WEAK AND NORM APPROXIMATE IDENTITIES ARE DIFFERENT

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An example is given of a convolution measure algebra which has a bounded weak approximate identity, but no norm approximate identity.

1. Introduction. Let $A$ be a commutative Banach algebra, $A'$ the dual space of $A$, and $\Delta A$ the maximal ideal space of $A$. A weak approximate identity for $A$ is a net $\{e(\lambda) : \lambda \in \Lambda\}$ in $A$ such that

$$\chi(e(\lambda)\alpha) \rightarrow \chi(\alpha)$$

for all $\alpha \in A$, $\chi \in \Delta A$. A norm approximate identity for $A$ is a net $\{e(\lambda) : \lambda \in \Lambda\}$ in $A$ such that

$$\| e(\lambda)a - a \| \rightarrow 0$$

for all $a \in A$. A net $\{e(\lambda) : \lambda \in \Lambda\}$ in $A$ is bounded and of norm $M$ if there exists a positive number $M$ such that $\| e(\lambda) \| \leq M$ for all $\lambda \in \Lambda$.

It is well known that if $A$ has a bounded weak approximate identity for which $f(e(\lambda)\alpha) \rightarrow f(\alpha)$ for all $f \in A'$ and $\alpha \in A$, then $A$ has a bounded norm approximate identity [1, Proposition 4, page 58]. However, the situation is different if weak convergence is with respect to $\Delta A$ and not $A'$. An example is given in § 2 of a Banach algebra $A$ which has a weak approximate identity, but does not have a norm approximate identity. This algebra provides a counterexample to a theorem of J. L. Taylor [4, Theorem 3.1], because it is proved in [3, Corollary 3.2] that the structure space of a convolution measure algebra $A$ has an identity if and only if $A$ has a bounded weak approximate identity of norm one.

2. The example. Throughout this paper the set of complex numbers is denoted $C$ and the set of real numbers $R$.

Let $S$ be a commutative semigroup, and $s_{\lambda}(S)$ the Banach space of all complex functions $\alpha: S \rightarrow C$ such that $\| \alpha \| = \sum_{x \in S} |\alpha(x)|$ is finite, made into a convolution algebra under the product

$$\alpha \ast \beta = \sum_{x \in S} \sum_{u \in S} \alpha(u)\beta(v)\delta_x,$$

where $\delta_x$ represents the point mass at $x \in S$, $\alpha = \sum_{x \in S} \alpha(x)\delta_x$ and $\beta = \sum_{x \in S} \beta(x)\delta_x$. A semicharacter on $S$ is a bounded nonzero function $\chi: S \rightarrow C$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$. The set of
all semicharacters is denoted \( \hat{S} \).

It has been shown in a previous paper [3] that if \( \zeta(S) \) is semisimple, then the existence of a bounded weak approximate identity of norm one in \( \zeta(S) \) is equivalent to the existence of a net \( \{u_d\} \) in \( S \) such that \( \chi(u_d) \to 1 \) for all \( \chi \in \hat{S} \). It has also been shown that the existence of a norm approximate identity bounded by 1 is equivalent to the existence of a net \( \{u_d\} \) in \( S \) with the following property: for each \( x \in S \), there exists \( d_x \) such that \( xu_d = x \) for all \( d \geq d_x \). For the particular semigroup \( S \) to follow, it will be shown that \( \zeta(S) \) does indeed have a bounded weak approximate identity, but does not have a norm approximate identity.

Let the set of integers be denoted by \( \mathbb{Z} \) and the set of positive integers by \( \mathbb{Z}^+ \). Further, let \( S = \{m/n: m, n \in \mathbb{Z}^+\} \) under addition. Then \( S \) is a cancellative semigroup and so \( \zeta(S) \) is semisimple [2]. If \( \chi \in \hat{S} \), then \( \chi \) is uniquely determined by its values on \( \{1/n: n \in \mathbb{Z}^+\} \). For if \( m \) is any positive integer, then for all \( n \in \mathbb{Z}^+ \), \( \chi(m/n) = \chi(1/n)^m \).

In fact \( \chi(1) = \chi(n/n) = \chi(1/n)^n \) for all \( n \in \mathbb{Z}^+ \), and so \( \chi(1/n) \) is an \( n \)th root of \( \chi(1) \). Now, each pair \( (k, z) \), where \( k \in \mathbb{Z} \) and \( z = re^{i\theta} \) with \( |z| \leq 1 \) and \( r, \theta \in \mathbb{R} \), determines a semicharacter \( \chi_{k,z} \) of \( S \) by defining

\[
\chi_{k,z}(m/n) = r^{m/n}e^{i(m\theta+2k\pi)/n}
\]

for all \( m/n \) in \( S \). It is clear that \( \chi_{k,z}(1/n) \to 1 \) for each \( \chi_{k,z} \in \hat{S} \). However, not all semicharacters have such a nice form. In constructing an arbitrary semicharacter \( \chi \), there are very few restrictions imposed upon how the \( n \)th root of \( \chi(1) \) is to be chosen. Thus, a more elaborate argument is required to obtain a weak approximate identity for \( \zeta(S) \).

**Lemma 2.1.** Let \( G \) be an infinite discrete group with identity \( e \). Then there exists a net \( \{g_\lambda\} \subset G \), \( g_\lambda \neq e \) for all \( \lambda \), such that \( \chi(g_\lambda) \to 1 \) for each \( \chi \in \hat{G} \).

**Proof.** Let \( \hat{G} \) be the Bohr compactification of \( G \). Then there is an algebra isomorphism \( \hat{i} \) of \( G \) onto a dense subset of \( \hat{G} \). Specifically, for each \( g \in \hat{G} \), there exists a net \( \{\hat{i}(g_\lambda): g_\lambda \in G\} \) such that \( \hat{i}(g_\lambda) \to g \); equivalently, \( \hat{\chi}(\hat{i}(g_\lambda)) \to \hat{\chi}(g) \) for each \( \chi \in \hat{G} \), where \( \hat{\chi} \) is the unique extension of \( \chi \in \hat{G} \) to \( \hat{\chi} \in \hat{G} \) [3]. Since \( \hat{G} \) is infinite and compact, the identity \( \hat{i}(e) \) of \( \hat{G} \) is not isolated in \( \hat{G} \). Hence, there is a net \( \{\hat{i}(g_\lambda): g_\lambda \in G\} \), \( g_\lambda \neq e \) for all \( \lambda \), such that \( \hat{i}(g_\lambda) \to \hat{i}(e) \). Therefore,

\[
\chi(g_\lambda) = \hat{\chi}(\hat{i}(g_\lambda)) \to \hat{\chi}(\hat{i}(e)) = 1 \quad \text{for each} \quad \chi \in \hat{G} .
\]

Let \( T = \{z \in \mathbb{C}: |z| = 1\} \) and \( D = \{z \in \mathbb{C}: |z| \leq 1\} \). Then the previous lemma yields the following number-theoretic result.
THEOREM 2.2. Let \( \{z_1, z_2, \ldots, z_p\} \subset T, p \in \mathbb{Z}^+ \). Then for each \( \varepsilon > 0 \) there exists \( m \in \mathbb{Z}^+ \) such that \( |(z_i)^m - 1| < \varepsilon \) for all \( i, 1 \leq i \leq p \).

Proof. Consider the group \( G = \mathbb{Z} \) under addition. Then \( \hat{G} = \{\chi_z : z \in T\} \), where \( \chi_z(n) = z^n \), \( n \in \mathbb{G} \). Now, let \( \varepsilon > 0 \) be given. By Lemma 2.1, there exists a net \( \{n_\lambda : \lambda \in \Lambda\} \subset G \), \( n_\lambda \neq 0 \) for all \( \lambda \), such that \( z^{n_\lambda} = \chi_z(n_\lambda) \to 1 \) for each \( \chi_z \in \hat{G} \). Without loss of generality, assume that \( n_\lambda \in \mathbb{Z}^+ \) for all \( \lambda \). Hence, given \( \{z_1, z_2, \ldots, z_p\} \subset T \), there exist \( \lambda_0, \lambda_1, \ldots, \lambda_p \) in \( \Lambda \) such that \( |z_i^{\lambda_i} - 1| < \varepsilon \) for all \( \lambda_0 \leq \lambda_i, 1 \leq i \leq p \). Thus, if \( \lambda_0 \in \Lambda \) is such that \( \lambda_0 = \lambda_i, 1 \leq i \leq p \), then with \( m = \lambda_0 \),

\[
|(z_i)^m - 1| < \varepsilon \quad \text{for} \quad i = 1, 2, \ldots, p .
\]

COROLLARY 2.3. Let \( \{z_1, z_2, \ldots, z_p\} \subset T, p \in \mathbb{Z}^+ \). Then for each \( \varepsilon, 0 < \varepsilon < 1 \), there exist neighborhoods \( U_1, U_2, \ldots, U_p \) and there exists \( m_0 \in \mathbb{Z}^+ \) such that

(1) \( z_i \in U_i \) and \( U_i \subset D, 1 \leq i \leq p \),

(2) \( |u - 1| < \varepsilon \) for all \( u \in U_i^{m_0} = \{w_1 w_2 \cdots w_{m_0} : w_j \in U_i\}, 1 \leq i \leq p \).

Proof. Let \( z_j = e^{i\theta_j}, 1 \leq j \leq p \). By Theorem 2.2, there exists \( m_0 \in \mathbb{Z}^+ \) such that \( |m_0 \theta_j (\mod 2\pi)| < \varepsilon/2 \) for all \( j \). Now, for each \( j \), let

\[
U_j = \left\{ w = |w| e^{i\varphi} \in D : |\omega - \theta_j| < \frac{\varepsilon}{4m_0} \quad \text{and} \quad |w| > \left[ 1 - \frac{\varepsilon}{4} \right]^{1/m_0} \right\} .
\]

Then if \( u \in U_j^{m_0}, u = w_1 w_2 \cdots w_{m_0}, w_k \in U_j \) for all \( k \), so that \( |w_1 + w_2 + \cdots + w_{m_0} - m_0 \theta_j| < \varepsilon/4 \) and \( |w_1| |w_2| \cdots |w_{m_0}| > 1 - \varepsilon/4 \). Thus, if \( u \in U_j^{m_0} \), then

\[
|u - 1| \leq |u - z_j^{m_0}| + |z_j^{m_0} - 1| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .
\]

After a technical lemma, the desired result will be proved. \( S \) continues to be the semigroup of positive rationals under addition.

LEMMA 2.4. Let \( \{\chi_1, \chi_2, \ldots, \chi_p\} \subset \hat{S}, p \in \mathbb{Z}^+ \). Then there exists a subsequence \( \{1/n_k : k \in \mathbb{Z}^+\} \) of \( \{1/n : n \in \mathbb{Z}^+\} \) and there exist \( z_1, z_2, \ldots, z_p \in T \) such that \( \chi_i(1/n_k) \to z_i \) for each \( i, 1 \leq i \leq p \).

Proof. Note that for each \( i \), \( \chi_i(1/n) \) is an \( n \)th root of \( \chi_i(1) \) and so \( |\chi_i(1/n)| \to 1 \) as \( n \to \infty \).
Now, \( \{ \chi_i(1/n) : n \in \mathbb{Z}^+ \} \) is a subset of the closed unit disk \( D \), and so by compactness has a convergent subsequence with limit \( z_i ; z_i \in T \) by the above remark. Further, if a subsequence \( \{ 1/n_k : i \in \mathbb{Z}^+ \} \) exists such that \( \chi_i(1/n_k) \to z_i \) for \( i = 1, 2, \ldots, j \), then by compactness \( \{ \chi_{j+i}(1/n_k) : i \in \mathbb{Z}^+ \} \) has a convergent subsequence \( \{ \chi_{j+i}(1/n_k) : k \in \mathbb{Z}^+ \} \) with limit \( z_{j+i} \in T \). Thus, the induction proof is complete.

**Theorem 2.5.** There exists a net \( \{ q_d : d \in \mathcal{D} \} \subseteq S \) such that \( \chi(q_d) \to 1 \) for each \( \chi \in \hat{S} \). Therefore, \( \varepsilon(S) \) has a weak approximate identity of norm one.

**Proof.** Let \( \mathcal{F}(\hat{S}) \) denote the collection of all finite subsets of \( \hat{S} \) and let \( \mathcal{D} = \mathbb{Z}^+ \times \mathcal{F}(\hat{S}) \) be directed by \( (n, A) \leq (m, B) \) if and only if \( n \leq m \) and \( A \subseteq B \).

Now, define a mapping \( d \mapsto q_d \) of \( \mathcal{D} \) into \( S \) as follows: For each \( d = (n, A) \), \( A = \{ \chi_i, \ldots, \chi_p \} \), fix a subsequence \( \{ 1/n_k : k \in \mathbb{Z}^+ \} \) such that \( \chi_i(1/n_k) \to z_i \in T \) for all \( i \). Then there exist \( m_0 \in \mathbb{Z}^+ \) and neighborhoods \( U_1, \ldots, U_p \) of \( z_1, \ldots, z_p \), respectively, such that \( |u - 1| < 1/n \) for all \( u \in U_i \), \( 1 \leq i \leq p \). Now, there exist \( K_i \in \mathbb{Z}^+ \) such that \( k \geq K_i \) implies \( \chi_i(1/n_k) \in U_i \) for \( 1 \leq i \leq p \). Hence, for each \( i, 1 \leq i \leq p \),

\[
|\chi_i(m_0/n_k) - 1| = |\chi_i(1/n_k)^{m_0} - 1| < \frac{1}{n}
\]

for all \( k \geq K_i \). Set \( K_0 = \max \{ K_i : i = 1, 2, \ldots, p \} \). Then define \( q_d = m_0/n_{K_0} \).

Finally, it remains to show that for each \( \chi \in \hat{S}, \chi(q_d) \to 1 \). So, let \( \varepsilon > 0 \) be given. Then choose \( n_0 \) such that \( (1/n_0) < \varepsilon \), and let \( A_0 = \{ \chi \} \). If \( d = (n, A) \geq (n_0, A_0) = d_0 \), then \( |\chi(q_d) - 1| < (1/n_0) < \varepsilon \).

**Corollary 2.6.** There exists a net \( \{ 1/n_d : d \in \mathcal{D} \} \subseteq \{ 1/n : n \in \mathbb{Z}^+ \} \) such that \( \chi(1/n_d) \to 1 \) for each \( \chi \in \hat{S} \).

**Proof.** Repeat the proofs of Lemma 2.4 and Theorem 2.5 with \( \{ 1/n : n \in \mathbb{Z}^+ \} \) replaced by \( \{ 1/n! : n \in \mathbb{Z}^+ \} \). Then in the proof of Theorem 2.5 choose \( K_0 \) such that

(1) \( K_0 \geq \max \{ K_i : i = 1, 2, \ldots, p \} \) and

(2) \( n_{K_0} \geq m_0 \). Thus, \( q_d = m_0/n_{K_0} \) is of the form \( 1/n_d \) for some \( n_d \in \mathbb{Z}^+ \).

Theorem 2.5 and Corollary 2.6 make it clear that \( \varepsilon(S) \) has a bounded weak approximate identity \( \{ \delta_{1/n_d} : d \in \mathcal{D} \} \) [3]. However, \( S \) does not have relative units. That is, given \( m/n \in S \), there is no \( v \in S \) such that \( v(m/n) = m/n \). Thus, \( \varepsilon(S) \) does not have a norm approximate identity, bounded or unbounded.
3. A general result. The same techniques developed in § 2 can be used to prove a useful result about weak approximate identities of norm one for a commutative Banach algebra.

**Theorem 3.1.** Let $A$ be a commutative Banach algebra. Then $A$ has a weak approximate identity of norm one if and only if there exists a net $\{v(\rho) : \rho \in \mathcal{S}\}$ in $A$, $|v(\rho)| \leq 1$ for all $\rho$, such that $|\chi(v(\rho))| \to 1$ for all $\chi \in \Delta A$.

**Proof.** If $A$ has a weak approximate identity of norm one, then there exists a net $\{v(\rho) : \rho \in \mathcal{S}\}$ in $A$, $|v(\rho)| \leq 1$ for all $\rho$, such that
\[
\chi(v(\rho)a) \to \chi(a) \quad \text{for all} \quad a \in A, \quad \chi \in \Delta A.
\]
Thus, for each $\chi \in \Delta A$, $\chi(a) \neq 0$ for some $a \in A$ implies that $\chi(v(\rho)) \to 1$ and hence $|\chi(v(\rho))| \to 1$.

Conversely, assume that $\{v(\rho)\}$ is such that $|\chi(v(\rho))| \to 1$ for each $\chi \in \Delta A$. Let $\mathcal{F}(\Delta A)$ be the collection of all finite subsets of $\Delta A$ and let $A = Z^+ \times \mathcal{F}(\Delta A)$ be directed by $(n, F) \leq (m, E)$ if and only if $n \leq m$ and $F \subseteq E$.

Then define a mapping $\lambda \mapsto e(\lambda)$ of $A$ into $A$ as follows: for each $\lambda = (n, F)$, where $n \in Z^+$ and $F = \{\chi_1, \chi_2, \ldots, \chi_r\}$, there exists by compactness of $D$ a subnet $\{v(\rho'_i)\}$ of $\{v(\rho)\}$ such that $\chi_i(\rho'_i) \to z_i \in T$ for $i, 1 \leq i \leq r$. By Corollary 2.3, there exists $m_0 \in Z^+$ and neighborhoods $U_i$ of $z_i$ in $D$ such that $|z - 1| < 1/n$ for all $z \in U_{i}^{m_0}$, $1 \leq i \leq r$. Now, let $\rho'_0$ be such that $\chi_i(v(\rho'_0)) \in U_i$ for all $i, 1 \leq i \leq r$, and define $e(\lambda) = v(\rho'_0)^{m_0}$. Note that for each $i$,
\[
|\chi_i(e(\lambda)) - 1| = |\chi_i(v(\rho'_0)^{m_0}) - 1| = |(\chi_i(v(\rho'_0)))^{m_0} - 1| < \frac{1}{n}.
\]
Thus, $\chi(e(\lambda)) \to 1$ for each $\chi \in \Delta A$ and hence $\chi(e(\lambda)a) \to \chi(a)$ for each $\chi \in \Delta A$, $a \in A$. Also, $||e(\lambda)|| = ||v(\rho'_0)^{m_0}|| \leq 1$ for all $\lambda \in A$.

**Corollary 3.2.** Let $S$ be a commutative semigroup for which $\mathcal{L}(S)$ is semisimple. Then $\mathcal{L}(S)$ has a weak approximate identity of norm one if and only if there exists a net $\{s(\rho) : (\rho) \in \mathcal{S}\}$ in $S$ such that $|\chi(s(\rho))| \to 1$ for all $\chi \in \hat{S}$.

**Proof.** The Banach algebra $\mathcal{L}(S)$ has a weak approximate identity of norm one if and only if there exists a net $\{s(\lambda) : \lambda \in \Lambda\}$ in $S$ such that $\chi(s(\lambda)) \to 1$ for all $\chi \in \hat{S}$ [3]. Thus, the proof is completed by applying Theorem 3.1 with $v(\rho) = \delta_{s(\rho)}$ for all $\rho$. 

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