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# WEAK AND NORM APPROXIMATE IDENTITIES ARE DIFFERENT

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**An example is given of a convolution measure algebra which has a bounded weak approximate identity, but no norm approximate identity.**

**1. Introduction.** Let $A$ be a commutative Banach algebra, $A'$ the dual space of $A$, and $\Delta A$ the maximal ideal space of $A$. A weak approximate identity for $A$ is a net $\{e(\lambda): \lambda \in \Lambda\}$ in $A$ such that

$$\chi(e(\lambda)a) \longrightarrow \chi(a)$$

for all $a \in A$, $\chi \in \Delta A$. A norm approximate identity for A is a net $\{e(\lambda): \lambda \in \Lambda\}$ in $A$ such that

$$\|e(\lambda)a - a\| \longrightarrow 0$$

for all $a \in A$. A net $\{e(\lambda): \lambda \in \Lambda\}$ in $A$ is bounded and of norm $M$ if there exists a positive number $M$ such that $\|e(\lambda)\| \leqq M$ for all $\lambda \in \Lambda$.

It is well known that if $A$ has a bounded weak approximate identity for which $f(e(\lambda)a) \to f(a)$ for all $f \in A'$ and $a \in A$, then $A$ has a bounded norm approximate identity [1, Proposition 4, page 58]. However, the situation is different if weak convergence is with respect to $\Delta A$ and not $A'$. An example is given in § 2 of a Banach algebra $A$ which has a weak approximate identity, but does not have a norm approximate identity. This algebra provides a counterexample to a theorem of J. L. Taylor [4, Theorem 3.1], because it is proved in [3, Corollary 3.2] that the structure space of a convolution measure algebra $A$ has an identity if and only if $A$ has a bounded weak approximate identity of norm one.

**2. The example.** Throughout this paper the set of complex numbers is denoted $\boldsymbol{C}$ and the set of real numbers $\boldsymbol{R}$.

Let $S$ be a commutative semigroup, and $\ell_1(S)$ the Banach space of all complex functions $\alpha: S \to \boldsymbol{C}$ such that $\|\alpha\| = \sum_{x \in S} |\alpha(x)|$ is finite, made into a convolution algebra under the product

$$\alpha * \beta = \sum_{x \in S} \sum_{\substack{u,v \\ uv=x}} \alpha(u)\beta(v)\delta_x ,$$

where $\delta_x$ represents the point mass at $x \in S$, $\alpha = \sum_{x \in S} \alpha(x)\delta_x$ and $\beta = \sum_{x \in S} \beta(x)\delta_x$. A semicharacter on $S$ is a bounded nonzero function $\chi: S \to \boldsymbol{C}$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$. The set of all semicharacters is denoted $\hat{S}$.

It has been shown in a previous paper [3] that if $\ell_1(S)$ is semisimple, then the existence of a bounded weak approximate identity of norm one in $\ell_1(S)$ is equivalent to the existence of a net $\{u_d\}$ in $S$ such that $\chi(u_d) \to 1$ for all $\chi \in \hat{S}$. It has also been shown that the existence of a norm approximate identity bounded by 1 is equivalent to the existence of a net $\{u_d\}$ in $S$ with the following property: for each $x \in S$, there exists $d_x$ such that $xu_d = x$ for all $d \geqq d_x$. For the particular semigroup $S$ to follow, it will be shown that $\ell_1(S)$ does indeed have a bounded weak approximate identity, but does not have a norm approximate identity.

Let the set of integers be denoted by $\boldsymbol{Z}$ and the set of positive integers by $\boldsymbol{Z}^+$. Further, let $S = \{m/n\colon m, n \in \boldsymbol{Z}^+\}$ under addition. Then $S$ is a cancellative semigroup and so $\ell_1(S)$ is semisimple [2]. If $\chi \in \hat{S}$, then $\chi$ is uniquely determined by its values on $\{1/n\colon n \in \boldsymbol{Z}^+\}$. For if $m$ is any positive integer, then for all $n \in \boldsymbol{Z}^+$, $\chi(m/n) = \chi(1/n)^m$. In fact $\chi(1) = \chi(n/n) = \chi(1/n)^n$ for all $n \in \boldsymbol{Z}^+$, and so $\chi(1/n)$ is an $n$th root of $\chi(1)$. Now, each pair $(k, z)$, where $k \in \boldsymbol{Z}$ and $z = re^{i\theta}$ with $|z| \leqq 1$ and $r, \theta \in \boldsymbol{R}$, determines a semicharacter $\chi_{k,z}$ of $S$ by defining

$$\chi_{k,z}(m/n) = r^{m/n} e^{im(\theta+2k\pi)/n}$$

for all $m/n$ in $S$. It is clear that $\chi_{k,z}(1/n) \to 1$ for each $\chi_{k,z} \in \hat{S}$. However, not all semicharacters have such a nice form. In constructing an arbitrary semicharacter $\chi$, there are very few restrictions imposed upon how the $n$th root of $\chi(1)$ is to be chosen. Thus, a more elaborate argument is required to obtain a weak approximate identity for $\ell_1(S)$.

LEMMA 2.1. *Let $G$ be an infinite discrete group with identity $e$. Then there exists a net $\{g_\lambda\} \subset G$, $g_\lambda \neq e$ for all $\lambda$, such that $\chi(g_\lambda) \to 1$ for each $\chi \in \hat{G}$.*

*Proof.* Let $\bar{G}$ be the Bohr compactification of $G$. Then there is an algebra isomorphism $i$ of $G$ onto a dense subset of $\bar{G}$. Specifically, for each $g \in \bar{G}$, there exists a net $\{i(g_\lambda)\colon g_\lambda \in G\}$ such that $i(g_\lambda) \to g$; equivalently, $\bar{\chi}(i(g_\lambda)) \to \bar{\chi}(g)$ for each $\chi \in \hat{G}$, where $\bar{\chi}$ is the unique extension of $\chi \in \hat{G}$ to $\bar{\chi} \in \hat{\bar{G}}$ [3]. Since $\bar{G}$ is infinite and compact, the identity $i(e)$ of $\bar{G}$ is not isolated in $\bar{G}$. Hence, there is a net $\{i(g_\lambda)\colon g_\lambda \in G\}$, $g_\lambda \neq e$ for all $\lambda$, such that $i(g_\lambda) \to i(e)$. Therefore,

$$\chi(g_\lambda) = \bar{\chi}(i(g_\lambda)) \longrightarrow \bar{\chi}(i(e)) = 1 \quad \text{for each} \quad \chi \in \hat{G}\,.$$

Let $T = \{z \in \boldsymbol{C}\colon |z| = 1\}$ and $D = \{z \in \boldsymbol{C}\colon |z| \leqq 1\}$. Then the previous lemma yields the following number-theoretic result.

THEOREM 2.2. *Let* $\{z_1, z_2, \cdots, z_p\} \subset T$, $p \in \boldsymbol{Z}^+$. *Then for each* $\varepsilon > 0$ *there exists* $m \in \boldsymbol{Z}^+$ *such that* $|(z_i)^m - 1| < \varepsilon$ *for all* $i, 1 \leqq i \leqq p$.

*Proof.* Consider the group $G = \boldsymbol{Z}$ under addition. Then $\hat{G} = \{\chi_z : z \in T\}$, where $\chi_z(n) = z^n$, $n \in G$. Now, let $\varepsilon > 0$ be given. By Lemma 2.1, there exists a net $\{n_\lambda : \lambda \in \Lambda\} \subset G$, $n_\lambda \neq 0$ for all $\lambda$, such that $z^{n_\lambda} = \chi_z(n_\lambda) \rightarrow 1$ for each $\chi_z \in \hat{G}$. Without loss of generality, assume that $n_\lambda \in \boldsymbol{Z}^+$ for all $\lambda$. Hence, given $\{z_1, z_2, \cdots, z_p\} \subset T$, there exist $\lambda_1, \lambda_2, \cdots, \lambda_p$ in $\Lambda$ such that $|z_i^{n_\lambda} - 1| < \varepsilon$ for all $\lambda \geqq \lambda_i$, $1 \leqq i \leqq p$. Thus, if $\lambda_0 \in \Lambda$ is such that $\lambda_0 \geqq \lambda_i$, $1 \leqq i \leqq p$, then with $m = n_{\lambda_0}$,

$$|(z_i)^m - 1| < \varepsilon \quad \text{for} \quad i = 1, 2, \cdots, p .$$

COROLLARY 2.3. *Let* $\{z_1, z_2, \cdots, z_p\} \subset T$, $p \in \boldsymbol{Z}^+$. *Then for each* $\varepsilon$, $0 < \varepsilon < 1$, *there exist neighborhoods* $U_1, U_2, \cdots, U_p$ *and there exists* $m_0 \in \boldsymbol{Z}^+$ *such that*

(1) $z_i \in U_i$ *and* $U_i \subset D$, $1 \leqq i \leqq p$,

(2) $|u - 1| < \varepsilon$ *for all*

$$u \in U_i^{m_0} = \{w_1 w_2 \cdots w_{m_0} : w_j \in U_i\} , \quad 1 \leqq i \leqq p .$$

*Proof.* Let $z_j = e^{i\theta_j}$, $1 \leqq j \leqq p$. By Theorem 2.2, there exists $m_0 \in \boldsymbol{Z}^+$ such that $|m_0\theta_j \pmod{2\pi}| < \varepsilon/2$ for all $j$. Now, for each $j$, let

$$U_j = \left\{w = |w|e^{i\omega} \varepsilon D : |\omega - \theta_j| < \frac{\varepsilon}{4m_0} \text{ and } |w| > \left[1 - \frac{\varepsilon}{4}\right]^{1/m_0}\right\} .$$

Then if $u \in U_j^{m_0}$, $u = w_1 w_2 \cdots w_{m_0}$, $w_k \in U_j$ for all $k$, so that $|\omega_1 + \omega_2 + \cdots + \omega_{m_0} - m_0\theta_j| < \varepsilon/4$ and $|w_1||w_2| \cdots |w_{m_0}| > 1 - \varepsilon/4$. Thus, if $u \in U_j^{m_0}$, then

$$|u - 1| \leqq |u - z_j^{m_0}| + |z_j^{m_0} - 1| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

After a technical lemma, the desired result will be proved. $S$ continues to be the semigroup of positive rationals under addition.

LEMMA 2.4. *Let* $\{\chi_1, \chi_2, \cdots, \chi_p\} \subset \hat{S}$, $p \in \boldsymbol{Z}^+$. *Then there exists a subsequence* $\{1/n_k : k \in \boldsymbol{Z}^+\}$ *of* $\{1/n : n \in \boldsymbol{Z}^+\}$ *and there exist* $z_1, z_2, \cdots, z_p \in T$ *such that* $\chi_i(1/n_k) \rightarrow z_i$ *for each* $i$, $1 \leqq i \leqq p$.

*Proof.* Note that for each $i$, $\chi_i(1/n)$ is an $n$th root of $\chi_i(1)$ and so $|\chi_i(1/n)| \rightarrow 1$ as $n \rightarrow \infty$.

Now, $\{\chi_1(1/n): n \in Z^+\}$ is a subset of the closed unit disk $D$, and so by compactness has a convergent subsequence with limit $z_1$; $z_1 \in T$ by the above remark. Further, if a subsequence $\{1/n_\ell: \ell \in Z^+\}$ exists such that $\chi_i(1/n_\ell) \to z_i$ for $i = 1, 2, \cdots, j$, then by compactness $\{\chi_{j+1}(1/n_\ell): \ell \in Z^+\}$ has a convergent subsequence $\{\chi_{j+1}(1/n_k): k \in Z^+\}$ with limit $z_{j+1} \in T$. Thus, the induction proof is complete.

THEOREM 2.5. *There exists a net* $\{q_d: d \in \mathscr{D}\} \subset S$ *such that* $\chi(q_d) \to 1$ *for each* $\chi \in \hat{S}$. *Therefore,* $\ell_1(S)$ *has a weak approximate identity of norm one.*

*Proof.* Let $\mathscr{F}(\hat{S})$ denote the collection of all finite subsets of $\hat{S}$ and let $\mathscr{D} = Z^+ \times \mathscr{F}(\hat{S})$ be directed by $(n, A) \leqq (m, B)$ if and only if $n \leqq m$ and $A \subset B$.

Now, define a mapping $d \mapsto q_d$ of $\mathscr{D}$ into $S$ as follows: For each $d = (n, A)$, $A = \{\chi_1, \cdots, \chi_p\}$, fix a subsequence $\{1/n_k: k \in Z^+\}$ such that $\chi_i(1/n_k) \to z_i \in T$ for all $i$. Then there exist $m_0 \in Z^+$ and neighborhoods $U_1, \cdots, U_p$ of $z_1, \cdots, z_p$, respectively, such that $|u - 1| < 1/n$ for all $u \in U_j^{m_0}$, $1 \leqq i \leqq p$. Now, there exist $K_i \in Z^+$ such that $k \geqq K_i$ implies $\chi_i(1/n_k) \in U_i$ for $1 \leqq i \leqq p$. Hence, for each $i$, $1 \leqq i \leqq p$,

$$|\chi_i(m_0/n_k) - 1| = |\chi_i(1/n_k)^{m_0} - 1| < \frac{1}{n}$$

for all $k \geqq K_i$. Set $K_0 = \max\{K_i: i = 1, 2, \cdots, p\}$. Then define $q_d = m_0/n_{K_0}$.

Finally, it remains to show that for each $\chi \in \hat{S}$, $\chi(q_d) \to 1$. So, let $\varepsilon > 0$ be given. Then choose $n_0$ such that $(1/n_0) < \varepsilon$, and let $A_0 = \{\chi\}$. If $d = (n, A) \geqq (n_0, A_0) = d_0$, then $|\chi(q_d) - 1| < (1/n_0) < \varepsilon$.

COROLLARY 2.6. *There exists a net* $\{1/n_d: d \in \mathscr{D}\} \subset \{1/n: n \in Z^+\}$ *such that* $\chi(1/n_d) \to 1$ *for each* $\chi \in \hat{S}$.

*Proof.* Repeat the proofs of Lemma 2.4 and Theorem 2.5 with $\{1/n: n \in Z^+\}$ replaced by $\{1/n!: n \in Z^+\}$. Then in the proof of Theorem 2.5 choose $K_0$ such that

(1) $K_0 \geqq \max\{K_i: i = 1, 2, \cdots, p\}$ and

(2) $n_{K_0} \geqq m_0$. Thus, $q_d = m_0/n_{K_0}!$ is of the form $1/n_d$ for some $n_d \in Z^+$.

Theorem 2.5 and Corollary 2.6 make it clear that $\ell_1(S)$ has a bounded weak approximate identity $\{\delta_{1/n_d}: d \in \mathscr{D}\}$ [3]. However, $S$ does not have relative units. That is, given $m/n \in S$, there is no $v \in S$ such that $v(m/n) = m/n$. Thus, $\ell_1(S)$ does not have a norm approximate identity, bounded or unbounded.

**3. A general result.** The same techniques developed in § 2 can be used to prove a useful result about weak approximate identities of norm one for a commutative Banach algebra.

THEOREM 3.1. *Let $A$ be a commutative Banach algebra. Then $A$ has a weak approximate identity of norm one if and only if there exists a net $\{v(\rho): \rho \in \mathscr{I}\}$ in $A$, $\|v(\rho)\| \leqq 1$ for all $\rho$, such that $|\chi(v(\rho))| \rightarrow 1$ for all $\chi \in \Delta A$.*

*Proof.* If $A$ has a weak approximate identity of norm one, then there exists a net $\{v(\rho): \rho \in \mathscr{I}\}$ in $A$, $\|v(\rho)\| \leqq 1$ for all $\rho$, such that

$$\chi(v(\rho)a) \longrightarrow \chi(a) \quad \text{for all} \quad a \in A\,, \quad \chi \in \Delta A\,.$$

Thus, for each $\chi \in \Delta A$, $\chi(a) \neq 0$ for some $a \in A$ implies that $\chi(v(\rho)) \rightarrow 1$ and hence $|\chi(v(\rho))| \rightarrow 1$.

Conversely, assume that $\{v(\rho)\}$ is such that $|\chi(v(\rho))| \rightarrow 1$ for each $\chi \in \Delta A$. Let $\mathscr{F}(\Delta A)$ be the collection of all finite subsets of $\Delta A$ and let $\Lambda = \boldsymbol{Z}^+ \times \mathscr{F}(\Delta A)$ be directed by $(n, F) \leqq (m, E)$ if and only if $n \leqq m$ and $F \subset E$.

Then define a mapping $\lambda \mapsto e(\lambda)$ of $\Lambda$ into $A$ as follows: for each $\lambda = (n, F)$, where $n \in \boldsymbol{Z}^+$ and $F = \{\chi_1, \chi_2, \cdots, \chi_r\}$, there exists by compactness of $D$ a subnet $\{v(\rho')\}$ of $\{v(\rho)\}$ such that $\chi_i(\rho') \rightarrow z_i \in T$ for $i, 1 \leqq i \leqq r$. By Corollary 2.3, there exists $m_0 \in \boldsymbol{Z}^+$ and neighborhoods $U_i$ of $z_i$ in $D$ such that $|z - 1| < 1/n$ for all $z \in U_i^{m_0}$, $1 \leqq i \leqq r$. Now, let $\rho_0'$ be such that $\chi_i(v(\rho_0')) \in U_i$ for all $i, 1 \leqq i \leqq r$, and define $e(\lambda) = v(\rho_0')^{m_0}$. Note that for each $i$,

$$\begin{aligned} |\chi_i(e(\lambda)) - 1| &= |\chi_i(v(\rho_0')^{m_0}) - 1| \\ &= |(\chi_i(v(\rho_0')))^{m_0} - 1| < \frac{1}{n}\,. \end{aligned}$$

Thus, $\chi(e(\lambda)) \rightarrow 1$ for each $\chi \in \Delta A$ and hence $\chi(e(\lambda)a) \rightarrow \chi(a)$ for each $\chi \in \Delta A$, $a \in A$. Also, $\|e(\lambda)\| = \|v(\rho_0')^{m_0}\| \leqq 1$ for all $\lambda \in \Lambda$.

COROLLARY 3.2. *Let $S$ be a commutative semigroup for which $\ell_1(S)$ is semisimple. Then $\ell_1(S)$ has a weak approximate identity of norm one if and only if there exists a net $\{s(\rho):(\rho) \in \mathscr{I})\}$ in $S$ such that $|\chi(s(\rho))| \rightarrow 1$ for all $\chi \in \hat{S}$.*

*Proof.* The Banach algebra $\ell_1(S)$ has a weak approximate identity of norm one if and only if there exists a net $\{s(\lambda): \lambda \in \Lambda\}$ in $S$ such that $\chi(s(\lambda)) \rightarrow 1$ for all $\chi \in \hat{S}$ [3]. Thus, the proof is completed by applying Theorem 3.1 with $v(\rho) = \delta_{s(\rho)}$ for all $\rho$.

## REFERENCES

1. F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, 1973.
2. E. Hewitt and H. S. Zuckerman, *The $\ell_1$-algebra of a commutative semigroup*, Trans. Amer. Math. Soc., **83** (1956), 70–97.
3. C. D. Lahr, *Approximate identities for convolution measure algebras*, Pacific J. Math., **47** (1973), 147–159.
4. J. L. Taylor, *The structure of convolution measure algebras*, Trans. Amer. Math. Soc., **119** (1965), 150–166.

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