AUTOMORPHISMS OF THE SEMIGROUP OF FINITE COMPLEXES OF A PERIODIC LOCALLY CYCLIC GROUP

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In this paper the automorphism group of the semigroup
of finite complexes of a periodic locally cyclic group is
determined.

1. Introduction. Let \( G \) be a group, written additively but not
assumed to be abelian, and let \( F(G) \) denote the collection of finite
nonempty subsets of \( G \). Then \( F(G) \) is a semigroup with respect to
the operation \( A + B = \{ a + b \mid a \in A \text{ and } b \in B \} \). The collection
of automorphisms of \( F(G) \) is a group under the operation of composi-
tion of functions and we shall denote this group by \( \text{Aut } F(G) \). The
automorphism group of \( G \) will be denoted by \( \text{Aut } G \). Since the
collection of singleton subsets of \( G \) is the group of units of \( F(G) \),
we frequently identify \( G \) with \( \{ \{ g \} \mid g \in G \} \). Each automorphism of
\( G \) induces an automorphism of \( F(G) \) in the natural way. These
elements of \( \text{Aut } F(G) \) will be called standard automorphisms. If
\( \alpha \in \text{Aut } G \) and \( \alpha^* \) is the standard automorphism of \( F(G) \) induced by
\( \alpha \), then the mapping which sends \( \alpha \) to \( \alpha^* \) is an isomorphism of
\( \text{Aut } G \) onto the collection of standard automorphisms of \( F(G) \).

Our interest in \( \text{Aut } F(G) \) comes from our study of retractable
groups. In [1] it was shown that the retractions of a torsion-free
abelian group \( G \) generate a large class of nonstandard automorphisms
of \( F(G) \). In particular, it was shown that \( \text{Aut } F(Z) \) is countably
infinite, where \( Z \) denotes the additive group of integers. Since \( \text{Aut } Z \)
has only two elements, it was natural to inquire if the semigroup of
complexes of a finite cyclic group admits nonstandard automorphisms.
For a natural number \( n \), let \( Z_n \) denote the group of integers modulo \( n \).
Clearly \( \text{Aut } F(Z_n) \) and \( \text{Aut } F(Z_2) \) have only standard automorphisms.
In §3 we exhibit nonstandard automorphisms for \( F(Z_3) \), \( F(Z_4) \), and
\( F(Z_5) \) and classify their corresponding automorphism groups. The
only automorphisms of \( F(Z_6) \) are standard. In Theorems 2 and 3
we prove that if \( n \geq 7 \), then \( \text{Aut } F(Z_n) \) has only standard auto-
morphisms and hence, \( \text{Aut } F(Z_n) \) is isomorphic to \( \text{Aut } Z_n \). If \( Q \)
denotes the additive group of rationals and \( G \) is a subgroup of \( Q/Z \),
we can use the preceding results to characterize \( \text{Aut } F(G) \) in terms
of \( \text{Aut } G \). It appears that the absence of retractions (retractable
groups are torsion-free) might restrict the number of nonstandard
automorphisms. In Theorem 4 we show that an automorphism of $F(G)$ is standard if and only if the automorphism is inclusion preserving.

In §2 (Theorem 1) we show that if $A \in F(\mathbb{Z}_n)$ and $\theta \in \text{Aut} F(\mathbb{Z}_n)$, then $|A| = |A\theta|$. This theorem was crucial for our work. Our results are computational in nature and are established through a sequence of lemmas. If $X$ and $Y$ are sets, then $X \setminus Y$ denotes the set of elements in $X$ but not in $Y$.

2. Preliminaries. We have been unable to determine if the elements of $\text{Aut} F(G)$ preserve the cardinality of subsets of $G$. The purpose of this section is to prove that the elements of $\text{Aut} F(\mathbb{Z}_n)$ do preserve set cardinality.

**Lemma 1.** Let $G$ be a finite group and let $A, B \in F(G)$ with $|A| < |B| < |G|$. Then there exists $g \in G \setminus \{0\}$ such that $|A + \{0, g\}| < |B + \{0, g\}|$ and $|B| < |B + \{0, g\}|$.

**Proof.** To prove the lemma it suffices to take $|B| = |A| + 1$. If $G \setminus A = \{x_1, \ldots, x_{t+1}\}$ and $G \setminus B = \{y_1, \ldots, y_t\}$, let

$$A^* = \{(x_i, x_j), g_{ij} \mid i, j \in \{1, \ldots, t + 1\}, i \neq j, \text{ and } -x_i + x_j = g_{ij}\}$$

and

$$B^* = \{(y_r, y_s), g_{rs} \mid r, s \in \{1, \ldots, t\}, r \neq s, \text{ and } -y_r + y_s = g_{rs}\}.$$  

Then $|A^*| = t(t + 1) > t(t - 1) = |B^*|$ and $0 \in A^* \pi_2 \cup B^* \pi_2$.

**Case 1.** There exists $g \in A^* \pi_2 \setminus B^* \pi_2$. Then $g = -x_i + x_j$ for some $i \neq j$. Since $x_i \in A$, $x_j = x_i + g \in A + g$ and since $x_i \in A$, $x_i \in A \cup (A + g) = A + \{0, g\}$. Thus $|A + \{0, g\}| < |G|$. If $y \in G \setminus B$, then there exists $z \in G$ such that $z + g = y$, and $g \neq 0$ implies that $z \neq y$. Since $g \in B^* \pi_2$, we have that $z \in G \setminus B$. Hence $z \in B$ and so $y \in B + g$. Therefore we have that

$$|A + \{0, g\}| < |G| = |B + \{0, g\}|$$

and

$$|B| < |B + \{0, g\}|.$$  

**Case 2.** $A^* \pi_2 \subseteq B^* \pi_2$. Since $|A^*| > |B^*|$, there exists $g \in A^* \pi_2$ such that if

$$J = \{(x_i, x_j), ((x_i, x_j), g) \in A^*\}$$

and

$$K = \{(y_r, y_s), ((y_r, y_s), g) \in B^*\},$$

then $|J| > |K|$. Let $J = \{(x_i, x_i), \ldots, (x_m, x_m)\}$ and $K = \{(y_1, y_1), \ldots, (y_t, y_t)\}$.
\[
\cdots, (y_{r_n}, y_{s_n}), \text{ where } m > n. \text{ If } 1 \leq k \leq m, \text{ then } x_{j_k} = x_{i_k} + g \text{ and since } x_{i_k}, x_{j_k} \in A, x_{j_k} \in A + \{0, g\}. \text{ If } x \in G\backslash (A \cup \{x_{j_1}, \ldots, x_{j_m}\}), \text{ then } x = z + g \text{ for some } z \in G, z \neq x. \text{ Since } x \in J_{\alpha_2}, \text{ we have that } z \in A. \text{ Thus } x \in A + \{0, g\}. \text{ Consequently, } A + \{0, g\} = G\backslash \{x_{j_1}, \ldots, x_{j_m}\}. \text{ Similarly, } B + \{0, g\} = G\backslash \{y_{s_1}, \ldots, y_{s_n}\} \text{ and so } |A + \{0, g\}| < |B + \{0, g\}|.
\]

Suppose (by way of contradiction) that \( |B| = |B + \{0, g\}| \). Then \( B = B + \{0, g\} = G\backslash \{y_{s_1}, \ldots, y_{s_n}\} \) and hence \( n = t \) and \( m = t + 1 \). Thus for each \( x_i, x_i + g = x_j \) for some \( x_j \). Then \( x_i + 2g = x_i + g = x_k \), for some \( k, 1 \leq k \leq t + 1 \). It follows, by induction, that \( x_t + \langle g \rangle \subseteq G\backslash A \) for each \( i, 1 \leq i \leq t + 1 \), and so \( G\backslash A \) is a union of cosets of \( \langle g \rangle \). Therefore, \( o(g) | (t + 1) \). Similarly, \( o(g) | t \) and this is a contradiction as \( g \neq 0 \). Hence, \( |B| < |B + \{0, g\}| \).

If \( A \) is an element of the semigroup \( F(G) \), define \( 1A = A \) and for \( n > 1 \), define \( nA = (n - 1)A + A \). Note that \( nA \) does not necessarily equal \( \{na \mid a \in A\} \).

**Lemma 2.** Let \( G \) be a finite abelian group. If \( A \in F(G) \) and \( |A + kA| = |kA| \) for some \( k \geq 1 \), then \( |lA + kA| = |kA| \) for all \( l \geq 1 \).

**Proof.** The equality holds for \( l = 1 \). Assume that \( l \geq 1 \) and that \( |lA + kA| = |kA| \). If \( a \in lA \), then \( |lA + kA| = | -a + lA + kA| \) and since \( kA \subseteq -a + lA + kA \), we have that \( kA = -a + lA + kA \). Therefore, \( |kA| = |A + kA| = |A + ( -a + lA + kA)| = |(l + 1)A + kA| \). By induction, the lemma holds for all \( l \geq 1 \).

The proof of the next lemma is straightforward and will be omitted.

**Lemma 3.** Let \( G \) be a group and let \( H \) be a finite subgroup of \( G \). If \( \theta \in \text{Aut } F(G) \), then

1. \( \theta F(H) \) is a semigroup isomorphism of \( F(H) \) onto \( F(H\theta) \);
2. \( H \) and \( H\theta \) are isomorphic subgroups of \( G \);
3. if \( \theta | G \) is the identity, then \( H = H\theta \).

**Lemma 4.** If \( G \) is a finite group, \( b \in G\backslash \{0\} \), and \( \theta \in \text{Aut } F(G) \), then \( |\{0, b\}\theta| = 2 \).

**Proof.** Let \( \theta \in \text{Aut } F(G) \) and \( \eta \) be the standard automorphism of \( F(G) \) induced by \( (\theta | G)^{-1} \). Since \( \eta \) preserves set cardinality, \( |A\theta\eta| = |A\theta| \) for all \( A \in F(G) \), and \( \theta\eta | G = \iota \) where \( \iota \) denotes the identity mapping of \( G \). Thus we may assume that \( \theta | G = \iota \). If \( n = o(b) \) and \( H = \langle b \rangle \), then \( H = (n - 1)\{0, b\} \). By Lemma 3, \( H = H\theta = (n - 1)(\{0, b\}\theta) \). Further, \( n - 1 \) is the smallest natural number such
that \((n-1)((0,b)\theta) = H\). If \(A = (0,b)\theta\), then \(A \subseteq H\) and \((n-1)A = H\). If \(B \subseteq G\), \(|B| > 2\), and \(k \geq 1\), then by induction and Lemma 2, \(|(k+1)B| > k+2\) or \(|(k+1)B| = |kB|\). In particular, if \(|A| > 2\), then since \(|(n-2)+1)A| = n\), we must have that \(n = |(n-1)A| = |(n-2)A|\). Thus \((n-2)A = H\), and this is a contradiction.

**Corollary 1.** Let \(G\) be a finite group and \(a, b \in G\) with \(a \neq b\). If \(\theta \in \text{Aut } F(G)\), then \(|\{a, b\}\theta| = 2\).

**Lemma 5.** If \(G\) is a finite group and \(A \in F(G)\) with \(|A| = |G| - 1\), then \(|A\theta| = |A|\) for all \(\theta \in \text{Aut } F(G)\).

**Proof.** As in Lemma 4, we may assume \(\theta|G = 1\). Let \(a, b \in G\) with \(a \neq b\). Since \(|\{a, b\}| + |A| = |G| + 1\), \(\{a, b\} + A = G\) [3, Theorem 1]. Suppose (by way of contradiction) that \(A\theta = B\), where \(|B| < |A|\). Then there exists \(x, y \in G\) with \(x \neq y\). Since \(x \notin \{0, x-y\} + B\), \(\{0, x-y\} \neq G\). By the preceding corollary, \(\{0, x-y\} = \{a, b\}\theta\) for some \(a, b \in G\) with \(a \neq b\). But then
\[
\{a, b\} + A = G = G\theta = (\{a, b\} + A)\theta = \{0, x-y\} + B \neq G ,
\]
a contradiction. Hence, \(|A\theta| = |A|\).

If \(A, B \in F(G)\) and \(A = g + B\) for some \(g \in G\), then \(A\) is said to be a (left) translate of \(B\). Clearly \(F(G)\) is the union of mutually disjoint translation classes. Moreover, if \(G\) is abelian and \(\theta \in \text{Aut } F(G)\), then \(\theta\) is completely determined by its action on the group of units of \(F(G)\) and a system of representatives for the translation classes.

**Lemma 6.** Let \(\theta \in \text{Aut } Z_n\) and \(\{0, 1\}\theta = g + \{0, 1\}\). If \(1 \leq a \leq n - 1\), then there exists \(h \in Z_n\) such that \(\{0, a\}\theta = h + \{0, a\}\).

**Proof.** The translation class of \(\{0, a\}\) is the same as the translation class of \(\{0, n-a\}\). Thus we may assume that \(a \leq n/2\). If \(n\) is even and \(a = n/2\), then \(\{0, a\}\) is the only subgroup of \(Z_n\) of order 2. By Lemma 3, \(\{0, a\}\theta = \{0, a\}\). Therefore, we may further assume that \(1 \leq a < n/2\). By induction we may assume that we have verified the lemma for all \(b\) such that \(1 \leq b < a < n/2\). If \(\{0, a\}\theta = y + \{0, t\}\), then, since \(\{0, t\} = t + \{0, n-t\}\), we may assume \(t \leq n/2\) and since \(\theta\) maps a translation class onto a translation class, \(a \leq t\). Now we have the equation \((a-1)\{0, 1\} + \{0, a\} = (2a-1)\{0, 1\}\) and, taking the image of both sides under \(\theta\), we obtain
\[
(a-1)g + (a-1)\{0, 1\} + y + \{0, t\} = (2a-1)g + (2a-1)\{0, 1\} .
\]
Hence, \(\{0, 1, \ldots, a-1, t, t+1, \ldots, t+a-1\}\) is a translate of
Since \( t + a - 1 < n - 1 \), \( t = a \) is the only possible solution.

**Theorem 1.** If \( A \in F(Z_n) \) and \( \theta \in \text{Aut} F(Z_n) \), then \( |A\theta| = |A| \).

**Proof.** Let \( \{0, 1\} \theta = g + \{0, b\} \). Since \((n - 1)\{0, 1\} = Z_n\), \( b \) is a generator for \( Z_n \). Let \( \Phi \) be the standard automorphism of \( F(Z_n) \) induced by the automorphism of \( Z_n \) that maps \( b \) to 1. Now \( \Phi \) preserves cardinality and hence, \( \theta \Phi \) will preserve cardinality if and only if \( \theta \) preserves cardinality. Thus we may assume that \( \{0, 1\} \theta = g + \{0, 1\} \). Suppose (by way of contradiction) that there exists \( B \in F(Z_n) \) such that \( |B\theta| \neq |B| \). We may assume that if \( A \in F(G) \) with \( |A| > |B| \), then \( |A\theta| = |A| \). Thus, \( |B\theta| < |B| \) and by Lemma 5, \( |B| < n - 1 \). By Lemma 1, there exists \( x \in Z_n \) such that \( |B| < |B + \{0, x\}| \) and \( |B\theta + \{0, x\}| < |B + \{0, x\}| \), and by Lemma 6, \( \{0, x\} \theta = h + \{0, x\} \), for some \( h \). Therefore,

\[
|\{(B + \{0, x\})\theta\}| = |B \theta + h + \{0, x\}|
= |B \theta + \{0, x\}| < |B + \{0, x\}|
= |(B + \{0, x\})\theta| .
\]

This is a contradiction and hence, \( |A\theta| = A \) for all \( A \in F(Z_n) \).

3. Determination of \( \text{Aut} F(Z_n) \). Let \( G \) be a group, \( H \) be the group of standard automorphisms of \( F(G) \), and \( K \) be the group of automorphisms of \( F(G) \) that are the identity on the group of units of \( F(G) \). Then \( K \) is a normal subgroup of \( \text{Aut} F(G) \), \( H \cap K = \{e\} \), and \( \text{Aut} F(G) = KH \). If \( \theta \in K \) and \( G \) is abelian, then \( \theta \) is uniquely determined by its action on a system of representatives of the translation classes of \( F(G) \). Clearly \( F(Z_n) \) admits only standard automorphisms. The verification of the following assertions are computational (some are lengthy) and will be omitted. If \( G = Z_3 \), then there exists \( \theta \in \text{Aut} F(G) \) with \( \theta|G = \iota, \{0, 1\} \theta = \{0, 2\}, \{0, 1, 2\} \theta = \{0, 1, 2\} \), \( K = \{\iota, \theta, \theta^2\} \), \( H = \{\iota, \beta\} \) where \( (1)\beta = 2 \), and \( \beta^{-1}\theta \beta = \theta^2 \). Thus, \( \text{Aut} F(G) = KH \) and is isomorphic to \( S_3 \), the symmetric group of degree 3. If \( G = Z_4 \), then there exists \( \theta \in \text{Aut} F(G) \) with \( \theta|G = \iota, \{0, 1\} \theta = \{0, 3\}, \{0, 2\} \theta = \{0, 2\}, \{0, 1, 2\} \theta = \{0, 2, 3\}, \{0, 1, 2, 3\} \theta = \{0, 1, 2, 3\} \), \( K = \{\iota, \theta, \theta^2, \theta^3\} \), \( H = \{\iota, \beta\} \), where \( (1)\beta = 3 \), and \( \beta^{-1}\theta \beta = \theta^3 \). Thus, \( \text{Aut} F(G) = KH \) and is isomorphic to \( D_4 \), the dihedral group of order 8. If \( G = Z_6 \), then there exists \( \theta \in F(G) \) with \( \theta|G = \iota, \{0, 1\} \theta = \{2, 4\}, \{0, 2\} \theta = \{3, 4\}, \{0, 1, 2\} \theta = \{1, 3, 4\}, \{0, 1, 2, 3\} \theta = \{2, 3, 4\}, \{0, 1, 2, 3, 4\} \theta = \{0, 1, 2, 3, 4\} \), \( K = \{\iota, \theta\} \), \( H = \{\iota, \gamma, \gamma^2, \gamma^3\} \), where \( (1)\gamma = 2 \), and \( \theta \gamma \gamma = \gamma \theta \). Thus, \( \text{Aut} F(G) = KH \) and is isomorphic to the direct product of \( Z_2 \) and \( Z_4 \). Finally, if \( G = Z_8 \), then \( K = \{\iota\} \)
and \( \text{Aut } F(G) \) is isomorphic to \( \text{Aut } G \).

The remaining portion of this paper is devoted to showing that \( \text{Aut } F(G) \) consists only of standard automorphisms if \( G \) is a subgroup of \( Q/Z \) and \( |G| \geq 7 \), and hence \( \text{Aut } F(G) \) is isomorphic to \( \text{Aut } G \). The proofs of the next three lemmas are straightforward and will be omitted.

**Lemma 7.** If \( A \in F(G) \) and \( L(A) = \{ g \mid g + A = A \} \), then \( L(A) \) is a subgroup of \( G \) and \( A \) is a union of right cosets of \( L(A) \). If \( G \) is finite, then the number of translates of \( A \) is the index of \( L(A) \) in \( G \).

**Lemma 8.** If \( G \) is a finite group, \( A \in F(G) \), and \( |A| = |G| - 1 \), then \( L(A) = \{ 0 \} \) and all subsets of \( G \) of cardinality \( |G| - 1 \) belong to the translation class of \( A \).

**Lemma 9.** If \( a \) is a generator of \( Z_n \) and \( A = k\{0, a\} \), where \( 1 < k \leq n - 2 \), then \( L(A) = \{ 0 \} \).

For the remainder of this paper we shall assume that \( n \geq 7 \).

**Lemma 10.** If \( \theta \in \text{Aut } F(Z_n) \) and \( \{0, 1\} \theta = \{0, 1\} \), then \( \{0, r\} \theta = \{0, r\} \) for every \( r \in Z_n \setminus \{0\} \) and \( \theta | Z_n \) is the identity.

**Proof.** We first assume that \( 1 < r \leq n/2 \). If \( r = n/2 \), then \( \{0, r\} \) is a subgroup of \( Z_n \) and by Lemma 3, \( \{0, r\} \theta = \{0, r\} \) since it is the only subgroup of order two. Thus we may suppose that \( 1 < r < n/2 \). By Lemma 6, \( \{0, r\} \theta = h + \{0, r\} \) for some \( h \in Z_n \). Now

\[
(r - 1)\{0, 1\} + \{0, r\} = (2r - 1)\{0, 1\}.
\]

If we apply \( \theta \) to each side of this equation, we have that

\[
(r - 1)\{0, 1\} + h + \{0, r\} = (2r - 1)\{0, 1\}.
\]

It follows from Lemma 9 that \( h = 0 \) and so \( \{0, r\} \theta = \{0, r\} \).

We now show that \( (1) \theta = 1 \). We do this by considering separately the cases where \( n \) is even and \( n \) is odd.

**Case 1.** \( n \) is even. Then \( \{0, 1\} + \{0, 1, 3\} = \{0, 1, 2, 3, 4\} = 4\{0, 1\} \). Applying \( \theta \) to this equation we have \( \{0, 1\} + \{a, b, c\} = 4\{0, 1\} \). It follows that \( \{a, b, c\} = \{0, 1, 3\} \) or \( \{a, b, c\} = \{0, 2, 3\} \). Now the following equalities hold:

\[
\frac{n - 4}{2}\{0, 2\} + \{0, 1, 3\} = (n - 1) + (n - 2)\{0, 1\}
\]
and

\[(2) \quad \frac{n - 4}{2} (0, 2) + (0, 2, 3) = 2 + (n - 2)(0, 1).\]

Suppose that \(\{0, 1, 3\} \theta = \{0, 2, 3\}\). Then, using (1), we obtain

\[
\frac{n - 4}{2} (0, 2) + (0, 2, 3) = (n - 1) \theta + (n - 2)(0, 1).
\]

Using this equation, equation (2), and Lemma 9, we have that \((n - 1) \theta = 2\) which is a contradiction since 2 is not a generator of \(Z_n\). Thus \(\{0, 1, 3\} \theta = \{0, 1, 3\}\). Applying \(\theta\) to equation (1) and by Lemma 9, we have \((n - 1) \theta = n - 1\) and hence, \((1) \theta = 1\).

**Case 2.** \(n\) is odd. In this case we have the equation

\[
1 + (n - 2)(0, 2) = (n - 2)(0, 1).
\]

By applying \(\theta\) to this equation we conclude that \((1) \theta = 1\).

Next suppose that \(n/2 < r \leq n\). Then \(\{0, r\} = r + \{0, n - r\}\), so that \(\{0, r\} \theta = r \theta + \{0, n - r\} \theta = r + \{0, n - r\} = \{0, r\}\).

**Lemma 11.** If \(\theta \in \text{Aut} \, F(Z_n)\) and \(\{0, 1\} \theta = \{0, 1\}\), then \(A \theta = A\) for every \(A \in F(Z_n)\).

**Proof.** By the preceding lemma, \(\{0, r\} \theta = \{0, r\}\) for every \(r \in Z_n \setminus \{0\}\) and \(\theta \mid Z_n\) is the identity. If \(A \in F(Z_n)\) and \(|A| = n - 1\), then \(A\) is a translate of \((n - 2)(0, 1)\) and so \(A \theta = A\). Suppose (by way of contradiction) that there exists \(A \in F(Z_n)\) such that \(A \theta \neq A\). Then \(|A| \leq n - 2\) and we may assume that if \(B \in F(G)\) with \(|A| < |B|\), then \(B \theta = B\). Let \(w \in A \theta \setminus A\) and \(u \in Z_n \setminus A\) with \(w \neq u\). Then \(w \in A + \{0, w - u\}\), but \(w \in A \theta + \{0, w - u\} = (A + \{0, w - u\}) \theta\). By the maximality of \(|A|\), \(|A| \geq |A + \{0, w - u\}| \geq |A|\) and so \(A = A + \{0, w - u\}\). Therefore, \(w - u \in L(A)\) and so \(|L(A)| \geq 2\). By Lemma 7, \(A = \bigcup_{i=1}^{t}(L(A) + a_i)\). Since subgroups of \(Z_n\) are fixed by \(\theta\), all cosets of \(L(A)\) are fixed by \(\theta\). Hence, \(t \geq 2\). Now \(L(A) + u \not\leq A\). Since we have shown that \(w - u \in L(A)\) for every \(u \in Z_n \setminus A\), we have that \(Z_n \setminus A\) is a single coset of \(L(A)\) and hence, \(A\) is the union of all but one coset of \(L(A)\). Let \(a\) be the smallest positive integer such that \(L(A) = \langle a \rangle\). Then \(a > 2\) and a system of representatives for the cosets of \(L(A)\) in \(Z_n\) is \(\{0, 1, \ldots, a - 1\}\). We may assume that \(\{a_1, \ldots, a_t\} \subseteq \{0, 1, \ldots, a - 1\}\). Let \(b \in \{0, 1, \ldots, a - 1\}\) such that \(L(A) + b = L(A) + w\). If \(x = (a - 1) - b\), then \(\{a_i + x, \ldots, a_t + x\}\) is a system of representatives for all but one coset of \(L(A)\). The
coset not included is $L(A) + (a - 1)$. Then $L(A) + \{0, 1, \cdots, a - 2\} = \bigcup_{i=1}^{l} (L(A) + a_i + x) = A + x$. Now $L(A)\theta = L(A)$ and

$$\{0, 1, \cdots, a - 2\}\theta = (a - 2)\{0, 1\}$$

and consequently, $(A + x)\theta = A + x$. Since $x\theta = x$, we have that $A\theta = A$, which is a contradiction. Thus the lemma is proven.

**Lemma 12.** If $\theta \in \text{Aut} F(Z_n)$ and $\{0, 1\}\theta = c + \{0, 1\}$ for some $c \in Z_n$, then $\{0, a\}\theta = ac + \{0, a\}$ for $1 \leq a < n/2$.

**Proof.** By Lemma 6, $\{0, a\}\theta = h + \{0, a\}$ for some $h \in Z_n$. If $2 \leq a < n/2$, then $(a - 1)\{0, 1\} + \{0, a\} = (2a - 1)\{0, 1\}$. If we apply $\theta$ to this equation we obtain $(a - 1)c + (a - 1)\{0, 1\} + h + \{0, a\} = (2a - 1)c + (2a - 1)\{0, 1\}$. Thus, $(a - 1)c + h \equiv (2a - 1)c \pmod{n}$, and so $h \equiv ac \pmod{n}$.

**Lemma 13.** Let $a \in Z_n$ with $2 \leq a \leq n/2$. Then

1. if $k$ and $l$ are positive integers such that $k\{0, 1\} + \{0, 1, a\} = l\{0, 1\}$, then $a - 2 \leq k$ and $2a - 2 \leq l$;
2. if $k$ and $l$ are positive integers such that $k\{0, 1\} + \{0, a - 1, a\} = l\{0, 1\}$, then $a - 2 \leq k$ and $2a - 2 \leq l$;
3. if $1 < b < a - 1$, then there exist positive integers $k$ and $l$ such that $k < a - 2$ and $k\{0, 1\} + \{0, b, a\} = l\{0, 1\}$;
4. if $x, y, z \in Z_n$ with $x < y < z < n$ and $(a - 2)\{0, 1\} + \{x, y, z\} = (2a - 2)\{0, 1\}$, then $x = 0$ and $z = a$;
5. $(a - 2)\{0, 1\} + \{0, 1, a\} = (2a - 2)\{0, 1\}$;
6. $(a - 2)\{0, 1\} + \{0, a - 1, a\} = (2a - 2)\{0, 1\}$.

**Proof.** Clearly (v) and (vi) hold. To see that (i) is true, we observe that if $1 \leq k < a - 2$, then $a - 1 \notin k\{0, 1\} + \{0, 1, a\}$ and so there is no natural number $l$ such that $k\{0, 1\} + \{0, 1, a\} = l\{0, 1\}$. Thus if $k$ and $l$ are natural numbers such that $k\{0, 1\} + \{0, 1, a\} = l\{0, 1\}$, then $a - 2 \leq k$ and so $2a - 2 \leq l$. The proof of (ii) is similar and will be omitted. For (iii), let $1 < b < a - 1$ and $k = \max\{b - 1, a - (b + 1)\}$. If $k = b - 1$, then $k\{0, 1\} + \{0, b, a\} = (a + b - 1)\{0, 1\}$ and $k = b - 1 < a - 2$. If $k = a - (b + 1)$, then $k\{0, 1\} + \{0, b, a\} = (2a - b - 1)\{0, 1\}$ and $k = a - (b + 1) < a - 2$. For (iv) we suppose that $x < y < z$ and $(a - 2)\{0, 1\} + \{x, y, z\} = (2a - 2)\{0, 1\}$. Then we have $\{x, x + 1, \cdots, x + a - 2\} \cup \{y, y + 1, \cdots, y + a - 2\} \cup \{z, z + 1, \cdots, z + a - 2\} = \{0, 1, \cdots, 2a - 2\}$. The elements from $2a - 1$ to $n - 1$ belong to $Z_n$ but not to the right hand side. The left hand side is the union of three consecutive listings and so the elements from $2a - 1$ to $n - 1$ must occur between $x + a - 2$ and $y, y + a - 2$.
and $z$, or $z + a - 2$ and $x$. The first two cases cannot occur as this would force $y$ or $z$ to be larger than $n - 1$. Thus, $z + a - 2 = 2a - 2$ and so $z = a$. Then $x = 0$.

**Lemma 14.** Let $a \in \mathbb{Z}_n$ with $1 < a \leq n/2$ and $\theta \in \text{Aut } F(\mathbb{Z}_n)$ such that $(0, 1)\theta = c + \{0, 1\}$. Then

(i) $\{0, 1, a\} \theta$ is in the translation class of either $\{0, 1, a\}$ or $\{0, a - 1, a\}$;

(ii) $\{0, a - 1, a\} \theta$ is in the translation class of either $\{0, 1, a\}$ or $\{0, a - 1, a\}$;

(iii) if $\{0, 1, a\} \theta$ is in the translation class of $\{0, 1, a\}$, then $\{0, 1, a\} \theta = ac + \{0, 1, a\}$ and $\{0, a - 1, a\} \theta = ac + \{0, a - 1, a\}$;

(iv) if $\{0, 1, a\} \theta$ is in the translation class of $\{0, a - 1, a\}$, then $\{0, 1, a\} \theta = ac + \{0, a - 1, a\}$ and $\{0, a - 1, a\} \theta = ac + \{0, 1, a\}$.

**Proof.** (i) Let $\{0, 1, a\} \theta = \{x, y, z\}$. By (v) of Lemma 13, $(a - 2)(0, 1) + \{0, 1, a\} = (2a - 2)(0, 1)$. If we apply $\theta$ to this equation we obtain the equation

$$(a - 2)c + (a - 2)(0, 1) + \{x, y, z\} = (2a - 2)c + (2a - 2)(0, 1).$$

Thus

$$(a - 2)(0, 1) + \{x - ac, y - ac, z - ac\} = (2a - 2)(0, 1).$$

Without loss of generality we may assume that $0 \leq x - ac < y - ac < z - ac < n$. By Lemma 13 (iv), $x - ac = 0$ and $z - ac = a$. Let $b = y - ac$. Suppose (by way of contradiction) that $1 < b < a - 1$. Then by Lemma 13 (iii), there are positive integers $k$ and $l$ such that $k < a - 2$ and $k(0, 1) + \{x - ac, y - ac, z - ac\} = l(0, 1)$. Thus

$$(a - 2)c + k(0, 1) + \{x, y, z\} = (2a - 2)c + l(0, 1).$$

Applying $\theta^{-1}$ to this last equation, we obtain an equation of the form

$$d + k(0, 1) + \{0, 1, a\} = f + l(0, 1).$$

Hence,

$$((a - 2) - k)(0, 1) + k(0, 1) + \{0, 1, a\}$$

$$= (f - d) + ((a - 2) - k + l)(0, 1)$$

and so

$$(a - 2)(0, 1) + \{0, 1, a\} = (f - d) + ((a - 2) - k + l)(0, 1).$$

Hence, $(2a - 2)(0, 1) = (f - d) + ((a - 2) - k + l)(0, 1)$. It follows
that $f \equiv d \pmod{n}$. Therefore, $k\{0, 1 \} + \{0, 1, a \} = \ell \{0, 1 \}$, but this contradicts Lemma 13 (i). Consequently, $\{0, 1, a \} \theta$ is in the class of $\{0, 1, a \}$ or $\{0, a - 1, a \}$.

The proof of (ii) is similar. Parts (iii) and (iv) then follow from what has been shown.

The proof of the next lemma is straightforward and will be omitted.

**Lemma 15.** If $n$ is even, $a \in \mathbb{Z}_n$, $a$ is odd, and $1 < a \leq n/2$, then

(i) $(n - 4)/2\{0, 2 \} + \{0, 1, a \} = (n - 1) + (n - 2)\{0, 1 \}$;

(ii) $(n - 4)/2\{0, 2 \} + \{0, a - 1, a \} = (a - 1) + (n - 2)\{0, 1 \}$.

**Lemma 16.** If $n$ is even, $\theta \in \text{Aut} F(\mathbb{Z}_n)$, and $\{0, 1 \} \theta = c + \{0, 1 \}$, then $\theta$ is a standard automorphism and $c = 0$ or $c = n - 1$.

**Proof.** By Lemma 15,

\[ 1 + \frac{n - 4}{2} \{0, 2 \} + \{0, 1, 3 \} = (n - 2)\{0, 1 \}. \]

By Lemma 14, $\{0, 1, 3 \} \theta = 3c + \{0, 1, 3 \}$ or $\{0, 1, 3 \} \theta = 3c + \{0, 2, 3 \}$.

**Case 1.** $\{0, 1, 3 \} \theta = 3c + \{0, 1, 3 \}$. Then $\{0, 2, 3 \} \theta = 3c + \{0, 2, 3 \}$.

If we apply $\theta$ to equation (1), then by Lemma 12, we obtain that

\[ (1) \theta + c + \frac{n - 4}{2} \{0, 2 \} + \{0, 1, 3 \} = (n - 2)c + (n - 2)\{0, 1 \} \]

so that

\[ (1) \theta + c + \frac{n - 4}{2} \{0, 2 \} + \{0, 1, 3 \} = (n - 2)\{0, 1 \}. \]

By equation (1), we have that $(1) \theta + c + (n - 1) + (n - 2)\{0, 1 \} = (n - 2)\{0, 1 \}$ and hence, $(1) \theta + c + n - 1 \equiv 0 \pmod{n}$. Consequently, $(1) \theta + c - 1 \equiv 0 \pmod{n}$. By Lemma 15, we have that

\[ (2) \frac{n - 4}{2} \{0, 2 \} + \{0, 2, 3 \} = 2 + (n - 2)\{0, 1 \}. \]

Applying $\theta$ to this equation we obtain that $-2(1) \theta + c + 2 \equiv 0 \pmod{n}$. Thus, $3c \equiv 0 \pmod{n}$. If $n = 8$, then $c = 0$ and by Lemma 11, $\theta = \iota$. Suppose that $n \geq 10$. Then by Lemma 15, we have that
\[ 1 + \frac{n-4}{2} \{0, 2\} + \{0, 1, 5\} = (n-2)\{0, 1\} \]

and
\[ \frac{n-4}{2} \{0, 2\} + \{0, 4, 5\} = 4 + (n-2)\{0, 1\} . \]

**Subcase 1.1.** \( \{0, 1, 5\} = 5c + \{0, 1, 5\} \). Then we have that
\[
(1)\theta + (n-4)c + \frac{n-4}{2} \{0, 2\} + 5c + \{0, 1, 5\}
\]
\[= (n-2)c + (n-2)\{0, 1\} . \]

Hence, \((1)\theta + 3c - 1 + (n-2)\{0, 1\} = (n-2)\{0, 1\} \) and so \((1)\theta + 3c - 1 \equiv 0 \pmod{n} \). Since \(3c \equiv 0 \pmod{n} \), we have that \((1)\theta = 1\) and \(c = 0\). Therefore, by Lemma 11, \( \theta = \iota \).

**Subcase 1.2.** \( \{0, 1, 5\} = 5c + \{0, 4, 5\} \). By an argument similar to the one given in Subcase 1.1, we obtain that \((1)\theta + 3c + 4 \equiv 0 \pmod{n} \). Since \(3c \equiv 0 \pmod{n} \), \((1)\theta \equiv (n-4) \pmod{n} \), but this is impossible as \(n-4\) is not a generator of \( \mathbb{Z}_n \).

**Case 2.** \( \{0, 1, 3\} = 3c + \{0, 2, 3\} \). Then by Lemma 15 and the same techniques as above, we obtain the congruences \((1)\theta + c + 2 \equiv 0 \pmod{n} \) and \(c - 2(1)\theta - 1 \equiv 0 \pmod{n} \). Then \(3c \equiv (n-3) \pmod{n} \). If \(n = 8\), then \(c \equiv 7 \pmod{8}\) and \((1)\theta = 7\). If \(\eta\) is the standard automorphism of \(F(Z_n)\) that takes 1 to \(-1\), then \(\{0, 1\} = \{0, 1\} \) and so, by Lemma 11, \(\theta = \iota\). Thus, \(\theta = \eta^{-1}\) and hence \(\theta\) is standard. If \(n \geq 10\), then, as in Case 1, \((1)\theta = n - 1\) and \(c = n - 1\). Thus, \(\theta\) is standard.

**Theorem 2.** If \(n\) is even, then \(\text{Aut } F(Z_n)\) consists only of standard automorphisms and so is isomorphic to \(\text{Aut } \mathbb{Z}_n\).

**Proof.** If \(\theta \in \text{Aut } F(Z_n)\), then \(\{0, 1\} = h + \{0, r\}\) for some \(h, r \in Z_n\). Let \(\eta\) be the standard automorphism of \(F(Z_n)\) that takes \(r\) to 1. Then \(\{0, 1\} = \{0, 1\} \) for some \(c \in Z_n\). By Lemma 16, \(\theta \eta = \psi\) is a standard automorphism and hence \(\theta = \psi \eta^{-1}\) is standard.

For \(n\) odd, we proceed almost as above.

**Lemma 17.** If \(n\) is odd, then
(i) \(\frac{n-3}{2} \{0, 2\} + \{0, 1, 3\} = (n-2)\{0, 1\}\);
(ii) \(\frac{n-3}{2} \{0, 2\} + \{0, 2, 3\} = 2 + (n-2)\{0, 1\}\).
THEOREM 3. If $n$ is odd, then $\text{Aut } F(Z_n)$ consists only of standard automorphisms and so is isomorphic to $\text{Aut } Z_n$.

Proof. If $\theta \in \text{Aut } F(Z_n)$, then $\{0, 1\} \theta = h + \{0, r\}$ for some $h, r \in Z_n$. Let $\gamma$ be the standard automorphism of $F(Z_n)$ that takes $r$ to 1. Then $\{0, 1\} \theta \gamma = c + \{0, 1\}$ for some $c \in Z_n$. If $\{0, 1, 3\} \theta \gamma = 3c + \{0, 1, 3\}$, then we apply $\theta \gamma$ to (i) of Lemma 17 and obtain the congruence $2c \equiv 0$ (modulo $n$) and so $c \equiv 0$ (modulo $n$). Thus, by Lemma 11, $\theta \gamma$ is a standard automorphism and consequently, $\theta$ is standard. The same conclusion holds if $\{0, 1, 3\} \theta \gamma = 3c + \{0, 2, 3\}$.

The following theorem gives a characterization of standard automorphisms for arbitrary groups. It was proven in [1, Theorem 5] and, for completeness, we repeat the proof here.

THEOREM 4. If $G$ is a group and $\theta \in \text{Aut } F(G)$, then $\theta$ is a standard automorphism if and only if $A, B \in F(G)$ with $A \subseteq B$ implies that $A\theta \subseteq B\theta$.

Proof. Clearly if $\theta$ is a standard automorphism, then $\theta$ preserves set containment. Conversely suppose that $\theta$ is inclusion preserving, let $\alpha = \theta|G$, and $\theta_a$ be the standard automorphism of $F(G)$ induced by $\alpha$. We proceed by induction on the cardinality of the sets in $F(G)$. If $A \in F(G)$ such that $|A| = 1$, then $A\theta = A\theta_a$. Assume that for all $A \in F(G)$ with $|A| \leq k$, $A\theta = A\theta_a$, and let $B \in F(G)$ with $|B| = k + 1$. If $D = B\theta_a$, then there exists $C \in F(G)$ such that $C\theta = D$. Since $\theta$ is inclusion preserving, if $b \in B$, then $b\alpha = b\theta \in B\theta$. Hence, $B\theta_a \subseteq B\theta$. If $x \in C$, then $x\alpha \in C\theta = D$. Thus, $x\alpha = b\alpha$ for some $b \in B$ and so $x = b$. Therefore, $C \subseteq B$. If $C \neq B$, then, by the inductive hypothesis, $C\theta = C\theta_a = D = B\theta_a$ and so $C = B$. Therefore, $C = B$ and so $B\theta = C\theta = D = B\theta_a$. Thus, $\theta$ is the standard automorphism $\theta_a$.

We now extend our results to a larger class of groups.

THEOREM 5. If $G$ is a subgroup of $Q/Z$ such that $|G| > 5$, then $\text{Aut } F(G)$ consists only of standard automorphisms and hence $\text{Aut } F(G)$ is isomorphic to $\text{Aut } G$.

Proof. If $G$ is finite, then $G$ is cyclic with $|G| > 5$ and so $\text{Aut } F(G)$ consists only of standard automorphisms. Suppose that $G$ is infinite and let $A, B \in F(G)$ with $A \subseteq B$, and let $\theta \in \text{Aut } F(G)$. Then there is a finite cyclic subgroup $H$ of $G$ such that $B \subseteq H$ and $|H| > 5$. Since $H$ is the only subgroup of $G$ of order $|H|$, we have, by Lemma 3, $H = H\theta$. Thus, $\theta|F(H) \in \text{Aut } F(H)$ and so $\theta|F(H)$ is a
standard automorphism of $F(H)$. Hence, $A\theta \subseteq B\theta$. By Theorem 4, $\theta$ is a standard automorphism of $F(G)$.

**COROLLARY 2.** If $P$ denotes the set of prime integers and $q \in P$, then $\text{Aut} F(Z(q^\infty))$ is isomorphic to $\text{Aut} Z(q^\infty)$ and $\text{Aut} F(Q/Z)$ is isomorphic to $\prod_{p \in P} \text{Aut} F(Z(p^\infty))$.

**Proof.** By [2, p. 221–222], $Q/Z$ is isomorphic to $\sum_{p \in P} Z(p^\infty)$ and $\text{Aut} Q/Z$ is isomorphic to $\prod_{p \in P} \text{Aut} Z(p^\infty)$. With these observations the corollary is an immediate consequence of the theorem.

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