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**A CHARACTERIZATION FOR COMPACT CENTRAL DOUBLE  
CENTRALIZERS OF  $C^*$ -ALGEBRAS**

SIN-EI TAKAHASI

# A CHARACTERIZATION FOR COMPACT CENTRAL DOUBLE CENTRALIZERS ON $C^*$ -ALGEBRAS

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**The purpose of this note is to give a characterization for compact central double centralizers on any  $C^*$ -algebra  $A$  in view of the Dixmier's representation theorem of central double centralizers on  $A$ . The proof makes use of the Urysohn's lemma for spectra of  $C^*$ -algebras and algebraic properties of a central double centralizer.**

Throughout the note,  $A$  denotes a  $C^*$ -algebra. Let  $\text{Prim } A$  denote the structure space of  $A$ , that is the set of all primitive ideals of  $A$ , with the hull-kernel topology. Let  $M(A)$  denote the double centralizer algebra of  $A$  and  $Z(M(A))$  the center of  $M(A)$ . Busby [1] has noted that the algebra  $C^b(\text{Prim } A)$  of all bounded continuous complex-valued functions on  $\text{Prim } A$  can be canonically identified with  $Z(M(A))$ , which is equivalent with a result of Dixmier ([5], Theorem 5). Moreover, we can regard the algebra  $Z(M(A))$  as the algebra of all bounded linear operators  $T$  on  $A$  such that  $(Tx)y = x(Ty)$  for all  $x, y \in A$ . In its final form, this identification  $\Phi$  between  $Z(M(A))$  and  $C^b(\text{Prim } A)$  can be described as follows: If  $T \in Z(M(A))$ , then  $Ta + P = \Phi(T)(P)(a + P)$  for all  $a \in A$  and  $P \in \text{Prim } A$ , where  $a + P$  for  $P \in \text{Prim } A$  denotes the canonical image of  $a$  in  $A/P$  (Dauns and Hofmann theorem [3] shows that every functions in  $C^b(\text{Prim } A)$  can be realized uniquely in this way). We will characterize the set of all compact central double centralizers on  $A$  in view of this representation theorem of  $Z(M(A))$ . Our characterization is similar to ones established by Kellogg [6] and Ching and Wong [2] for  $H^*$ -algebras, and this is also a generalization of one proved by Rowlands [7] for dual  $B^*$ -algebras.

Let  $Z_c(M(A))$  denote the compact central double centralizers on  $A$ . If  $LC(A)$  is the algebra of all compact operators on  $A$ , then  $Z_c(M(A)) = Z(M(A)) \cap LC(A)$ , so that  $Z_c(M(A))$  is a closed ideal of  $Z(M(A))$ . Let  $I_c$  be the set of all functions  $f$  in  $C^b(\text{Prim } A)$  such that for any closed compact subset  $K$  in  $\text{supp}(f)$ ,  $A/I_K$  is finite dimensional. Here  $\text{supp}(f)$  denotes the set of all  $P \in \text{Prim } A$  such that  $f(P) \neq 0$ , and  $I_K$  denotes a closed two-sided ideal of  $A$  with  $\text{Prim}(A/I_K) \simeq K$  (cf. [4], §3.2). Note that if  $K$  is the empty set, then  $A/I_K$  is zero-dimensional, so that  $I_c$  contains the zero function. Now  $I_c$  is a closed ideal in  $C^b(\text{Prim } A)$ . For since  $\text{supp}(f) \supset \text{supp}(fg)$  for each  $f, g$  in  $C^b(\text{Prim } A)$ ,  $I_c$  is an ideal in  $C^b(\text{Prim } A)$ . Let  $\{f_n\}$  be a sequence of functions in  $I_c$  which converges uniformly

to a function  $f$  in  $C^b(\text{Prim } A)$ . Let  $K$  be any nonempty closed compact subset in  $\text{supp}(f)$ . Set

$$\delta = \inf \{|f(P)| : P \in K\}.$$

Then  $\delta > 0$  and  $\|f_N - f\| < \delta$  for sufficiently large number  $N$ . This implies  $K \subset \text{supp}(f_N)$ . Then  $A/I_K$  is finite dimensional since  $f_N \in I_C$ . Hence  $f \in I_C$  and so  $I_C$  is uniformly closed. Let  $C_0(\text{Prim } A)$  be the set of all bounded continuous complex-valued functions on  $\text{Prim } A$  which vanish at infinity. Let  $I_{C_0} = I_C \cap C_0(\text{Prim } A)$ . Then  $I_{C_0}$  is a closed ideal of  $C^b(\text{Prim } A)$ .

We now show that these ideals  $Z_C(M(A))$  and  $I_{C_0}$  can be canonically identified and thus obtain a characterization for  $Z_C(M(A))$ .

**THEOREM 1.**  $Z_C(M(A))$  is isometrically \*-isomorphic to  $I_{C_0}$ .

To show the above theorem, we need the following Urysohn's lemma for arbitrary  $C^*$ -algebras.

**LEMMA 2** ([8], Theorem). Let  $\hat{A}$  be the spectrum of  $A$  and let  $S_1, S_2$  be two nonempty closed subsets in  $\hat{A}$ . Then the following two conditions are equivalent

(i)  $S_1 \cap S_2 = \emptyset$ .

(ii) For any element  $a \geq 0$  in  $A$  there exists an element  $x$  in  $A$  such that  $0 \leq x \leq a$ ,  $\pi(x) = 0$  for all  $\pi \in S_1$ , and  $\pi(x) = \pi(a)$  for all  $\pi \in S_2$ .

*Proof of Theorem 1.* Let  $\Phi$  be the canonical \*-isomorphism of  $Z(M(A))$  onto  $C^b(\text{Prim } A)$  as be stated above. We will show that  $\Phi(Z_C(M(A))) = I_{C_0}$  going through three steps.

(I)  $\Phi(Z_C(M(A))) \supset I_{C_0}$ . Let  $f \in I_{C_0}$  and  $\varepsilon > 0$  be chosen arbitrarily. Set

$$K_\varepsilon = \{P \in \text{Prim } A : |f(P)| \geq \varepsilon\}$$

and

$$F_\varepsilon = \{P \in \text{Prim } A : |f(P)| \leq \varepsilon/2\}.$$

Let  $\{u_\lambda\}$  be a positive approximate identity for  $A$  (in the sense of Appendix B29 in [4]). By Lemma 2, for each  $\lambda$  there exists an element  $x_{\lambda,\varepsilon}$  in  $A$  such that  $0 \leq x_{\lambda,\varepsilon} \leq u_\lambda$ ,  $x_{\lambda,\varepsilon} + P = u_\lambda + P$  for all  $P \in K_\varepsilon$  and  $x_{\lambda,\varepsilon} + P = 0$  for all  $P \in F_\varepsilon$ . Set  $T = \Phi^{-1}(f)$ , so that  $T$  is a central double centralizer on  $A$ . Moreover, set

$$T_{\lambda,\varepsilon}(a) = T(x_{\lambda,\varepsilon}a)$$

for each  $\lambda$  and  $a \in A$ . Then  $T_{\lambda,\varepsilon}$  is a bounded linear operator on  $A$ .

We will show that  $T_{\lambda,\varepsilon}$  is an element of  $LC(A)$ . Let  $\text{supp}(Tx_{\lambda,\varepsilon})$  be the set of all  $P \in \text{Prim } A$  such that  $Tx_{\lambda,\varepsilon} \notin P$ . Since  $Tx_{\lambda,\varepsilon} \in T(P) \subset P$  for all  $P \in F_\varepsilon$ , we have  $F_\varepsilon$  is included  $\text{Prim}(A) \setminus \text{supp}(Tx_{\lambda,\varepsilon})$ . This implies that

$$\text{cl}(\text{supp}(Tx_{\lambda,\varepsilon})) \subset \text{cl}(\text{Prim}(A) \setminus F_\varepsilon) \subset K_{\varepsilon/2},$$

where  $\text{cl}$  denotes closure in the hull-kernel topology. Since  $K_{\varepsilon/2}$  is compact, it follows that  $\text{cl}(\text{supp}(Tx_{\lambda,\varepsilon}))$  is a closed compact subset in  $\text{supp}(f)$ . Let  $I_{\lambda,\varepsilon}$  is a closed two-sided ideal of  $A$  such that  $\text{Prim}(A/I_{\lambda,\varepsilon}) \simeq \text{cl}(\text{supp}(Tx_{\lambda,\varepsilon}))$ . Then  $A/I_{\lambda,\varepsilon}$  is finite dimensional since  $f \in I_C$ . Let  $\{a_n\}$  be a sequence of  $A$  with  $\|a_n\| \leq 1$  for all  $n = 1, 2, \dots$ . Then  $\{a_n + I_{\lambda,\varepsilon}\}$  is also a bounded sequence in  $A/I_{\lambda,\varepsilon}$ , so that there exists a convergent subsequence  $\{a_{n_j} + I_{\lambda,\varepsilon}\}$ . We now have

$$\begin{aligned} & \|T_{\lambda,\varepsilon}(a_{n_j}) - T_{\lambda,\varepsilon}(a_{n_k})\| \\ &= \sup \{ \|(Tx_{\lambda,\varepsilon})(a_{n_j} - a_{n_k}) + P\| : P \in \text{Prim } A \} \\ &= \sup \{ \|(Tx_{\lambda,\varepsilon} + P)(a_{n_j} - a_{n_k} + P)\| : P \in \text{cl}(\text{supp}(Tx_{\lambda,\varepsilon})) \} \\ &\leq \sup \{ \|T\| \|a_{n_j} - a_{n_k} + P\| : P \in \text{cl}(\text{supp}(Tx_{\lambda,\varepsilon})) \} \\ &= \|T\| \| (a_{n_j} + I_{\lambda,\varepsilon}) - (a_{n_k} + I_{\lambda,\varepsilon}) \| \end{aligned}$$

for all  $j, k = 1, 2, \dots$ . Then  $\{T_{\lambda,\varepsilon}(a_{n_j})\}$  is Cauchy and hence converges in  $A$ . Thus  $T_{\lambda,\varepsilon}$  is compact for each  $\lambda$ . Now since  $f \in I_C$  and  $K_\varepsilon$  is a closed compact subset in  $\text{supp}(f)$ , it follows that  $A/I_{K_\varepsilon}$  is finite dimensional  $C^*$ -algebra and hence  $\{u_\lambda + I_{K_\varepsilon}\}$  converges to the identity  $1_\varepsilon$  of  $A/I_{K_\varepsilon}$ . Then there exists a  $\lambda_\varepsilon$  such that  $\|1_\varepsilon - (u_{\lambda_\varepsilon} + I_{K_\varepsilon})\| < \varepsilon$ . Set  $T_\varepsilon = T_{\lambda_\varepsilon,\varepsilon}$  and  $x_\varepsilon = x_{\lambda_\varepsilon,\varepsilon}$ . For any  $a \in A$  we further set

$$\begin{aligned} \alpha &= \sup \{ \|(Ta - x_\varepsilon Ta) + P\| : P \in K_\varepsilon \}, \\ \beta &= \sup \{ \|T(a - x_\varepsilon a) + P\| : P \in \text{Prim}(A) \setminus K_\varepsilon \}. \end{aligned}$$

Since  $x_\varepsilon + P = u_{\lambda_\varepsilon} + P$  for all  $P \in K_\varepsilon$ , we have

$$\begin{aligned} \alpha &= \sup \{ \|(Ta + P) - (u_{\lambda_\varepsilon} + P)(Ta + P)\| : P \in K_\varepsilon \} \\ &= \|(1_\varepsilon - (u_{\lambda_\varepsilon} + I_{K_\varepsilon}))(Ta + I_{K_\varepsilon})\| \\ &\leq \|Ta\| \varepsilon. \end{aligned}$$

We further have

$$\begin{aligned} \beta &= \sup \{ \|f(P)\| \|(a - x_\varepsilon a) + P\| : P \in \text{Prim}(A) \setminus K_\varepsilon \} \\ &\leq (\|a\| + \|u_{\lambda_\varepsilon}\| \|a\|) \varepsilon \\ &\leq 2 \|a\| \varepsilon. \end{aligned}$$

Therefore  $\|Ta - T_\varepsilon a\| \leq \alpha + \beta \leq (\|Ta\| + 2\|a\|)\varepsilon$  for all  $a \in A$ , so that  $\|T - T_\varepsilon\| \leq (\|T\| + 2)\varepsilon$ . Since  $T_\varepsilon$  is compact and  $\varepsilon$  is arbitrary,  $T$  is also compact and (I) is proved.

(II)  $\Phi(Z_c(M(A))) \subset I_c$ . Let  $f \in \Phi(Z_c(M(A)))$  and  $T \in Z_c(M(A))$  with  $f = \Phi(T)$ . Suppose that  $f \notin I_c$ , so that there exists a non-empty closed compact subset  $K$  in  $\text{supp}(f)$  such that  $A/I_K$  is infinite dimensional. Then there exist elements  $a_n$  in  $A$  such that  $\|a_n + I_K\| = 1$  ( $n = 1, 2, \dots$ ) and  $\|(a_n + I_K) - (a_m + I_K)\| \geq 1/2$  ( $n \neq m$ ). We can assume that  $\|a_n\| \leq 2$  ( $n = 1, 2, \dots$ ). Set

$$\delta = \inf \{ |f(P)| : P \in K \}.$$

Then  $\delta > 0$  since  $K$  is compact and we have

$$\begin{aligned} \|Ta_n - Ta_m\| &\geq \sup \{ |f(P)| \|(a_n - a_m) + P\| : P \in K \} \\ &\geq \sup \{ \|(a_n - a_m) + P\| \delta : P \in K \} \\ &= \|(a_n + I_K) - (a_m + I_K)\| \delta \\ &\geq \delta/2 \end{aligned}$$

for all distinct numbers  $n, m$ . Then  $\{Ta_n\}$  contains no convergent subsequence. But this is impossible since  $T$  is compact and (II) is proved.

(III)  $\Phi(Z_c(M(A))) \subset C_0(\text{Prim } A)$ . Let  $T \in Z_c(M(A))$  and  $\varepsilon > 0$ . Set

$$f = \Phi(T) \text{ and } K_\varepsilon = \{P \in \text{Prim } A : |f(P)| \geq \varepsilon\}.$$

We only show that  $K_\varepsilon$  is compact. Let  $I_{K_\varepsilon}$  be a closed two-sided ideal of  $A$  with  $\text{Prim}(A/I_{K_\varepsilon}) \simeq K_\varepsilon$ , as be stated above. Suppose that  $A/I_{K_\varepsilon}$  is infinite dimensional. Then, as in the proof of (II), there exist elements  $a_n$  in  $A$  such that  $\|a_n\| \leq 2$ ,  $\|a_n + I_{K_\varepsilon}\| = 1$  ( $n = 1, 2, \dots$ ) and  $\|(a_n + I_{K_\varepsilon}) - (a_m + I_{K_\varepsilon})\| \geq 1/2$  ( $n \neq m$ ). By the same computation in the proof of (II), we have  $\|Ta_n - Ta_m\| \geq \varepsilon/2$ , so that  $\{Ta_n\}$  contains no convergent subsequence, which contradicts  $T$  is compact. Thus  $A/I_{K_\varepsilon}$  is a finite dimensional  $C^*$ -algebra. Then  $A/I_{K_\varepsilon}$  can be canonically identified with its enveloping von Neumann algebra. Suppose that  $\text{Prim}(A/I_{K_\varepsilon})$  contains an infinite countable subset  $\{P_1, P_2, \dots\}$ . Let  $\pi_i$  be a nonzero irreducible representation of  $A/I_{K_\varepsilon}$  with  $P_i = \text{Ker } \pi_i$  and  $\xi_i$  a norm one element in the Hilbert space associated with  $\pi_i$  for each  $i$ . Set

$$f_i(x + I_{K_\varepsilon}) = (\pi_i(x + I_{K_\varepsilon})\xi_i | \xi_i) \quad (i = 1, 2, \dots)$$

for each  $x + I_{K_\varepsilon} \in A/I_{K_\varepsilon}$ . Since  $\pi_i \neq \pi_j$  ( $i \neq j$ ), it follows that  $\|f_i - f_j\| = 2$  ( $i \neq j$ ) (cf. [4], 2.12.1). Let  $p_i$  denote the support of  $f_i$  for each  $i$ . Then  $\{p_i\}$  are mutually orthogonal (cf. [4], 12.3.1). But this is impossible since each  $p_i$  is an element in  $A/I_{K_\varepsilon}$  and so

$\text{Prim}(A/I_{K_\varepsilon})$  is finite set. Then  $K_\varepsilon$  is also a finite set, so that it is compact and (III) is proved.

We will next show that a result of Rowlands ([7], Theorem 2) is a special case of Theorem 1. Let  $\Omega(A)$  be the space of minimal closed two-sided ideals of  $A$  with its discrete topology, in case  $A$  is dual. Let  $\{I_\lambda; \lambda \in A\}$  be the family of all minimal closed two-sided ideals of  $A$  and  $A_0 = \{\lambda \in A: I_\lambda \text{ is infinite dimensional}\}$ . Let  $I_0$  be the set of all functions  $f$  in the algebra  $C^b(\Omega(A))$  of all bounded complex-valued functions on  $\Omega(A)$  such that  $f(I_\lambda) = 0$  for all  $\lambda \in A_0$ ; if  $A_0 = \emptyset$ , let  $I_0 = C^b(\Omega(A))$ . Let  $C_0(\Omega(A))$  be the subalgebra of  $C^b(\Omega(A))$  which consists of functions vanishing at infinity.

**COROLLARY 3** ([7], Theorem 2). *If  $A$  is a dual  $C^*$ -algebra, then  $Z_C(M(A))$  is isometrically  $*$ -isomorphic to  $I_0 \cap C_0(\Omega(A))$ .*

*Proof.* By ([4], 10.10.6),  $\text{Prim } A$  is discrete. For each  $P \in \text{Prim } A$ , we define a function  $\delta_P$  on  $\text{Prim } A$  by the equation:  $\delta_P(P) = 1$  and  $\delta_P(Q) = 0$  if  $Q \neq P$ , and set  $\mu(P) = \Phi^{-1}(\delta_P)(A)$ . Then we can easily see that  $P \rightarrow \mu(P)$  is a bijection of  $\text{Prim } A$  onto  $\Omega(A)$ . Let  $\mu^*$  be the dual map of  $\mu$ . Then  $\mu^*$  is a isometric  $*$ -isomorphism of  $C^b(\Omega(A))$  onto  $C^b(\text{Prim } A)$ . By the definitions of  $I_C$  and  $I_0$ , we see that  $\mu^*(I_0 \cap C_0(\Omega(A))) = I_{C_0}$ . Set  $\Psi(T) = (\mu^*)^{-1}(\Phi(T))$  for each  $T \in Z_C(M(A))$ . Then  $\Psi(Z_C(M(A))) = I_0 \cap C_0(\Omega(A))$  by Theorem 1 and the corollary is proved.

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