THE SECOND DUAL OF $C(X)$

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In this paper, we undertake a study of the order dual, denoted \( M \), of the radon measures of compact support on a locally compact space \( X \). In the case that \( X \) is realcompact, \( M \) is the second (order) dual of the space of continuous functions on \( X \), \( C(X) \). We define the sublattice of semi-continuous elements, \( S(X) \), and prove that each member of \( M \) is dominated by a member of \( S(X) \). It follows that the ideal generated by \( S(X) \) in \( M \) is all of \( M \). On the other hand, the ideal generated by \( C(X) \) in \( M \) is all of \( M \) if and only if \( X \) is a cb-space.

Finally, we show that \( S(X) \) and \( C(X) \) can be identified in \( M \) as certain spaces of multiplication operators which are continuous with respect to certain weak topologies. This extends the work of J. Mack, who first characterized \( M \) as the (continuous) multiplication operators on the Radon measures.

Introduction. In [3] Kaplan considered \( C^k(X) = C^k \), the continuous functions of compact support on a locally compact space, and its order dual \( L^k \) (the space of Radon measures). In the process, he singled out \( \cup L(K) \), the ideal of those measures having compact support. It is the order dual of this space, denoted \( M \), in which we will be interested. In the case that \( X \) is realcompact, \( M \) is the second dual of the space of continuous functions and therefore of particular interest.

\( M \) has already been studied by Mack [5], who characterized it as the set of (order) continuous multiplication operators on \( L^k \). It is our purpose to extend his work. In considering the case where \( X \) is compact, Kaplan studied various sublattices of \( M \) including what he called the semi-continuous elements \( S(X) \). We will extend the study to our more general setting and show that \( S(X) \) and \( C(X) \) can be identified in \( M \) as spaces of multiplication operators on \( L^k \), continuous with respect to certain weak topologies. Thus we will relate the work of the two authors.

1. Preliminaries. The information and results summarized here will be used frequently in the rest of the paper. We assume a knowledge of the basic results on Riesz spaces.

1.1. A subset \( B \) of a Riesz space (vector lattice) \( E \) is called
bounded if it is contained in some interval $[a, b] = \{c \in E: a \leq c \leq b\}$. 

$E$ is called (Dedekind) complete if the supremum, $\vee B$, and the infimum, $\wedge B$, exist for all bounded sets. If $E$ and $F$ are vector lattices, a map from $E$ to $F$ is called bounded if it transforms bounded sets into bounded sets. A linear mapping is called positive if it maps the positive cone $E_+$ into $F_+$. If $F$ is a complete vector lattice, then a linear mapping from $E$ to $F$ is bounded if and only if it is the difference of two positive mappings, and the set of all such mappings is a complete vector lattice. The set of bounded linear functionals is denoted $E^b$. If $A \subset E^b$ is directed upward, that is for each $f_1$ and $f_2$ in $A$, there exists $f^* \in A$ such that $f^* \geq f_1$ and $f^* \geq f_2$, then $f = \vee A$ if and only if $\langle a, f \rangle = \vee \{\langle a, f_a \rangle | f_a \in A\}$ for all $a$ in $E_+$. Given a subset $B$ of $E^b$ one may adjoin to it all suprema of finite subsets and the resulting set will be directed upward and have the same supremum, if it exists, as the subset $B$.

1.2. Given a directed net $\{a_\alpha\}$ in $E$, $a_\alpha \uparrow a$ means $a_\alpha \geq a_\beta$ for $\alpha \geq \beta$ and $a = \vee a_\alpha$, $a_\alpha \downarrow a$ is defined similarly. A net $\{a_\alpha\}$ converges to $a$ if there exists a net $\{b_\alpha\}$ such that $b_\alpha \downarrow 0$ and $|a - a_\alpha| \leq b_\alpha$ for all $\alpha$. We will write in this case $a_\alpha \rightarrow a$. A linear functional $\phi$ is called continuous if $a_\alpha \rightarrow a$ implies $\langle a_\alpha, \phi \rangle \rightarrow \langle a, \phi \rangle$. $E^*$ denotes the space of continuous linear functionals.

1.3. A subset $A$ is closed if $\{a_\alpha\} \subset A$ and $a_\alpha \rightarrow a$ imply $a \in A$. An ideal is a linear subspace $I$ of $E$ such that $a \in I$ and $|b| \leq |a|$ imply $b \in I$. If $A$ is a subspace of $E$, the ideal generated by $A$, $I(A) = \{b \in E: |b| \leq |a| \text{ for some } a \in A\}$. If $E = I \oplus J$, then $a_\alpha$ will denote the component of $a \in E$ in $I$. $I$ will be called a band. Given a subset $A$ in $E$, $A'$ will be the set of elements disjoint from $A$: $A' = \{x: |x| \wedge |a| = 0 \text{ for all } a \in A\}$. $A'$ is a closed ideal and if $E = I \oplus J$, $J = I'$, so bands are closed ideals. In a complete space, closed ideals are bands (Riesz). Finally, if $E = I \oplus J$, then $E^b = J^1 \oplus I^1$ where $I^1$ has the usual definition. It is also true that $I^b = J^1 = (I')^1 = I'^1$. If $\phi \in E^b$, then the component of $\phi$ in $I'^1$ is given by $\langle \phi|_{I'^1}, \mu \rangle = \vee \{\langle \phi, \nu \rangle | 0 \leq \nu \leq \mu, \nu \in I\}$ for all $\mu \in E_+$.

1.4. In this paper, $X$ is locally compact, $C = C(X)$ is the space of continuous functions, $C^b = C^b(X)$ is the subset of those having compact support. $1_x$ will denote the function identically equal to one on $X$. If $f \in C(X)$, the symbols $S(f)$ and $\text{coz } f$ represent respectively, $cl_x \{x: f(x) \neq 0\}$ and $\{x: f(x) \neq 0\}$. $L^b_k(X) = C^b_k$, the space of Radon measures. Unless there is danger of confusion, we will not indicate the underlying space in the above notation. If $\mu \in L^b_k$ and if $\sup |\langle h, \mu \rangle |: h \in C_k$ and $|h| \leq 1$ exists and is finite then $\mu$ is called
a bounded Radon measure and the supremum is defined to be $||\mu||$.
All measures of compact support are bounded. For compact sets, we
follow Kaplan’s notation: if $K$ is a compact set, $C(K)$ is the Banach
lattice of continuous functions, $L(K)$ its dual and $M(K)$ the bidual.

1.5. Let $K$ be a compact subset of $X$. In general, $C(K)$ cannot
be identified with an ideal in $C_k$. Its dual, however, is a direct
summand of $L_k$. Indeed, let $I = \{f \in C_k: f|_K = 0\}$. Then $C(K)$ can be
identified with the quotient space $C_k/I$. It follows that $L(K) = I^\perp$
and since $L_k$ is complete, $L_k = L(K) \oplus I^\perp$. If $M_k$ is the (continuous)
second dual of $C_k$, we also have $M_k = M(K) \oplus L(K)^\perp$.

1.6. The set $\bigcup L(K)$ as $K$ ranges over all compact subsets of
$X$ is an ideal in $L_k$ [3, (4.2)]. Let $M = M(X) = (\bigcup L(K))^b$. If $X$
is realcompact, $M$ is the second dual of $C$. If $f \in M$ and
sup\{|$\langle f, \mu \rangle$; $\mu \in \bigcup L(K)$, $||\mu|| \leq 1$\} exists and is finite, then $f$ is called
bounded and the supremum is denoted $||f||$. Since for each com-
 pact set $K$, $L(K)$ is a closed ideal in $\bigcup L(K)$, $M(X) = M(K) \oplus L(K)^\perp$.
For each compact set $K$, $M(K)$ consists of bounded elements.

1.7. Let $(L_k)_0$ be the closed ideal generated by $X$ when considered
as a subset of $L_k$ and $(L_k)_i$, its complementary ideal. Clearly, $(L_k)_0$
consists of the purely atomic Radon measures on $X$. If $x \in X$, we
will represent the atomic measure at $x$ by $x$. For any subset $A \subset L_k$,
we let $A_i$ be the projection on $(L_k)_i$. Then since $\bigcup L(K) = (\bigcup L(K))_0 \oplus$
$(\bigcup L(K))_i$, we have $M = M_0 \oplus M_i$ where $M_i = (\bigcup L(K))_i$ in $M$. $M_0$
is lattice and ring isomorphic to the locally bounded functions on $X$.
[5, (5.7)]. For convenience, if $x$ is the atomic measure at $x$ and
$f \in M$, we will usually write $\langle f, x \rangle$ as $f_0(x)$.

2. The ideals $M(K)$. Since for each compact set $K$, $M = M(K) \oplus$
$L(K)^\perp$ in $M$, the problem of identifying $C(K)$ with an ideal in $C_k$ is
partially alleviated. Indeed, since $C \subset M$, $C(K)$ can be identified with
$C_{M(K)}$, the projection of $C$ on $M(K)$.

**Proposition 2.1.** $C_{M(K)} = (C_k)_{M(K)}$ for every compact subset $K \subset X$.

**Proof.** Clearly $(C_k)_{M(K)} \subset C_{M(K)}$. So let $g_{M(K)} \in C_{M(K)}$. We show
there exists $h \in C_{M(K)}$ such that $h_{M(K)} = g_{M(K)}$. Choose $h \in C_k$ such that
$h = g$ on $K$. This can be done since $K$ is compact. Then $g_{M(K)} = h_{M(K)}$.
Indeed, let $\mu \in \bigcup L(K)_+$. Since $M(K) = L(K)^\perp$, (1, 3) gives $\langle g_{M(K)}, \mu \rangle =$
sup\{|$\langle g, \nu \rangle$; $0 \leq \nu \leq \mu$ and $\nu \in L(K)$\}. The result follows since $\langle g, \nu \rangle =$
$\langle h, \nu \rangle$ for every such measure $\nu$ and hence
\[
\langle g_{M(K)}, \mu \rangle = \sup \{\langle h, \nu \rangle; 0 \leq \nu \leq \mu, \nu \in L(K) = \langle h_{M(K)}, \mu \rangle.
\]
Since \( C(K) \) can be identified with \( C_{M(K)} \), the vague topology on \( L(K), \sigma(L(K), C(K)) \), is the same as \( \sigma(L(K), C_{M(K)}) \) which equals \( \sigma(L(K), (C_{K})_{M(K)}) \) by the above. The following is easily checked.

**Proposition 2.2.** The following topologies on \( L(K) \) are equivalent:

(a) \( \sigma(L_k, C_k)_{|L(K)} \)
(b) \( \sigma(L(K), C_k) \)
(c) \( \sigma(L(K), (C_k)_{M(K)}) \)
(d) \( \sigma(L(K), C(K)) \).

3. \( M(X) \) as multiplication operators on \( L_k \). A bounded operator on a vector lattice \( E \) is called a multiplication operator if each closed ideal is invariant with respect to the operator. Mack has shown each \( f \in M \) defines an order continuous multiplication operator on \( L_k \) by the following definition: for \( \mu \in L_k \) and \( h \in C_k \), \( \langle h, f^\dagger \mu \rangle = \langle f, h^\dagger \mu \rangle \) where \( h^\dagger \mu \) is the element of \( \cup L(K) \) defined by \( \langle g, h^\dagger \mu \rangle = \langle gh, \mu \rangle \) for all \( g \in C(X) \). Indeed he was able to show every such operator arises in this way.

**Theorem 3.1** (Mack [5, (4.4)]). \( M \) is lattice isomorphic with the vector lattice of all multiplication operators on \( L_k \).

If \( \sigma(E, F) \) is a weak topology on a vector lattice \( E \), we say a linear operator \( T \) from \( E \) to itself is \( \sigma(E, F) \) continuous if \( \{\mu_\alpha\} \rightarrow 0 \) \( \sigma(E, F) \) implies \( \{T\mu_\alpha\} \rightarrow 0 \) \( \sigma(E, F) \). We now determine those elements of \( M \) for which \( f^t \) is a \( \sigma(L_k, C_k) \) continuous operator.

**Theorem 3.2.** Let \( f \in M \). Then \( f^t \) is a \( \sigma(L_k, C_k) \) continuous operator on \( L_k \) if and only if \( f \in C(X) \).

**Proof.** Suppose \( f \in C(X) \) and assume \( \langle h, \mu_\alpha \rangle \rightarrow 0 \) for all \( h \in C_k \). We must show \( \langle g, f^t \mu_\alpha \rangle \rightarrow 0 \) for all \( g \in C_k \). But this is clear because \( \langle g, f^t \mu_\alpha \rangle = \langle f, g^t \mu_\alpha \rangle = \langle fg, \mu_\alpha \rangle \rightarrow 0 \) since \( fg \in C_k \).

For the converse we need two lemmas

**Lemma 3.3.** Suppose \( X \) is compact and \( f \in M(X) \). If \( f^t \) is a \( \sigma(L, C) \) continuous operator on \( L \), then \( f \in C(X) \).

**Proof.** We show \( f \) is \( \sigma(L, C) \) continuous on \( L \). Suppose \( \{\mu_\alpha\} \subset L(X) \) and \( \langle h, \mu_\alpha \rangle \rightarrow 0 \) for all \( h \in C \). We show \( \langle f, \mu_\alpha \rangle \rightarrow 0 \). But \( \langle f, \mu_\alpha \rangle = \langle f, 1^t \mu_\alpha \rangle = \langle 1, f^t \mu_\alpha \rangle \rightarrow 0 \) since \( f^t \) is a \( \sigma(L, C) \) continuous operator.
Lemma 3.4. Let \( f \in M \). If for each compact set \( K \), \( f_{M(K)} \in C(K) \), then \( f \in C(X) \).

Proof. We first show that \( f_0 = g_0 \) for some \( g \in C \). We then show \( f = g \). Let \( p \in X \). Since \( X \) is locally compact, \( p \) has a compact neighborhood \( K \). Since \( f_{M(K)} \in C(K) \), \( (f_{M(K)})_0 \in (C_{M(K)})_0 \) so \( f_0 \) is continuous on a neighborhood of \( p \). Since \( p \) was arbitrary, \( f_0 \) is continuous as a function on \( X \). Therefore, let \( g \) be the continuous function such that \( g_0 = f_0 \). We claim that \( f = g \). Let \( \mu \in \cup L(K) \) and \( S = S(\mu) \). Since \( S \) is compact, \( f_{M(S)} \in C(S) \). Furthermore, \( g_{M(S)} = f_{M(S)} \) \( [1, (5.4)] \). So we have \( \langle f, \mu \rangle = \langle f_{M(S)}, \mu \rangle = \langle g, \mu \rangle \). To complete the proof of the proposition:

By Lemma 3.4 it suffices to show \( f_{M(K)} \in C(K) \) for all compact sets \( K \) in \( X \). By Lemma 3.3 it then suffices to show \( f_{M(K)}^t \) is a \( \sigma(L(K), C(K)) \) continuous multiplication operator on \( L(K) \). So let \( \{\mu_a\} \subset L(K) \) and \( \langle h, \mu_a \rangle \to 0 \) for all \( h \in C(K) \). We must show \( \langle h, f_{M(K)}^t \mu_a \rangle \to 0 \) for all \( h \in C(K) \).

By Proposition 2.2 \( \langle h, f_{M(K)}^t \mu_a \rangle \to 0 \) for all \( h \in C(K) \) if and only if \( \langle g, f_{M(K)}^t \mu_a \rangle \to 0 \) for all \( g \in C_h \). Since by hypothesis \( f^t \) is a \( \sigma(L_h, C_h) \) continuous operator, \( \langle g, f^t \mu_a \rangle \to 0 \) for all \( g \in C_h \). Now \( g^t \mu_a \in L(K) \), so by (1.5) we have \( \langle g, f^t \mu_a \rangle = \langle f, g^t \mu_a \rangle = \langle f_{M(K)}^t, g^t \mu_a \rangle \). Thus \( \langle g, f_{M(K)}^t \mu_a \rangle = \langle f_{M(K)}, g^t \mu_a \rangle \to 0 \) for all \( g \in C_h \).

4. The semi-continuous elements. We now proceed in a manner analogous to Kaplan's for the compact case and employ some methods from integration theory. Unless otherwise indicated, all infima and suprema will be taken in \( M \).

Definition 4.1. An element \( f \in M \) is usc if for each real number \( r \) there exists a subset \( A_r \) of \( C \) such that \( f \wedge r1_X = \wedge A_r \).

Remark 4.2. Clearly if \( f \in C \), then \( f \) is usc. Furthermore if \( f = \wedge f_a \) for some collection \( \{f_a\} \subset C \), then \( f \) is usc. If \( X \) is compact, then this definition is equivalent to that of Kaplan.

If \( K \) is a compact space, we follow Kaplan's notation and let \( S(K) \) be the sublattice generated by the usc elements in \( M(K) \). In our more general case we still have the result that \( S(K) \) is a subset of \( M(K) \) for every compact set \( K \subset X \). We show now that \( f \in M \) is usc exactly in the case that \( f_{M(K)} \) is a usc element in \( M(K) \) for every compact set \( K \subset X \). For this we need several lemmas.

Lemma 4.3. If \( B \subset C(X) \) and \( f = \wedge B \), then \( f_{M(K)} \in S(K) \). Indeed,
$f_{M(K)}$ is a usc element in $M(K)$.

Proof. If $f = \land B$, then $f_{M(K)} = \land \{g_{M(K)} : g \in B\}$. Since $C(K)$ is identified with the projection of $C(X)$ on $M(K)$, the result follows immediately.

We say a real valued function on $X$ is upper semi-continuous if $\{x : f(x) < r\}$ is open for every real number $r$. The following is easily proved.

**Lemma 4.4.** If $f \in M$ is usc, then $f_0$ is an upper semi-continuous function on $X$.

**Lemma 4.5.** Let $f : X \to R$ be a function such that for each compact set $K$ there exist positive upper semi-continuous function $f^K_1$ so that $f^K_1 = f^K_1 - f^K_2$. Then $f = f_1 - f_2$ where each $f_i$ is a positive upper semi-continuous function.

Proof. Let $f_i(x) = \land \{u_i(x) : f(x) = u_i(x) - u_2(x)\}$ on a neighborhood of $x$ for some positive upper semi-continuous functions $u_i$. Then $f_i$ is well defined, positive and upper semi-continuous. Letting $f_2(x) = f_i(x) - f(x)$ we have $f_2(x) = \land \{u_i(x) : f(x) = u_i(x) - u_2(x)\}$ on a neighborhood of $x$ for some positive upper semi-continuous functions $u_i$. Then $f(x) = \land \{u_2(x) : f(x) = u_i(x) - u_2(x)\}$ on a neighborhood of $x$ for some positive upper semi-continuous functions $u_i$. So $f_2$ is also a positive upper semi-continuous function.

**Lemma 4.6.** Let $f : X \to R$ be a locally bounded upper semi-continuous function. Then there exists a usc element $g \in M$ so that $g_0 = f$.

Proof. For each $n \in \mathbb{N}$, let $g_n = \land H_{f,n} = \land \{h \in C(X) : h \geq f \land n1_x\}$. Then $H_{f,n}$ is filtering downward and to show $g_n$ is well defined, it suffices to show $\langle g_n, \mu \rangle$ is finite for each $\mu \in \cup L(K)_+$. [See 1.1 and make appropriate changes.] This follows because $f$ is locally bounded and hence bounded below on compact sets. If $\mu \in \cup L(K)_+$ and $K' = S(\mu)$, choose a natural number $r$ so that $f|K' \leq -r1_x|K'$. Then $\langle h, \mu \rangle \geq \langle -r1_x, \mu \rangle > -\infty$ for each $h \in H_{f,n}$ and the infimum exists. Similarly, to show $g = \lor g_n$ exists, we choose an arbitrary $\mu \in \cup L(K)_+$ and a compact set $K'$ so that $S(\mu) \subseteq \text{int } K'$. Then if $r \in \mathbb{N}$ is such that $f|K' \leq r$, we claim that for $n > r$, $\langle g_n, \mu \rangle = \langle g_r, \mu \rangle$. Indeed it is clear that $\langle g_n, \mu \rangle \leq \langle g_r, \mu \rangle$. For the opposite inequality, let $\varepsilon > 0$ be given and choose $h_i \in H_{f,r}$ so that $\langle g_r, \mu \rangle \geq \langle h_i, \mu \rangle - \varepsilon$. Let $h_2 \in C(X)$ be chosen so that $h_2 = h_i$ on $S(\mu)$ and $h_2 = n1_x$ on $X \setminus K'$. If $h = h_1 \lor h_2$ then $h \geq f \land n1_x$ and $\langle h, \mu \rangle = \langle h_r, \mu \rangle$. Thus $\langle g_r, \mu \rangle \geq \langle g_n, \mu \rangle$.
THE SECOND DUAL OF C(X) 243

Furthermore, $g$ is usc, for if $n \in \mathbb{N}$, we claim $g \wedge n1_x = \bigwedge \{h \in C(X); h \geq g \wedge n1_x\}$. It suffices to show $g \wedge n1_x \geq \bigwedge \{h \in C(X); h \geq g \wedge n1_x\}$ and consequently that for each $\mu \in \bigcup L(K)_+$, $\langle g \wedge n1_x, \mu \rangle \geq \bigwedge \{\langle h, \mu \rangle; h \in C(X)\text{ and } h \geq g \wedge n1_x\}$. Let $\mu \in \bigcup L(K)_+$ and $\varepsilon > 0$. By definition of the infimum in $M[1, (2.1)]$ there exist $\mu_1$ and $\mu_2 \in \bigcup L(K)_+$ so $\mu = \mu_1 + \mu_2$ and $\langle g \wedge n1_x, \mu \rangle \geq \langle g, \mu_1 \rangle + \langle n1_x, \mu_2 \rangle - 1/2\varepsilon$. Let $K'$ and $H$ be compact sets such that $S(\mu) \subseteq \text{Int } H \subseteq \text{Int } K' \subseteq K'$. As above, if $f|_{K'} \leq r$, then $\langle g, \mu_1 \rangle = \langle g_r, \mu_1 \rangle$. Choose $h_1 \in C(X)$ such that $h_1 \geq f \wedge r1_x$ and such that $\langle g_r, \mu_1 \rangle \geq \langle h_1, \mu_1 \rangle - \varepsilon/2$ so that $\langle g \wedge n1_x, \mu \rangle \geq \langle h_1, \mu_1 \rangle + \langle n1_x, \mu_2 \rangle - \varepsilon$. Let $h_2 \in C(X)$ be chosen so that $h_2 = h_1$ on $S(\mu_1)$ and $h_2 = n1_x$ on $X \setminus \text{Int } H$ and let $h_3 = h_1 \vee h_2$. We claim $h_3 \geq g \wedge n1_x$. Indeed if $\nu \in \bigcup L(K)_+$ and $\Phi \in C_0$ such that $\Phi = 1$ on $H$ and $\Phi = 0$ on $X \setminus \text{Int } K'$ then $\nu = (\Phi\nu) + (1_x - \Phi)\nu$. Thus $\langle h_3, (\Phi\nu) \rangle + \langle h_3, (1_x - \Phi)\nu \rangle = \langle h_3, \nu \rangle$. Now $h_3 \geq h_1$ implies $\langle h_3, (\Phi\nu) \rangle \geq \langle g_r, (\Phi\nu) \rangle$ and $S((1_x - \Phi)\nu) \subseteq X \setminus \text{Int } H$ implies $\langle h_3, (1_x - \Phi)\nu \rangle \geq \langle n1_x, (1_x - \Phi)\nu \rangle$. Thus $\langle h_3, \nu \rangle \geq \langle g_r, (\Phi\nu) \rangle + \langle n1_x, (1_x - \Phi)\nu \rangle \geq \langle g, (\Phi\nu) \rangle + \langle n1_x, (1_x - \Phi)\nu \rangle \geq \langle g \wedge n1_x, \nu \rangle$. Finally since $\nu$ was arbitrary we have $h_3 \geq g \wedge n1_x$ and $h_3 \wedge n1_x \geq g \wedge n1_x$. As a result,

$$\langle g \wedge n1_x, \mu \rangle \geq \langle h_1, \mu \rangle + \langle n1_x, \mu_2 \rangle - \varepsilon \geq \langle h_3, \mu \rangle + \langle n1_x, \mu_2 \rangle - \varepsilon$$

Since $\varepsilon$ was arbitrary we have the desired result.

Finally, we check that $g_\delta(x) = \hat{f}(x)$ for each $x \in X$. Let $x \in X$, $K'$ a compact neighborhood of $x$ and $\varepsilon > 0$. Suppose $f|_{K'} \leq r$ for some natural number $r$. Then $g(x) = g_\delta(x) = (g_r)_\delta(x)$. Since $f \wedge r1_x$ is upper semi-continuous there exists $h \in C(X)$ such that $h \geq f \wedge r1_x$ and $h(x) < f \wedge r1_x(x) + \varepsilon$. Thus $(g_r)_\delta(x) \leq h(x) \leq f \wedge r1_x(x) + \varepsilon = \hat{f}(x) + \varepsilon$. Since the other inequality is clear, the proof is complete.

**Lemma 4.7.** If $g$ is usc in $M$, then $g_M(K)$ is usc in $M(K)$ for every compact set $K \subseteq X$.

**Proof.** Let $K$ be a compact set and $r$ be an integer so that $g_M(K) \leq r1_K$. Then $g_M(K) = (g \wedge r1_x)_M(K)$. The result now follows from the definition of usc elements Lemma (4.3).

**Theorem 4.8.** Let $f \in M$. Then $f$ is usc if and only if $f_M(K)$ is a usc element in $M(K)$ for every compact set $K$.

**Proof.** Suppose $f_M(K)$ is a usc element in $M(K)$ for every compact
set \( K \). Consider \( f_0 \). Since \( f_{M(K)} \) is bounded and usc in \( M(K) \), \( f_0 \) is locally bounded and upper semi-continuous as a function on \( X \). By Lemma 4.6, there exists a usc element \( g \) in \( M \) so that \( g_0 = f_0 \).

We show \( g = f \). This follows immediately from the fact that for each compact set \( K \), \( f_{M(K)} \) and \( g_{M(K)} \) are usc members of \( M(K) \) (Lemma 4.7). Since \( (g_{M(K)})_0 = (f_{M(K)})_0 \) it follows from the compact case that \( g_{M(K)} = f_{M(K)} \). Since this is true for each compact set \( K \), it follows that \( f = g \).

**Proposition 4.9.** If \( f \) and \( g \) are usc elements in \( M \), then so is \( f + g \).

*Proof.* Let \( K \) be a compact set. Then \( (f + g)_{M(K)} = f_{M(K)} + g_{M(K)} \) is the sum of two usc elements in \( M(K) \) and hence is usc. The result now follows from Theorem 4.8.

Similarly, the following propositions are easily verified using corresponding results for the compact case.

**Proposition 4.10.** If \( f \) and \( g \) are usc elements in \( M \), then \( f \land g \) and \( f \lor g \) are also. If \( a > 0 \), \( af \) is usc.

**Proposition 4.11.** If \( A \) is a subset of usc elements in \( M \) and \( f = \bigwedge A \), then \( f \) is usc.

Let \( S = S(X) \) denote the linear subspace of \( M \) generated by the positive usc elements. It follows from (4.9) and (4.10) that each element in \( S \) can be written as \( f - g \) where \( f \) and \( g \) are positive usc. The fact that \( S \) is a sublattice follows from the fact that \( (f_1 - g_1) \land (f_2 - g_2) = (f_1 + g_2) \land (f_2 + g_1) - (g_1 + g_2) \) and (4.10).

Of course for each compact set \( K \), \( S(K) \), the semi-continuous elements studied by Kaplan [1] is a subset of \( M(K) \) and hence of \( M \). In the following we assume a knowledge of the compact case.

**Proposition 4.12.** If \( f \in M \), then \( f \in S \) if and only if \( f_{M(K)} \in S(K) \) for each compact set \( K \).

*Proof.* If \( f \in S \), then \( f = f_i - f_s \) where each \( f_i \) is positive usc. But then \( f_{M(K)} = (f_i)_{M(K)} - (f_s)_{M(K)} \in S(K) \) by Lemma 4.7. Conversely, if \( f_{M(K)} \in S(K) \) for each compact set \( K \), we need only observe that in \( S(K) \) element can be written as the difference of positive usc elements and the proof follows as in Theorem 4.8 using Lemma 4.5.

**Proposition 4.13.** If \( f \) and \( g \) are members of \( S \), then \( f \leq g \) if
and only if \( f_0 \leq g_0 \).

\[ \text{Proof.} \] It suffices to show \( f_0 \leq g_0 \) implies \( f \leq g \). Let \( \mu \in \bigcup L(K)_+ \) and \( K' = S(\mu) \). Then \( f_0 \leq g_0 \) implies \( (f_{M(K')})_0 \leq (g_{M(K')})_0 \). By Proposition 4.12 \( f_{M(K')} \) and \( g_{M(K')} \) are members of \( S(K') \) and consequently using results from the compact case, \( f_{M(K')} \leq g_{M(K')} \). Thus \( \langle f, \mu \rangle = \langle f_{M(K')}, \mu \rangle \leq \langle g_{M(K')}, \mu \rangle = \langle g, \mu \rangle \). Since this is true for every \( \mu \in \bigcup L(K)_+ f \leq g \).

\[ \text{COROLLARY 4.14.} \] If \( f \) and \( g \) are members of \( S \), then \( f = g \) is equivalent to \( f_0 = g_0 \). In particular, the projection of \( S \) onto \( S_0 \) is one-to-one.

\[ \text{PROPOSITION 4.15.} \] \( S(K) = S(X)_{M(K)} \) for each compact set \( K \).

\[ \text{Proof.} \] By (4.12), \( S(X)_{M(K)} \subset S(K) \). It now suffices to show that if \( f \) is a positive use element in \( S(K) \), then \( f = g_{M(K)} \) for some \( g \in S(X) \). Suppose \( \{f_a\} \subset C(K) \) and \( f = \bigwedge f_a \) in \( M(K) \). Let \( g_a \in C_{k+} \) be such that \( g_a|_K = f_a \). Then \( g = \bigwedge g_a \) exists in \( M \) and is use. By Lemma 4.7, \( g_{M(K)} \in S(K) \). Thus, since \( (g_{M(K)})_0 = f_0 \) it follows from Corollary 4.14 applied to the compact case that \( g_{M(K)} = f \).

Since \( S(X) \) is not an ideal, it is not obvious that projections onto the ideals \( M(K) \) are still members of \( S(X) \). We consider this next.

\[ \text{LEMMA 4.16.} \] Let \( K \) be a compact subset of \( X \) and \( \{f_a\} \) a net in \( C_k \) such that \( \alpha \leq \beta \) implies \( f_\alpha(x) \geq f_\beta(x) \) for all \( x \) and \( f_\alpha(x) \downarrow 0 \) for all \( x \) in \( X \setminus K \). Then \( \bigwedge f_a \in M(K) \).

\[ \text{Proof.} \] Since \( M(K) = L(K)'^+ \) [see (1.3) and (1.5)], it suffices to show \( \langle \bigwedge f_a, \nu \rangle = \bigwedge \langle f_a, \nu \rangle = 0 \) for all \( \nu \in L(K)' \). Assume the contrary and let \( \nu \in L(K)' \), \( \nu \geq 0 \) be such that \( \langle f_a, \nu \rangle \geq r > 0 \) for all \( \alpha \). Let \( \bigwedge f_a \nu = \mu \). That is, for

\[ h \in C(X)_+ \langle h, \mu \rangle = \langle h, \bigwedge f_a \nu \rangle = \bigwedge \langle h, f_a \nu \rangle = \bigwedge \langle hf_a, \nu \rangle. \]

Now, \( \mu \) is not identically zero. Indeed, \( \| \mu \| = \langle 1_X, \mu \rangle = \bigwedge \langle f_a, \nu \rangle \geq r \). Furthermore, \( S(\mu) \subset K \), for if \( h \in C(X) \), \( h|_K = 0 \), then \( f_a h \downarrow 0 \) for all \( x \in X \) and since \( \nu \) has compact support, \( \langle h, \mu \rangle = \bigwedge \langle hf_a, \nu \rangle = 0 \) by the usual argument. Thus \( \mu \in L(K) \) and \( \mu \bigwedge \nu = 0 \). However, let \( b = \| f_a \| \vee 1 \) for some arbitrary \( \alpha \). Then for

\[ h \in C_{k+} \langle h, \mu \rangle = \bigwedge \langle hf_a, \nu \rangle \leq \langle hf_a, \nu \rangle \leq \langle b < h, \nu \rangle = \langle h, b \nu \rangle \]

so \( \mu \leq b \nu \) and \( \mu \bigwedge \nu \geq \mu \bigwedge b^{-1} \mu = b^{-1} \mu \neq 0 \) which is a contradiction.
COROLLARY 4.17. With the above hypothesis $\land f_\alpha \in S(K)$ and is positive usc.

PROPOSITION 4.18. Let $f \in C(X)_+$ and $K$ be a compact subset of $X$. Then $f_{M(K)} \in S(X)$ and indeed $f_{M(K)}$ is positive usc.

Proof. Let $\{f_\alpha\} = \{f_\alpha \in C_+: f_\alpha|_K = f|_K\}$. We direct the index set as follow $\alpha \leq \beta$ if $f_\alpha(x) \geq f_\beta(x)$ for all $x \in X$. It is easy to check that this definition satisfies the conditions for a directed set. Furthermore, since for each $x \in X \setminus K$, there exists $f_{a_0}$ such that $f_{a_0}|_K = f|_K$ and $f_{a_0}(x) = 0$ it is clear that $f_\alpha(x) \downarrow 0$ for all $x$ in $X \setminus K$. Let $g = \bigwedge f_\alpha$. By Corollary 4.17 $g \in S(K)$ and is positive usc. We show $f_{M(K)} = g$. But this is clear, for if $\mu \in L(K)_+$ then $g(f_{M(K)}, \mu) = g(f_\alpha, \mu) = g(f, \mu)$ for all $\alpha$. Thus $g(f_{M(K)}, \mu) = \bigwedge (f_\alpha, \mu) = \langle g, \mu \rangle$ and consequently $f_{M(K)} = g$ and is positive usc.

PROPOSITION 4.19. Let $f \in S(X)$, then $f_{M(K)} \in S(X)$ for each compact set $K$. In particular, if $f$ is a positive use element, then for each real number $r$, there exists a collection $B_r \subset C_+$ so that $f_{M(K)} \land r1_X = \bigwedge B_r$.

Proof. Assume $f$ is positive usc. Then for every real number $r$, there is a collection $A_r \subset C(X)_+$ so that $f \land r1_X = \bigwedge A_r$. Then

$$f_{M(K)} \land r1_X = f_{M(K)} \land (r1_X)_{M(K)} = (f \land r1_X)_{M(K)} = (\bigwedge A_r)_{M(K)} = \bigwedge \{g_{M(K)}: g \in A_r\}.$$  

If $g \in A_r$, then by the argument in Proposition 4.18 $g_{M(K)}$ is the infimum of a collection $A_g \subset C_+$. Thus we have: $f_{M(K)} \land r1_X = \bigwedge \{g_{M(K)}: g \in A_r\} = \bigwedge \{h \in \bigcup \{A_g: g \in A_r\}\}$. The result now follows by choosing $B_r = \bigcup \{A_g: g \in A_r\}$.

If $f$ is an arbitrary element of $S$, then $f = g - h$ where $g$ and $h$ are positive usc. Then $f_{M(K)} = g_{M(K)} - h_{M(K)} \in S(X)$.

COROLLARY 4.20. If $f$ is positive use and $K$ is a compact set, then there exists a collection $A_K \subset C_+$ so that $f_{M(K)} = \bigwedge A_K$.

Proof. Since $f_{M(K)}$ is bounded, there exists a real number $r$ so that $f_{M(K)} = f_{M(K)} \land r1_X$. The result now follows immediately from (4.19).

5. $S(X)$ as multiplication operators. Let $S_h(X) = \{f \in S(X): f = f_{M(K)}$ for some compact subset $K\}$. $S_h(X)$ can also be regarded as the union of all $S(K)$ as $K$ ranges over the compact subset of $X$. 
It is easy to see that \( f \in S_k(X) \) if and only if \( f_0 \) (as a function on \( X \)) is the difference of two positive upper semi-continuous functions with compact support. It is also clear that \( S_k(X) \subset \bigcup M(K) \subset M_k \) and that it is a sub vector lattice containing \( C_k \). As such, it is separating on \( L_k \) and determines a Hausdorff weak topology on \( L_k \), namely \( \sigma(L_k, S_k) \).

We have already considered \( M \) as multiplication operators on \( L_k \). Indeed if \( f \in M \) and \( \mu \in L_k(X)_+ \langle h, f^*\mu \rangle = \langle f, h^*\mu \rangle \) for all \( h \in C_{k+} \). Now consider the special case that \( f \in S(X) \). Suppose \( f \) is usc. Choose an integer \( r \) so that \( \langle f, h^*\mu \rangle = \langle f \land r1_x, h^*\mu \rangle \). Since \( f \) is usc there exists a collection \( \{f_a\} \subset C(X) \) such that \( f \land r1_x = \bigwedge \{f_a\} \). Thus

\[
\langle f, h^*\mu \rangle = \langle \land f_a, h^*\mu \rangle = \bigwedge \langle f_a, h^*\mu \rangle = \bigwedge \langle hf_a, \mu \rangle = \langle \land hf_a, \mu \rangle .
\]

Observe that \( \land hf_a \in S_k(X) \).

We have already shown that \( S(K) = S(X)_{M(K)} \). We now show that these are the same as \( (S_k)_{M(K)} \).

**Proposition 5.1.** \( S(X)_{M(K)} = (S_k)_{M(K)} \).

**Proof.** Clearly \( (S_k)_{M(K)} \subset S_{M(K)} \). Thus let \( g_{M(K)} \in S_{M(K)} \) and \( g \) be positive usc. We show there exists \( h \in S_k \) so that \( h_{M(K)} = g_{M(K)} \). Let \( r \) be a real number so that \( g_{M(K)} = (g \land r1_x)_{M(K)} \). By hypothesis, there exists \( \{g_a\} \subset C(X)_+ \) so that \( g \land r1_x = \bigwedge g_a \). Let \( H \) be a compact neighborhood of \( K \) and \( \{h_a\} \subset C(X) \) so that \( h_{a|K} = g_a \) and \( S(h_a) \subset H \). Let \( h = \bigwedge h_a \). Clearly \( h_{M(K)} = (g \land r1_x)_{M(K)} = g_{M(K)} \) and since \( h = h_{M(H)} \), \( h \in S_k \).

This proposition and the previous remark make it easy to verify the following:

**Proposition 5.2.** On \( L(K) \) the following topologies are equivalent:

1. \( \sigma(L_k, S_k)_{L(K)} \)
2. \( \sigma(L(K), (S_k)_{M(K)}) \)
3. \( \sigma(L(K), S(X)_{M(K)}) \)
4. \( \sigma(L(K), S(K)) \)

**Lemma 5.3.** Let \( X \) be compact and \( f \in M \). If \( f^t \) is \( \sigma(L, S) \) continuous on \( L \) then \( f \in S \).

**Proof.** The proof of (3.3) carries over by replacing \( C \) with \( S \).

**Theorem 5.4.** Let \( f \in M \), then \( f^t \) is a \( \sigma(L_k, S_k) \) continuous multiplication operator on \( L_k \) if and only if \( f \in S \).
Proof. Let $f \in S$ and $\{\mu_a\} \subset L_k$ such that $\langle h, \mu_a \rangle \to 0$ for all $h \in S_k$. We show $\langle h, f^i \mu_a \rangle \to 0$ for all $h \in S_k$. Without loss of generality, we may assume $f$ is positive and usc. Since $h \in S_k$, $h = h_1 - h_2$ where the $h_i$ are positive usc elements in $S_k$ so we may also assume $h$ is positive usc.

By Corollary 4.20, there exists a collection $\{h_\gamma\} \subset C_{k+}$ such that $h = \bigwedge h_\gamma$. Thus $\langle h, f^i \mu_a \rangle = \bigwedge \langle h_\gamma, f^i \mu_a \rangle = \bigwedge \langle f, h_i^\gamma \mu_a \rangle$. We may as well assume there is a compact set $K$ such that $S(h_\gamma) \subset K$ for all $\gamma$. Choose an integer $r$ so that $\langle f, h_i^\gamma \mu_a \rangle = \langle f, r 1_x, h_i^\gamma \mu_a \rangle$

for all $\alpha$ and $\gamma$. This can be done since $S(h_i^\gamma \mu_a) \subset K$. By assumption, there exists a collection $\{f_\beta\} \subset C(X)$ so that $f \wedge r 1_x = \bigwedge f_\beta$. Thus for each $\alpha$, $\langle h, f^i \mu_a \rangle = \bigwedge \langle f \wedge r 1_x, h_i^\gamma \mu_a \rangle = \bigwedge \langle f_\beta h_\gamma, \mu_a \rangle = \bigwedge \langle f_\beta h_\gamma, \mu_a \rangle = \bigwedge \langle f_\beta h_\gamma, \mu_a \rangle$. We observe that $\bigwedge f_\beta h_\gamma \in S_k$ and hence $\langle h, f^i \mu_a \rangle = \langle f_\beta h_\gamma, \mu_a \rangle \to 0$.

For the converse we merely adapt the proof of Theorem 3.2 replacing $C(K)$ with $S(K)$ and the references to 2.2, 3.3 and 3.4 with 5.2, 5.3 and 4.12.

6. The ideals generated by $S(X)$ and $C(X)$. We have now singled out two sublattices in $M$, namely $S$ and $C$. In the case that $X$ is compact, the constant function $1_x$ is a strong order unit in the normed vector space $M$. That is, it is a positive element of unit norm such that $f \in M(X)$ and $||f|| \leq 1$ imply $|f| \leq 1_x$. Since $1_x$ is a member of both $S$ and $C$, it can be shown that the ideals these sublattices generate, denoted $I(C)$ and $I(S)$, must be all of $M$. In the more general case we are considering, however, there may not be a strong order unit. Unless $X$ is pseudocompact, $1_x$ is only a weak order unit: that is, a positive element such that for each $f \in M$, $|f| \wedge 1_x = 0$ implies $f = 0$. It can be shown that under this condition, the closed ideals generated by $S$ and $C$ are all of $M$, although not necessarily the ideals themselves. We give necessary and sufficient conditions under which $I(S)$ (respectively $I(C)$) is all of $M$.

**Theorem 6.1.** $I(S) = M$. Indeed, each element of $M$ is dominated by positive usc element.

**Proof.** Let $f \in M$. Since $f = f^+ - f^- \leq f^+ + f^- = |f|$, it suffices to assume $f \geq 0$. For each $n \in N$, let $g_n = \bigwedge \{h \in C(X): h \geq f \wedge n 1_x\}$. To show $g = \bigvee g_n$ exists it suffices to show $\bigvee \langle g_n, \mu \rangle < \infty$ for each $\mu \in \bigcup L(K)_+$ (1.1). Let $\mu \in \bigcup L(K)$ and $K'$ be a compact set so that $S(\mu)$ is contained in the interior of $K'$. Assume $|f_{M(K')}| \leq$
We can prove as in Lemma 4.6 that for all \( n > r \), \( \langle g_n, \mu \rangle = \langle g_r, \mu \rangle \), and hence that the supremum in question is finite. Also \( \langle g, \mu \rangle = \langle g_r, \mu \rangle \geq \langle f \wedge r1_K, \mu \rangle = \langle f \wedge r1_{M(K)}, \mu \rangle = \langle f, \mu \rangle \). It remains to show \( g \) is usc. The proof in Lemma 4.6 applies if the condition \( f|_x \leq r \) is replaced by \( |f_{M(K)}| \leq r1_{K'} \).

**Definition 6.2.** A topological space \( X \) is called a cb-space if and only if for each locally bounded function \( h \), there exists \( f \in C(X) \) such that \( |h| \leq f \).

**Theorem 6.3 (Mack [4, Theorem 1]).** \( X \) is a cb space if and only if each countable increasing cover of \( X \) has a countable refinement by cozero sets; that is, sets of the form, \( \text{coz } f \) for some \( f \in C(X) \).

**Lemma 6.4.** Let \( f \in M_+ \) and \( K \) be the collection of all open relatively compact subsets of \( X \). Let \( U_n = \bigcup \{ K \in K : |f_{M(K)}| \leq n \} \). Let \( \mu \in \bigcup L(K)_+ \) and \( K_0 = S(\mu) \). If \( K_0 \subset U_n \) then \( \langle f, \mu \rangle \leq \langle n1_x, \mu \rangle = n\|\mu\| \).

**Proof.** Since \( X \) is locally compact, \( K \) is a cover of \( X \) and the compactness of \( K_0 \) implies there exist subset \( K_i, \cdots, K_m \) in \( U_n \) such that \( K_0 \subset K_1 \cup K_2 \cdots \cup K_m \). Let \( h_i, i = 1, \cdots, m \) be a partition of unity on \( K_0 \) subordinate to the cover \( \{K_i\}_{i=1}^m \). That is, \( h_i \in C_b \), \( 0 \leq h_i \leq 1 \), \( S(h_i) \subset K_i \) and \( \sum_{i}^{m} h_i(x) = 1 \) for all \( x \in K_0 \). Then

\[
\langle f, \mu \rangle = \langle f, \sum_{i}^{m} h_i \mu \rangle = \sum_{i}^{m} \langle f, h_i \mu \rangle
\]

and since \( S(h_i \mu) \subset K_i \) this last expression is dominated by

\[
\sum_{i}^{m} |f_{M(K_i)}||h_i \mu| \leq \sum_{i}^{m} n |h_i \mu| = \langle n, \sum_{i}^{m} h_i \mu \rangle = n\|\mu\|.
\]

**Theorem 6.5.** \( I(C) = M \) if and only if \( X \) is a cb-space.

**Proof.** Assume first that \( I(C) = M \). Let \( h \) be locally bounded and real valued on \( X \). We must show there exists \( f \in C(X) \) such that \( |h| \leq f \). Assume \( h \geq 0 \). Since \( h \) is locally bounded, it determines a function \( g \in M_+ \) such that \( g_0(x) = h(x) \) for all \( x \in X \). By hypothesis and (1.3) there exists \( p \in C(X) \) such that \( p \geq g \). Thus \( p_0 \geq g_0 \) and hence \( f(x) = p_0(x) \) is the required function.

For the converse, let \( f \in M_+ \). We show there exists \( g \in C(X) \) such that \( g \geq f \). Let \( K \) and \( \{U_n\} \) be defined as in Lemma 6.4. Then the collection \( \{U_n\} \) is an increasing cover of \( X \) by open sets. Since \( X \) is a cb-space and the countable union of cozero sets is again
a cozero set, we have by Theorem (6.3) a family \( \{g_n\} \subset C(X) \) such that \( \text{coz } g_n \subset U_n \) and \( \{\text{coz } g_n\} \) is a cover of \( X \). Assume \( g_i \geq 0 \). Define \( f_n = \sum_{i=1}^{n} 1_{x \in g_n} \). Given \( x \in X \), there exists an \( i \) such that \( g_i(x) > 0 \). Therefore, there exists a \( j \) such that \( g_j(y) > j^{-1} \) for each \( y \) in a neighborhood of \( x \). Therefore \( f_n \) vanishes on that neighborhood for \( n \geq i \lor j \). This implies \( \sum f_n \) is locally finite and thus \( g = 2 + \sum f_n \in C(X) \). We claim \( f \neq g \). Indeed, let \( \mu \in \bigcup L(K) \) and \( K_0 = S(\mu) \). Let \( W_n = U_n \cap \{x: g(x) > n\} \). Since \( g \) is continuous \( W_n \) is open. Furthermore, \( \{W_n\} \) is a cover. Indeed, let \( x \in X \) and \( n_0 = \bigwedge \{n: x \in U_n\} \). Then \( g_m(x) = 0 \) for \( m = 1, \ldots, n_0 - 1 \) and \( x \in W_{n_0} \).

Since \( K_0 \) is compact, there exists a natural number \( s \) such that \( K_0 \subset W_1 \cup \cdots \cup W_s \). For \( i = 1, \ldots, s \), choose \( h_i \in C_k \) such that \( 0 \leq h_i \leq 1 \), \( S(h_i) \subset W_i \) and \( \sum h_i(x) = 1 \) for all \( x \in K_0 \). Then \( \langle f, \mu \rangle = \langle f, (\sum h_i)^{\sharp} \mu \rangle \leq \sum h_i(x) = 1 \) by Lemma (6.4). For \( i = 1, \ldots, s \), choose \( w_i \in C_k \) such that \( 0 \leq w_i \leq 1 \), \( w_i(x) = 1 \) for all \( x \in S(h_i \mu) \) and \( S(w_i) \subset W_i \). So \( \langle f, \mu \rangle \leq \sum i w_i \), \( h_i^{\sharp} \mu \) and \( 0 \leq i w_i \leq 1 \) with \( S(i w_i) \subset W_i \). Since \( g > i \) on \( W_i \), this implies \( i w_i \leq g \). Finally this gives \( \langle f, \mu \rangle \leq \sum i \langle g, h_i^{\sharp} \mu \rangle = \langle g, (\sum h_i)^{\sharp} \mu \rangle = \langle g, \mu \rangle \).

**APPENDIX.** If one uses the definition of usc element as given in this paper, it would be natural to define an lsc element in \( M \) as one for which \( -f \) is usc. However, bearing in mind the properties possessed by lower semi-continuous functions, it is also natural to define an lsc element as one for which \( f = \bigvee f_a \) for some collection \( \{f_a\} \subset C(X) \). These are not compatible definitions. In a manner similar to that used above, we can show (using the second definition) that the linear sublattice \( T \) formed by the positive lsc elements consists of all those members of \( M \) which can be written as the difference of positive lsc elements. If \( X \) is compact, then this sublattice is exactly the same as \( S(X) \). In general, however, \( S \) and \( T \) are not the same.

We say a real valued function on \( X \) is lower semi-continuous if \( \{x: f(x) > r\} \) is open for each real \( r \).

The following is easily checked:

**Lemma 7.1.** If \( f \in M \) is lsc, the \( f_0 \) is a lower semi-continuous function on \( X \).

**Proposition 7.2.** Let \( f: X \to R \) be a positive lower semi-continuous function and locally bounded. Then \( h = \bigvee \{f_a \in C(X): 0 \leq f_a \leq f\} \) exists in \( M \), \( h \) is lsc and \( h_0(x) = f(x) \) for all \( x \in X \).

**Proof.** Let \( \mathscr{F} = \{f_a \in C(X): 0 \leq f_a \leq f\} \). Then \( \mathscr{F} \) determines an ascending net and we may write \( \bigvee \mathscr{F} = \bigvee f_a \). It suffices to
prove that for each \( \mu \in \bigcup L(K)_+ \), \( \langle f_\alpha, \mu \rangle \) exists and is finite. Since \( f \) is locally bounded and \( S(\mu) \) is compact, \( \sup \{ f(x) : x \in S(\mu) \} \) is some finite number \( r \). Then for each \( \alpha \), \( \langle f_\alpha, \mu \rangle = \langle f_\alpha \wedge r1_x, \mu \rangle + \langle (f_\alpha - r1_x)^+, \mu \rangle = \langle f_\alpha \wedge r1_x, \mu \rangle \) since \( (f_\alpha - r1_x)^+ = 0 \) on \( S(\mu) \). So \( \langle f_\alpha, \mu \rangle \leq r \| \mu \| \) for each \( \alpha \), giving an increasing set of real numbers which is bounded above. Hence the supremum in question exists and is finite. By definition \( h \) is lse and the last assertion follows from the fact that \( f \) is lower semi-continuous.

**Corollary 7.3.** Let \( f \in M_0 \) and \( f \geq 0 \). If as a function on \( X \), \( f \) is lower semi-continuous, then there exists a unique lse element \( g \in M \) such that \( g_\circ = f \).

**Proof.** We merely observe that the elements of \( M_0 \) are locally bounded as functions on \( X \). (1.7). The uniqueness follows from the fact that for lse elements \( f = g \) if and only if \( f_\circ = g_\circ \). (Argument follows as in [1]).

**Definition 7.4.** A space \( X \) is countably paracompact if and only if each countable open cover has a locally finite refinement.

**Theorem 7.5** (Mack [4, Theorem 10]). \( X \) is countably paracompact if and only if for each locally bounded function \( h \) defined on \( X \) there exists a locally bounded lower semi-continuous function \( g \) such that \( |h| \leq g \).

**Theorem 7.6.** \( I(T) = M \) if and only if \( X \) is countably paracompact.

**Proof.** Assume first that the ideal generated by \( T \), \( I(T) = M \). By Theorem 7.5 it suffices to show that for each locally bounded function \( h \) on \( X \) there exists a locally bounded lower semi-continuous function \( g \) such that \( |h| \leq g \). Now \( M_0 \) is isomorphic to the locally bounded functions on \( X \). Therefore, \( |h| \) determines a member of \( M_0 \). By hypothesis and (1.3), there exists \( g \in T_0 \) such that \( 0 \leq |h| \leq g \). By definition \( g = g_1 - g_2 \) where each \( g_i \) is a positive, locally bounded lse. The result follows from the observation that \( 0 \leq |h| \leq g_1 - g_2 \leq g_1 \).

For the converse, let \( f \in M_+ \). By (1.3) it suffices to show that there exists an lse element \( g \) such that \( g \geq f \). Let \( X^c \) and \( U_n \) be as in the Lemma 6.4 for the given \( f \). Since \( X \) is countably paracompact, there exists a refinement \( \{ V_n \} \) such that \( V_n \subset U_n \) [4, Theorem 10]. Let \( p(x) = \inf \{ n : x \in V_n \} \). Then \( p(x) \) is positive, locally bounded and lower semi-continuous. By Proposition 7.2 \( p \) determines a member \( g \) of \( T \) by \( \langle g, \mu \rangle = \bigvee \{ \langle g_\alpha, \mu \rangle : g_\alpha \in C(X) \text{ and } 0 \leq g_\alpha \leq p \} \). We claim
\[ f \leq g. \] Indeed, let \( \mu \in \bigcup L(K) \) and \( K_0 = S(\mu) \). Let \( W_n = U_n \cap \{ x: p(x) > n - 1 \} \) for \( n = 1, 2, \cdots \). Then \( W_n \) is open since \( p \) is lower semi-continuous. If \( x \in X \) and \( p(x) = n_0 \), then \( x \in W_{n_0} \). Thus \( \{ W_n \} \) is a cover of \( X \). Since \( K_0 \) is compact, there exists an integer \( m \) such that \( K_0 \subseteq W_1 \cup \cdots \cup W_m \). There also exist functions \( h_i \in C_k \) such that \( 0 \leq h_i \leq 1 \), \( S(h_i) \subseteq W_i \) and \( \sum_i h_i(x) = 1 \) for all \( x \in K_0 \). Then we have \( \langle f, \mu \rangle = \langle f, (\sum_i h_i)^* \mu \rangle = \sum_i \langle f, h_i^* \mu \rangle \leq \sum_i \langle h_i, h_i^* \mu \rangle \) since the fact that \( S(h_i^* \mu) \subseteq U_i \) allows us to apply Lemma 6.4. Now, for each \( i \), choose \( g_i \in C_k \) such that \( 0 \leq g_i \leq 1 \), \( g_i = 1 \) on \( S(h_i) \) and \( S(g_i) \subseteq W_i \). Then \( \langle i, h_i^* \mu \rangle = \langle g_i, h_i^* \mu \rangle \). Now \( 0 \leq g_i \leq i \) and \( S(g_i, i) \subseteq W_i \). But \( p \geq i \) on \( W_i \) so \( i g_i \leq p \). Thus we finally have \( \langle f, \mu \rangle \leq \sum_i \langle i, h_i^* \mu \rangle = \sum_i \langle g_i, h_i^* \mu \rangle \leq \sum_i \sup \{ \langle h_a, h_i^* \mu \rangle : h_a \in C(X), 0 \leq h_a \leq p \} = \sum_i \langle g, h_i^* \mu \rangle = \langle g, (\sum_i h_i)^* \mu \rangle = \langle g, \mu \rangle \).

This last result gives a good way to tell when the two sub-lattices \( S(X) \) and \( T(X) \) are identical.

**Theorem 7.7.** \( T = S \) if and only if \( X \) is countably paracompact.

**Proof.** If \( T = S \), then \( I(T) = I(S) = M \) by Theorem 6.1 and thus by Theorem 7.6, \( X \) is countably paracompact.

Assume next that \( X \) is countably paracompact. It will suffice to show that each positive lsc element can be written as the difference of positive usc elements and that each positive usc element can be written as the difference of positive lsc elements. Suppose \( f \) is lsc. Then by Theorem 6.1, there is a usc element \( g \) so that \( f \leq g \). It follows that \( f = g - (g - f) \) and hence it suffices to show \( g - f \) is usc. Let \( \{ f_\alpha \} \subseteq C(X) \) be such that \( f = \bigvee f_\alpha \). Then \( g - f = g - \bigvee f_\alpha = g + \bigwedge (g - f_\alpha) = g - f_\alpha \). Now \( g - f_\alpha \) is usc for each \( \alpha \) so by (4.11) \( g - f \) is usc.

Next suppose \( f \) is usc. Since \( f_0 \) is locally bounded on \( X \) and \( X \) is countably paracompact, Theorem 7.5 implies there is a lower semi-continuous function \( g \) so that \( f_0 \leq g \). By Corollary 7.3, there exists an lsc element \( G \) so that \( G_0 = g \). We have already shown that the lsc elements are members of \( S \). By Proposition 4.13 it follows that \( f \leq G \). Thus \( f = G - (G - f) \) and it will be sufficient to show that \( G - f = \bigvee \{ h \in C(X) : h \leq G - f \} \). Again, since the other inequality is clear it remains to show that \( \langle G - f, \mu \rangle \leq \bigvee \{ \langle h, \mu \rangle : h \in C(X), h \leq G - f \} \) for each \( \mu \in \bigcup L(K)_+ \). Let \( \varepsilon > 0 \) be given and \( \mu \in \bigcup L(K)_+ \). Let \( K_0 = S(\mu) \) and \( H \) be a compact set so that \( K_0 \) is contained in the interior of \( H \). Choose \( n \) so that \( f_{M(\mu)} = f \land n_{M(\mu)} \). Since \( G \) is lsc and \( f \) is usc, there exist subsets \( A \) and \( B \) of \( C(X) \) such that \( G = \bigvee A \) and \( f \land n_{1X} = \bigwedge B \). Choose \( f_1 \in A \) so that \( f_1 \leq G \) and \( \langle f_1, \mu \rangle \geq
\langle G, \mu \rangle - \varepsilon \text{ and } f_2 \in B \text{ so that } f_2 \geq f \wedge n1_X \text{ and }
\langle f_2, \mu \rangle \leq \langle f \wedge n1_X, \mu \rangle + \varepsilon .

Then \( f_1 - f_2 \leq G - f \wedge n1_X \). Let \( \Phi \in C_\kappa \) be such that \( \Phi = 1 \) on \( K_0 \) and \( S(\Phi) \subseteq H \). Then if \( h = \Phi(f_1 - f_2) \), we have \( h|_H \leq f_1 - f_2|_H \leq G - f \wedge n1_X|_H = G - f|_H \). Since \( S(h) \subseteq H \), we have \( h \leq G - f \). Then \( h|_{X_0} = (f_1 - f_2)|_{X_0} \) implies \( \langle h, \mu \rangle = \langle f_1 - f_2, \mu \rangle = \langle f_1, \mu \rangle - \langle f_2, \mu \rangle \geq \langle G, \mu \rangle - \varepsilon - \langle f \wedge n1_X, \mu \rangle - \varepsilon = \langle G - f \wedge n1_X, \mu \rangle - 2\varepsilon = \langle G - f, \mu \rangle - 2\varepsilon \). The result now follows.

REFERENCES


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<table>
<thead>
<tr>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kazuo Anzai and Shiro Ishikawa, On common fixed points for several continuous affine mappings</td>
<td>1</td>
</tr>
<tr>
<td>Bruce Alan Barnes, When is a representation of a Banach **-algebra Naimark-related to a **-representation</td>
<td>5</td>
</tr>
<tr>
<td>Richard Dowell Byrd, Justin Thomas Lloyd, Franklin D. Pedersen and James Wilson Stepp, Automorphisms of the semigroup of finite complexes of a periodic locally cyclic group</td>
<td>27</td>
</tr>
<tr>
<td>Donald S. Coram and Paul Frazier Duvall, Jr., Approximate fibrations and a movability condition for maps</td>
<td>41</td>
</tr>
<tr>
<td>Kenneth R. Davidson and Che-Kao Fong, An operator algebra which is not closed in the Calkin algebra</td>
<td>57</td>
</tr>
<tr>
<td>Garret J. Etgen and James Pawlowski, A comparison theorem and oscillation criteria for second order differential systems</td>
<td>59</td>
</tr>
<tr>
<td>Philip Palmer Green, C*-algebras of transformation groups with smooth orbit space</td>
<td>71</td>
</tr>
<tr>
<td>Charles Allen Jones and Charles Dwight Lahr, Weak and norm approximate identities are different</td>
<td>99</td>
</tr>
<tr>
<td>G. K. Kalisch, On integral representations of piecewise holomorphic functions</td>
<td>105</td>
</tr>
<tr>
<td>Y. Kodama, On product of shape and a question of Sher</td>
<td>115</td>
</tr>
<tr>
<td>Heinz K. Langer and B. Textorius, On generalized resolvents and Q(-)functions of symmetric linear relations (subspaces) in Hilbert space</td>
<td>135</td>
</tr>
<tr>
<td>Albert Edward Livingston, On the integral means of univalent, meromorphic functions</td>
<td>167</td>
</tr>
<tr>
<td>Wallace Smith Martindale, III and Susan Montgomery, Fixed elements of Jordan automorphisms of associative rings</td>
<td>181</td>
</tr>
<tr>
<td>R. Kent Nagle, Monotonicity and alternative methods for nonlinear boundary value problems</td>
<td>197</td>
</tr>
<tr>
<td>Richard John O’Malley, Approximately differentiable functions: the r topology</td>
<td>207</td>
</tr>
<tr>
<td>Mangesh Bhalchandra Rege and Kalathoor Varadarajan, Chain conditions and pure-exactness</td>
<td>223</td>
</tr>
<tr>
<td>Christine Ann Shannon, The second dual of C(X)</td>
<td>237</td>
</tr>
<tr>
<td>Sin-ei Takahasi, A characterization for compact central double centralizers of C*-algebras</td>
<td>255</td>
</tr>
<tr>
<td>Theresa Phillips Vaughan, A note on the Jacobi-Perron algorithm</td>
<td>261</td>
</tr>
<tr>
<td>Arthur Anthony Yanushka, A characterization of PSp(2m, q) and PΩ(2m + 1, q) as rank 3 permutation groups</td>
<td>273</td>
</tr>
</tbody>
</table>