Pacific Journal of Mathematics

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Vol. 72, No. 1 January 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol. 72, No. 1, 1977

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The purpose of this note is to give a characterization for compact central double centralizers on any C^*-algebra A in view of the Dixmier's representation theorem of central double centralizers on A. The proof makes use of the Urysohn's lemma for spectra of C^*-algebras and algebraic properties of a central double centralizer.

Throughout the note, A denotes a C^*-algebra. Let Prim A denote the structure space of A, that is the set of all primitive ideals of A, with the hull-kernel topology. Let $M(A)$ denote the double centralizer algebra of A and $Z(M(A))$ the center of $M(A)$. Busby [1] has noted that the algebra $C^b(\text{Prim}\, A)$ of all bounded continuous complex-valued functions on Prim A can be canonically identified with $Z(M(A))$, which is equivalent with a result of Dixmier ([5], Theorem 5). Moreover, we can regard the algebra $Z(M(A))$ as the algebra of all bounded linear operators T on A such that $(Tx)y = x(Ty)$ for all $x, y \in A$. In its final form, this identification Φ between $Z(M(A))$ and $C^b(\text{Prim}\, A)$ can be described as follows: If $T \in Z(M(A))$, then $Ta + P = \Phi(T)(P)(a + P)$ for all $a \in A$ and $P \in \text{Prim}\, A$, where $a + P$ for $P \in \text{Prim}\, A$ denotes the canonical image of a in A/P (Dauns and Hofmann theorem [3] shows that every functions in $C^b(\text{Prim}\, A)$ can be realized uniquely in this way). We will characterize the set of all compact central double centralizers on A in view of this representation theorem of $Z(M(A))$. Our characterization is similar to ones established by Kellogg [6] and Ching and Wong [2] for H^*-algebras, and this is also a generalization of one proved by Rowlands [7] for dual B^*-algebras.

Let $Z_C(M(A))$ denote the compact central double centralizers on A. If $LC(A)$ is the algebra of all compact operators on A, then $Z_C(M(A)) = Z(M(A)) \cap LC(A)$, so that $Z_C(M(A))$ is a closed ideal of $Z(M(A))$. Let I_C be the set of all functions f in $C^b(\text{Prim}\, A)$ such that for any closed compact subset K in $\text{supp}\,(f)$, A/I_K is finite dimensional. Here $\text{supp}\,(f)$ denotes the set of all $P \in \text{Prim}\, A$ such that $f(P) \neq 0$, and I_K denotes a closed two-sided ideal of A with $\text{Prim}\,(A/I_K) \simeq K$ (cf. [4], § 3.2). Note that if K is the empty set, then A/I_K is zero-dimensional, so that I_C contains the zero function. Now I_C is a closed ideal in $C^b(\text{Prim}\, A)$. For since $\text{supp}\,(f) \supset \text{supp}\,(fg)$ for each f, g in $C^b(\text{Prim}\, A)$, I_C is an ideal in $C^b(\text{Prim}\, A)$. Let $\{f_n\}$ be a sequence of functions in I_C which converges uniformly to a function f in $C^b(\text{Prim } A)$. Let K be any nonempty closed compact subset in $\text{supp}(f)$. Set

$$\delta = \inf \{|f(P)| : P \in K\} .$$

Then $\delta > 0$ and $\|f_N - f\| < \delta$ for sufficiently large number N. This implies $K \subset \text{supp}(f_N)$. Then A/I_K is finite dimensional since $f_N \in I_C$. Hence $f \in I_C$ and so I_C is uniformly closed. Let $C_0(\text{Prim } A)$ be the set of all bounded continuous complex-valued functions on Prim A which vanish at infinity. Let $I_{C0} = I_C \cap C_0(\text{Prim } A)$. Then I_{C0} is a closed ideal of $C^b(\text{Prim } A)$.

We now show that these ideals $Z_C(M(A))$ and I_{C0} can be canonically identified and thus obtain a characterization for $Z_C(M(A))$.

THEOREM 1. *$Z_C(M(A))$ is isometrically *-isomorphic to I_{C0}.*

To show the above theorem, we need the following Urysohn's lemma for arbitrary C^*-algebras.

LEMMA 2 ([8], Theorem). *Let $\hat{A}$ be the spectrum of A and let S_1, S_2 be two nonempty closed subsets in $\hat{A}$. Then the following two conditions are equivalent*

(i) $S_1 \cap S_2 = \emptyset$.

(ii) *For any element $a \geqq 0$ in A there exists an element x in A such that $0 \leqq x \leqq a$, $\pi(x) = 0$ for all $\pi \in S_1$, and $\pi(x) = \pi(a)$ for all $\pi \in S_2$.*

Proof of Theorem 1. Let Φ be the canonical *-isomorphism of $Z(M(A))$ onto $C^b(\text{Prim } A)$ as be stated above. We will show that $\Phi(Z_C(M(A))) = I_{C0}$ going through three steps.

(I) $\Phi(Z_C(M(A))) \supset I_{C0}$. Let $f \in I_{C0}$ and $\varepsilon > 0$ be chosen arbitrarily. Set

$$K_\varepsilon = \{P \in \text{Prim } A: |f(P)| \geqq \varepsilon\}$$

and

$$F_\varepsilon = \{P \in \text{Prim } A: |f(P)| \leqq \varepsilon/2\} .$$

Let $\{u_\lambda\}$ be a positive approximate identity for A (in the sense of Appendice B29 in [4]). By Lemma 2, for each λ there exists an element $x_{\lambda,\varepsilon}$ in A such that $0 \leqq x_{\lambda,\varepsilon} \leqq u_\lambda$, $x_{\lambda,\varepsilon} + P = u_\lambda + P$ for all $P \in K_\varepsilon$ and $x_{\lambda,\varepsilon} + P = 0$ for all $P \in F_\varepsilon$. Set $T = \Phi^{-1}(f)$, so that T is a central double centralizer on A. Moreover, set

$$T_{\lambda,\varepsilon}(a) = T(x_{\lambda,\varepsilon} a)$$

for each λ and $a \in A$. Then $T_{\lambda,\varepsilon}$ is a bounded linear operator on A.

We will show that $T_{\lambda,\varepsilon}$ is an element of $LC(A)$. Let $\operatorname{supp}(Tx_{\lambda,\varepsilon})$ be the set of all $P \in \operatorname{Prim} A$ such that $Tx_{\lambda,\varepsilon} \notin P$. Since $Tx_{\lambda,\varepsilon} \in T(P) \subset P$ for all $P \in F_\varepsilon$, we have F_ε is included $\operatorname{Prim}(A)\backslash\operatorname{supp}(Tx_{\lambda,\varepsilon})$. This implies that

$$\operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon})) \subset \operatorname{cl}(\operatorname{Prim}(A)\backslash F_\varepsilon) \subset K_{\varepsilon/2} ,$$

where cl denotes closure in the hull-kernel topology. Since $K_{\varepsilon/2}$ is compact, it follows that $\operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon}))$ is a closed compact subset in $\operatorname{supp}(f)$. Let $I_{\lambda,\varepsilon}$ is a closed two-sided ideal of A such that $\operatorname{Prim}(A/I_{\lambda,\varepsilon}) \simeq \operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon}))$. Then $A/I_{\lambda,\varepsilon}$ is finite dimensional since $f \in I_C$. Let $\{a_n\}$ be a sequence of A with $\|a_n\| \leqq 1$ for all $n = 1, 2, \cdots$. Then $\{a_n + I_{\lambda,\varepsilon}\}$ is also a bounded sequence in $A/I_{\lambda,\varepsilon}$, so that there exists a convergent subsequence $\{a_{n_j} + I_{\lambda,\varepsilon}\}$. We now have

$$\begin{aligned}
&\|T_{\lambda,\varepsilon}(a_{n_j}) - T_{\lambda,\varepsilon}(a_{n_k})\| \\
&\quad= \sup\{\|(Tx_{\lambda,\varepsilon})(a_{n_j} - a_{n_k}) + P\| : P \in \operatorname{Prim} A\} \\
&\quad= \sup\{\|(Tx_{\lambda,\varepsilon} + P)(a_{n_j} - a_{n_k} + P)\| : P \in \operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon}))\} \\
&\quad\leqq \sup\{\|T\|\,\|a_{n_j} - a_{n_k} + P\| : P \in \operatorname{cl}(\operatorname{supp}(Tx_{\lambda,\varepsilon}))\} \\
&\quad= \|T\|\,\|(a_{n_j} + I_{\lambda,\varepsilon}) - (a_{n_k} + I_{\lambda,\varepsilon})\|
\end{aligned}$$

for all $j, k = 1, 2, \cdots$. Then $\{T_{\lambda,\varepsilon}(a_{n_j})\}$ is Cauchy and hence converges in A. Thus $T_{\lambda,\varepsilon}$ is compact for each λ. Now since $f \in I_C$ and K_ε is a closed compact subset in $\operatorname{supp}(f)$, it follows that A/I_{K_ε} is finite dimensional C^*-algebra and hence $\{u_\lambda + I_{K_\varepsilon}\}$ converges to the identity 1_ε of A/I_{K_ε}. Then there exists a λ_ε such that $\|1_\varepsilon - (u_{\lambda_\varepsilon} + I_{K_\varepsilon})\| < \varepsilon$. Set $T_\varepsilon = T_{\lambda_\varepsilon,\varepsilon}$ and $x_\varepsilon = x_{\lambda_\varepsilon,\varepsilon}$. For any $a \in A$ we further set

$$\alpha = \sup\{\|(Ta - x_\varepsilon Ta) + P\| : P \in K_\varepsilon\} ,$$

$$\beta = \sup\{\|T(a - x_\varepsilon a) + P\| : P \in \operatorname{Prim}(A)\backslash K_\varepsilon\} .$$

Since $x_\varepsilon + P = u_{\lambda_\varepsilon} + P$ for all $P \in K_\varepsilon$, we have

$$\begin{aligned}
\alpha &= \sup\{\|(Ta + P) - (u_{\lambda_\varepsilon} + P)(Ta + P)\| : P \in K_\varepsilon\} \\
&= \|(1_\varepsilon - (u_{\lambda_\varepsilon} + I_{K_\varepsilon}))(Ta + I_{K_\varepsilon})\| \\
&\leqq \|Ta\|\,\varepsilon .
\end{aligned}$$

We further have

$$\begin{aligned}
\beta &= \sup\{|f(P)|\,\|(a - x_\varepsilon a) + P\| : P \in \operatorname{Prim}(A)\backslash K_\varepsilon\} \\
&\leqq (\|a\| + \|u_{\lambda_\varepsilon}\|\,\|a\|)\varepsilon \\
&\leqq 2\|a\|\,\varepsilon .
\end{aligned}$$

Therefore $\|Ta - T_\varepsilon a\| \leqq \alpha + \beta \leqq (\|Ta\| + 2\|a\|)\varepsilon$ for all $a \in A$, so that $\|T - T_\varepsilon\| \leqq (\|T\| + 2)\varepsilon$. Since T_ε is compact and ε is arbitrary, T is also compact and (I) is proved.

(II) $\Phi(Z_C(M(A))) \subset I_C$. Let $f \in \Phi(Z_C(M(A)))$ and $T \in Z_C(M(A))$ with $f = \Phi(T)$. Suppose that $f \notin I_C$, so that there exists a non-empty closed compact subset K in supp (f) such that A/I_K is infinite dimensional. Then there exist elements a_n in A such that $\|a_n + I_K\| = 1$ ($n = 1, 2, \cdots$) and $\|(a_n + I_K) - (a_m + I_K)\| \geqq 1/2$ ($n \neq m$). We can assume that $\|a_n\| \leqq 2$ ($n = 1, 2, \cdots$). Set

$$\delta = \inf\{|f(P)|\colon P \in K\} .$$

Then $\delta > 0$ since K is compact and we have

$$\begin{aligned} \|Ta_n - Ta_m\| &\geqq \sup\{|f(P)|\,\|(a_n - a_m) + P\|\colon P \in K\} \\ &\geqq \sup\{\|(a_n - a_m) + P\|\delta\colon P \in K\} \\ &= \|(a_n + I_K) - (a_m + I_K)\|\delta \\ &\geqq \delta/2 \end{aligned}$$

for all distinct numbers n, m. Then $\{Ta_n\}$ contains no convergent subsequence. But this is imposible since T is compact and (II) is proved.

(III) $\Phi(Z_C(M(A))) \subset C_0(\mathrm{Prim}\, A)$. Let $T \in Z_C(M(A))$ and $\varepsilon > 0$. Set

$$f = \Phi(T) \text{ and } K_\varepsilon = \{P \in \mathrm{Prim}\, A\colon |f(P)| \geqq \varepsilon\} .$$

We only show that K_ε is compact. Let I_{K_ε} be a closed two-sided ideal of A with $\mathrm{Prim}\,(A/I_{K_\varepsilon}) \simeq K_\varepsilon$, as be stated above. Suppose that A/I_{K_ε} is infinite dimensional. Then, as in the proof of (II), there exist elements a_n in A such that $\|a_n\| \leqq 2$, $\|a_n + I_{K_\varepsilon}\| = 1$ ($n = 1, 2, \cdots$) and $\|(a_n + I_{K_\varepsilon}) - (a_m + I_{K_\varepsilon})\| \geqq 1/2$ ($n \neq m$). By the same computation in the proof of (II), we have $\|Ta_n - Ta_m\| \geqq \varepsilon/2$, so that $\{Ta_n\}$ contains no convergent subsequence, which contradicts T is compact. Thus A/I_{K_ε} is a finite dimensinal C^*-algebra. Then A/I_{K_ε} can be canonically identified with its enveloping von Neumann algebra. Suppose that $\mathrm{Prim}\,(A/I_{K_\varepsilon})$ contains an infinite countable subset $\{P_1, P_2, \cdots\}$. Let π_i be a nonzero irreducible representation of A/I_{K_ε} with $P_i = \mathrm{Ker}\, \pi_i$ and ξ_i a norm one element in the Hilbert space associated with π_i for each i. Set

$$f_i(x + I_{K_\varepsilon}) = (\pi_i(x + I_{K_\varepsilon})\xi_i | \xi_i) \quad (i = 1, 2, \cdots)$$

for each $x + I_{K_\varepsilon} \in A/I_{K_\varepsilon}$. Since $\pi_i \not\simeq \pi_j$ ($i \neq j$), it follows that $\|f_i - f_j\| = 2$ ($i \neq j$) (cf. [4], 2.12.1). Let p_i denote the support of f_i for each i. Then $\{p_i\}$ are mutually orthogonal (cf. [4], 12.3.1). But this is imposible since each p_i is an element in A/I_{K_ε} and so

$\mathrm{Prim}\,(A/I_{K_\varepsilon})$ is finite set. Then K_ε is also a finite set, so that it is compact and (III) is proved.

We will next show that a result of Rowlands ([7], Theorem 2) is a special case of Theorem 1. Let $\Omega(A)$ be the space of minimal closed two-sided ideals of A with its discrete topology, in case A is dual. Let $\{I_\lambda: \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A and $\Lambda_0 = \{\lambda \in \Lambda: I_\lambda$ is infinite dimensional$\}$. Let I_0 be the set of all functions f in the algebra $C^b(\Omega(A))$ of all bounded complex-valued functions on $\Omega(A)$ such that $f(I_\lambda) = 0$ for all $\lambda \in \Lambda_0$; if $\Lambda_0 = \emptyset$, let $I_0 = C^b(\Omega(A))$. Let $C_0(\Omega(A))$ be the subalgebra of $C^b(\Omega(A))$ which consists of functions vanishing at infinity.

COROLLARY 3 ([7], Theorem 2). *If A is a dual C^*-algebra, then $Z_C(M(A))$ is isometrically *-isomorphic to $I_0 \cap C_0(\Omega(A))$.*

Proof. By ([4], 10.10.6), $\mathrm{Prim}\,A$ is discrete. For each $P \in \mathrm{Prim}\,A$, we define a function δ_P on $\mathrm{Prim}\,A$ by the equation: $\delta_P(P) = 1$ and $\delta_P(Q) = 0$ if $Q \neq P$, and set $\mu(P) = \Phi^{-1}(\delta_P)(A)$. Then we can easily see that $P \rightarrow \mu(P)$ is a bijection of $\mathrm{Prim}\,A$ onto $\Omega(A)$. Let μ^* be the dual map of μ. Then μ^* is a isometric *-isomorphism of $C^b(\Omega(A))$ onto $C^b(\mathrm{Prim}\,A)$. By the definitions of I_C and I_0, we see that $\mu^*(I_0 \cap C_0(\Omega(A))) = I_{C0}$. Set $\Psi(T) = (\mu^*)^{-1}(\Phi(T))$ for each $T \in Z_C(M(A))$. Then $\Psi(Z_C(M(A))) = I_0 \cap C_0(\Omega(A))$ by Theorem 1 and the corollary is proved.

The author wishes to thank the referee for his useful comments.

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Received October 13, 1976 and in revised form March 10, 1977.

IBARAKI UNIVERSITY
MITO, IBARAKI, JAPAN

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 72, No. 1 January, 1977