affirmative answer when the B^*-algebra is strongly ammenable [3]. An affirmative answer is provided by the author if either the representation is algebraically irreducible [1, Prop. 2.2], or if the representation is irreducible and contains in its image a nonzero operator with finite dimensional range [1, Cor. 2.3]. The question can be weakened to ask only that the given representation be Naimark-related to a

*-representation. Corollary 4 and [2, Theorem 3] provide partial answers to this version of the question.

In view of results such as those cited above concerning similarity or Naimark-relatedness of a representation to a *-representation when the given algebra is a B^*-algebra, it is of interest to determine conditions which imply that a representation π of a Banach *-algebra A extends to a continuous representation of $\bar{A}$ (clearly this is the case if and only if π is γ-continuous).

Question 3. Under what conditions is a Banach representation of a Banach *-algebra γ-continuous?

A minimal necessary condition for a representation π to be γ-continuous is that $\ker(\pi)$ be γ-closed. That this condition need not suffice for π to be γ-continuous follows from the example in §6. The work of T. Palmer [11] provides an equivalent condition that π be γ-continuous that may prove useful, namely, that the image under π of the group of unitaries of A (assuming A has an identity) be bounded. In the case that (π, X) is an algebraically irreducible Banach representation of A and X is not a Hilbert space in an equivalent norm, then a result of the author [1, Prop. 2.2] shows that π cannot extend to a continuous representation of $\bar{A}$.

Finally, we state a general question about which there seems to be little information available.

Question 4. Let A be a Banach *-algebra, and let π be a continuous irreducible Banach representation of A. If $\ker(\pi)$ is the kernel of some *-representation of A, is π Naimark-related to a *-representation of A?

Added in proof. In several places we have used the inequality $\gamma(f) \leqq \|f\|$ for f in a Banach *-algebra A. This inequality does not hold in general. However, using results in [11] it is not difficult to verify that there exists a constant $K > 0$ such that $\gamma(f) \leqq K\|f\|$ for all $f \in A$. This inequality suffices in all our arguments.

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Received September 28, 1976. This research was partially supported by NSF grant MCS76-06421.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 72, No. 1 January, 1977