A CHARACTERIZATION FOR COMPACT CENTRAL DOUBLE CENTRALIZERS OF C*-ALGEBRAS

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A CHARACTERIZATION FOR COMPACT CENTRAL DOUBLE CENTRALIZERS ON $C^*$-ALGEBRAS

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The purpose of this note is to give a characterization for compact central double centralizers on any $C^*$-algebra $A$ in view of the Dixmier's representation theorem of central double centralizers on $A$. The proof makes use of the Urysohn's lemma for spectra of $C^*$-algebras and algebraic properties of a central double centralizer.

Throughout the note, $A$ denotes a $C^*$-algebra. Let $\text{Prim } A$ denote the structure space of $A$, that is the set of all primitive ideals of $A$, with the hull-kernel topology. Let $M(A)$ denote the double centralizer algebra of $A$ and $Z(M(A))$ the center of $M(A)$. Busby [1] has noted that the algebra $C^0(\text{Prim } A)$ of all bounded continuous complex-valued functions on $\text{Prim } A$ can be canonically identified with $Z(M(A))$, which is equivalent with a result of Dixmier ([5], Theorem 5). Moreover, we can regard the algebra $Z(M(A))$ as the algebra of all bounded linear operators $T$ on $A$ such that $(Tx)y = x(Ty)$ for all $x, y \in A$. In its final form, this identification $\Phi$ between $Z(M(A))$ and $C^0(\text{Prim } A)$ can be described as follows: If $T \in Z(M(A))$, then $Ta + P = \Phi(T)(a + P)$ for all $a \in A$ and $P \in \text{Prim } A$, where $a + P$ for $P \in \text{Prim } A$ denotes the canonical image of $a$ in $A/P$ (Dauns and Hofmann theorem [3] shows that every functions in $C^0(\text{Prim } A)$ can be realized uniquely in this way).

We will characterize the set of all compact central double centralizers on $A$ in view of this representation theorem of $Z(M(A))$. Our characterization is similar to ones established by Kellogg [6] and Ching and Wong [2] for $H^*$-algebras, and this is also a generalization of one proved by Rowlands [7] for dual $B^*$-algebras.

Let $Z_c(M(A))$ denote the compact central double centralizers on $A$. If $LC(A)$ is the algebra of all compact operators on $A$, then $Z_c(M(A)) = Z(M(A)) \cap LC(A)$, so that $Z_c(M(A))$ is a closed ideal of $Z(M(A))$. Let $I_c$ be the set of all functions $f$ in $C^0(\text{Prim } A)$ such that for any closed compact subset $K$ in $\text{supp } (f)$, $A/I_K$ is finite dimensional. Here $\text{supp } (f)$ denotes the set of all $P \in \text{Prim } A$ such that $f(P) \neq 0$, and $I_K$ denotes a closed two-sided ideal of $A$ with $\text{Prim } (A/I_K) \simeq K$ (cf. [4], §3.2). Note that if $K$ is the empty set, then $A/I_K$ is zero-dimensional, so that $I_c$ contains the zero function. Now $I_c$ is a closed ideal in $C^0(\text{Prim } A)$. For since $\text{supp } (f) \supset \text{supp } (fg)$ for each $f, g$ in $C^0(\text{Prim } A)$, $I_c$ is an ideal in $C^0(\text{Prim } A)$. Let $\{ f_n \}$ be a sequence of functions in $I_c$ which converges uniformly...
to a function $f$ in $C^b(\text{Prim } A)$. Let $K$ be any nonempty closed compact subset in $\text{supp } (f)$. Set

$$\delta = \inf \{ |f(P)| : P \in K \} .$$

Then $\delta > 0$ and $\| f_N - f \| < \delta$ for sufficiently large number $N$. This implies $K \subseteq \text{supp } (f_N)$. Then $A/I_K$ is finite dimensional since $f_N \in I_c$. Hence $f \in I_c$ and so $I_c$ is uniformly closed. Let $C_0(\text{Prim } A)$ be the set of all bounded continuous complex-valued functions on $\text{Prim } A$ which vanish at infinity. Let $I_{c_0} = I_c \cap C_0(\text{Prim } A)$. Then $I_{c_0}$ is a closed ideal of $C^b(\text{Prim } A)$.

We now show that these ideals $Z_c(M(A))$ and $I_{c_0}$ can be canonically identified and thus obtain a characterization for $Z_c(M(A))$.

**Theorem 1.** $Z_c(M(A))$ is isometrically $^*$-isomorphic to $I_{c_0}$.

To show the above theorem, we need the following Urysohn’s lemma for arbitrary $C^*$-algebras.

**Lemma 2 ([8], Theorem).** Let $\hat{A}$ be the spectrum of $A$ and let $S_1$, $S_2$ be two nonempty closed subsets in $\hat{A}$. Then the following two conditions are equivalent

(i) $S_1 \cap S_2 = \emptyset$.

(ii) For any element $a \geq 0$ in $A$ there exists an element $x$ in $A$ such that $0 \leq x \leq a$, $\pi(x) = 0$ for all $\pi \in S_1$, and $\pi(x) = \pi(a)$ for all $\pi \in S_2$.

**Proof of Theorem 1.** Let $\Phi$ be the canonical $^*$-isomorphism of $Z(M(A))$ onto $C^b(\text{Prim } A)$ as be stated above. We will show that $\Phi(Z_c(M(A))) = I_{c_0}$ going through three steps.

(I) $\Phi(Z_c(M(A))) \supset I_{c_0}$. Let $f \in I_{c_0}$ and $\varepsilon > 0$ be chosen arbitrarily. Set

$$K_\varepsilon = \{ P \in \text{Prim } A : |f(P)| \geq \varepsilon \}$$

and

$$F_\varepsilon = \{ P \in \text{Prim } A : |f(P)| \leq \varepsilon/2 \} .$$

Let $\{ u_\lambda \}$ be a positive approximate identity for $A$ (in the sense of Appendix B29 in [4]). By Lemma 2, for each $\lambda$ there exists an element $x_{\lambda, \varepsilon}$ in $A$ such that $0 \leq x_{\lambda, \varepsilon} \leq u_\lambda$, $x_{\lambda, \varepsilon} + P = u_\lambda + P$ for all $P \in K$, and $x_{\lambda, \varepsilon} + P = 0$ for all $P \in F_\varepsilon$. Set $T = \Phi^{-1}(f)$, so that $T$ is a central double centralizer on $A$. Moreover, set

$$T_{\lambda, \varepsilon}(a) = T(x_{\lambda, \varepsilon} a)$$

for each $\lambda$ and $a \in A$. Then $T_{\lambda, \varepsilon}$ is a bounded linear operator on $A$. 
We will show that $T_{x,\varepsilon}$ is an element of $LC(A)$. Let $\text{supp}(T_{x,\varepsilon})$ be the set of all $P \in \text{Prim} A$ such that $T_{x,\varepsilon} \in P$. Since $T_{x,\varepsilon} \in T(P) \subset P$ for all $P \in F_{i}$, we have $F_{i}$ is included $\text{Prim} (A) \setminus \text{supp}(T_{x,\varepsilon})$. This implies that

$$\text{cl} (\text{supp} (T_{x,\varepsilon})) \subset \text{cl} (\text{Prim} (A) \setminus F_{i}) \subset K_{\varepsilon/2},$$

where $\text{cl}$ denotes closure in the hull-kernel topology. Since $K_{\varepsilon/2}$ is compact, it follows that $\text{cl} (\text{supp} (T_{x,\varepsilon}))$ is a closed compact subset in $\text{supp} (f)$. Let $I_{x,\varepsilon}$ is a closed two-sided ideal of $A$ such that $\text{Prim} (A/I_{x,\varepsilon}) \cong \text{cl} (\text{supp} (T_{x,\varepsilon}))$. Then $A/I_{x,\varepsilon}$ is finite dimensional since $f \in I_{\varepsilon}$. Let $\{a_{n}\}$ be a sequence of $A$ with $\|a_{n}\| \leq 1$ for all $n = 1, 2, \cdots$. Then $\{a_{n} + I_{x,\varepsilon}\}$ is also a bounded sequence in $A/I_{x,\varepsilon}$, so that there exists a convergent subsequence $\{a_{n_{j}} + I_{x,\varepsilon}\}$. We now have

$$\|T_{x,\varepsilon}(a_{n_{j}}) - T_{x,\varepsilon}(a_{n_{k}})\| = \sup \{\|T_{x,\varepsilon}(a_{n_{j}} - a_{n_{k}}) + P\| : P \in \text{Prim} A\}$$

$$= \sup \{\|T_{x,\varepsilon} + P(a_{n_{j}} - a_{n_{k}} + P)\| : P \in \text{cl} (\text{supp} (T_{x,\varepsilon}))\}$$

$$\leq \sup \{\|T\| \|a_{n_{j}} - a_{n_{k}} + P\| : P \in \text{cl} (\text{supp} (T_{x,\varepsilon}))\}$$

$$= \|T\| \|a_{n_{j}} + I_{x,\varepsilon} - (a_{n_{k}} + I_{x,\varepsilon})\|$$

for all $j, k = 1, 2, \cdots$. Then $\{T_{x,\varepsilon}(a_{n_{j}})\}$ is Cauchy and hence converges in $A$. Thus $T_{x,\varepsilon}$ is compact for each $\lambda$. Now since $f \in I_{\varepsilon}$ and $K_{\varepsilon}$ is a closed compact subset in $\text{supp} (f)$, it follows that $A/I_{K_{\varepsilon}}$ is finite dimensional $C^{*}$-algebra and hence $\{u_{\lambda} + I_{K_{\varepsilon}}\}$ converges to the identity $1_{\varepsilon}$ of $A/I_{K_{\varepsilon}}$. Then there exists a $\lambda_{\varepsilon}$ such that $\|1_{\varepsilon} - (u_{\lambda_{\varepsilon}} + I_{K_{\varepsilon}})\| < \varepsilon$. Set $T_{\varepsilon} = T_{x,\varepsilon}$ and $x_{\varepsilon} = x_{x,\varepsilon}$. For any $a \in A$ we further set

$$\alpha = \sup \{\|T(a - x_{\varepsilon}a) + P\| : P \in K_{\varepsilon}\},$$

$$\beta = \sup \{\|T(a - x_{\varepsilon}a) + P\| : P \in \text{Prim} (A) \setminus K_{\varepsilon}\}. $$

Since $x_{\varepsilon} + P = u_{\varepsilon} + P$ for all $P \in K_{\varepsilon}$, we have

$$\alpha = \sup \{\|T(a + P) - (u_{\varepsilon} + P)(T_{\varepsilon}a + P)\| : P \in K_{\varepsilon}\}$$

$$= \|1_{\varepsilon} - (u_{\lambda_{\varepsilon}} + I_{K_{\varepsilon}})(T_{\varepsilon}a + I_{K_{\varepsilon}})\|$$

$$\leq \|T_{\varepsilon}\| \varepsilon.$$ 

We further have

$$\beta = \sup \{|f(P)| \|a - x_{\varepsilon}a\| + P\| : P \in \text{Prim}(A) \setminus K_{\varepsilon}\}$$

$$\leq (\|a\| + \|u_{\varepsilon}\| \|a\|) \varepsilon$$

$$\leq 2 \|a\| \varepsilon.$$
Therefore \( \| T\alpha - T_x\alpha \| \leq \alpha + \beta \leq (\| T\alpha \| + 2 \| \alpha \|)\varepsilon \) for all \( \alpha \in A \), so that \( \| T - T_x \| \leq (\| T \| + 2)\varepsilon \). Since \( T \) is compact and \( \varepsilon \) is arbitrary, \( T \) is also compact and (I) is proved.

(II) \( \Phi(Z_c(M(A))) \subseteq I_c \). Let \( f \in \Phi(Z_c(M(A))) \) and \( T \in Z_c(M(A)) \) with \( f = \Phi(T) \). Suppose that \( f \notin I_c \), so that there exists a non-empty closed compact subset \( K \) in \( \text{supp}(f) \) such that \( A/I_K \) is infinite dimensional. Then there exist elements \( a_\alpha \) in \( A \) such that 
\[
\| a_n + I_K \| = 1 \quad (n = 1, 2, \ldots ) \quad \text{and} \quad \| (a_n + I_K) - (a_m + I_K) \| \geq 1/2 \quad (n \neq m).
\]
We can assume that \( \| a_n \| \leq 2 \quad (n = 1, 2, \ldots ) \). Set
\[
\delta = \inf \{ \| f(P) \| : P \in K \}.
\]
Then \( \delta > 0 \) since \( K \) is compact and we have
\[
\| T\alpha_n - T\alpha_m \| \geq \sup \{ \| f(P) \| \| (a_n - a_m) + P \| : P \in K \} \\
\geq \sup \{ \| (a_n - a_m) + P \| \delta : P \in K \} \\
= \| (a_n + I_K) - (a_m + I_K) \| \delta \\
\geq \delta/2
\]
for all distinct numbers \( n, m \). Then \( \{ T\alpha_n \} \) contains no convergent subsequence. But this is impossible since \( T \) is compact and (II) is proved.

(III) \( \Phi(Z_c(M(A))) \subseteq C_0(\text{Prim } A) \). Let \( T \in Z_c(M(A)) \) and \( \varepsilon > 0 \). Set
\[
f = \Phi(T) \quad \text{and} \quad K_\varepsilon = \{ P \in \text{Prim } A : |f(P)| \geq \varepsilon \}.
\]
We only show that \( K_\varepsilon \) is compact. Let \( I_{K_\varepsilon} \) be a closed two-sided ideal of \( A \) with \( \text{Prim } (A/I_{K_\varepsilon}) \simeq K_\varepsilon \), as he stated above. Suppose that \( A/I_{K_\varepsilon} \) is infinite dimensional. Then, as in the proof of (II), there exist elements \( a_\alpha \) in \( A \) such that
\[
\| a_n \| \leq 2, \quad \| a_n + I_{K_\varepsilon} \| = 1 \quad (n = 1, 2, \ldots )
\]
and 
\[
\| (a_n + I_{K_\varepsilon}) - (a_m + I_{K_\varepsilon}) \| \geq 1/2 \quad (n \neq m).
\]
By the same computation in the proof of (II), we have \( \| T\alpha_n - T\alpha_m \| \geq \varepsilon/2 \), so that \( \{ T\alpha_n \} \) contains no convergent subsequence, which contradicts \( T \) is compact. Thus \( A/I_{K_\varepsilon} \) is a finite dimensional \( C^* \)-algebra. Then \( A/I_{K_\varepsilon} \) can be canonically identified with its enveloping von Neumann algebra. Suppose that \( \text{Prim } (A/I_{K_\varepsilon}) \) contains an infinite countable subset \( \{ P_1, P_2, \ldots \} \). Let \( \pi_i \) be a nonzero irreducible representation of \( A/I_{K_\varepsilon} \) with \( P_i = \text{Ker } \pi_i \) and \( \xi_i \) a norm one element in the Hilbert space associated with \( \pi_i \) for each \( i \). Set
\[
f_i(x + I_{K_\varepsilon}) = (\pi_i(x + I_{K_\varepsilon})\xi_i | \xi_i) \quad (i = 1, 2, \ldots )
\]
for each \( x + I_{K_\varepsilon} \in A/I_{K_\varepsilon} \). Since \( \pi_i \neq \pi_j \) \( (i \neq j) \), it follows that \( \| f_i - f_j \| = 2 \) \( (i \neq j) \) (cf. [4], 2.12.1). Let \( p_i \) denote the support of \( f_i \) for each \( i \). Then \( \{ p_i \} \) are mutually orthogonal (cf. [4], 12.3.1). But this is impossible since each \( p_i \) is an element in \( A/I_{K_\varepsilon} \) and so
Prim \((A/I_{K_{\varepsilon}})\) is finite set. Then \(K_{\varepsilon}\) is also a finite set, so that it is compact and (III) is proved.

We will next show that a result of Rowlands ([7], Theorem 2) is a special case of Theorem 1. Let \(\Omega(A)\) be the space of minimal closed two-sided ideals of \(A\) with its discrete topology, in case \(A\) is dual. Let \(\{I_i : \lambda \in \Lambda\}\) be the family of all minimal closed two-sided ideals of \(A\) and \(\Lambda_0 = \{\lambda \in \Lambda : I_i\) is infinite dimensional\}. Let \(I_0\) be the set of all functions \(f\) in the algebra \(C^b(\Omega(A))\) of all bounded complex-valued functions on \(\Omega(A)\) such that \(f(I_i) = 0\) for all \(\lambda \in \Lambda_0\); if \(\Lambda_0 = \emptyset\), let \(I_0 = C^b(\Omega(A))\). Let \(C_0(\Omega(A))\) be the subalgebra of \(C^b(\Omega(A))\) which consists of functions vanishing at infinity.

**Corollary 3 ([7], Theorem 2).** If \(A\) is a dual C*-algebra, then 
\[Z_C(M(A)) \text{ is isometrically } *\text{-isomorphic to } I_0 \cap C_0(\Omega(A)).\]

**Proof.** By ([4], 10.10.6), \(\text{Prim } A\) is discrete. For each \(P \in \text{Prim } A\), we define a function \(\delta_P\) on \(\text{Prim } A\) by the equation: \(\delta_P(P) = 1\) and \(\delta_P(Q) = 0\) if \(Q \neq P\), and set \(\mu(P) = \Phi^{-1}(\delta_P)(A)\). Then we can easily see that \(P \rightarrow \mu(P)\) is a bijection of \(\text{Prim } A\) onto \(\Omega(A)\). Let \(\mu^*\) be the dual map of \(\mu\). Then \(\mu^*\) is a isometric \(*\)-isomorphism of \(C^b(\Omega(A))\) onto \(C^b(\text{Prim } A)\). By the definitions of \(I_0\) and \(I_0\), we see that \(\mu^*(I_0 \cap C_0(\Omega(A))) = I_0\). Set \(\Psi(T) = (\mu^*)^{-1}(\Phi(T))\) for each \(T \in Z_C(M(A))\). Then \(\Psi(Z_C(M(A))) = I_0 \cap C_0(\Omega(A))\) by Theorem 1 and the corollary is proved.

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