A CHARACTERIZATION OF $PSp(2m, q)$ AND $P\Omega(2m + 1, q)$ AS RANK 3 PERMUTATION GROUPS

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This paper characterizes the projective symplectic groups $PSp(2m, q)$ and the projective orthogonal groups $PΩ(2m+1, q)$ as the only transitive rank 3 permutation groups $G$ of a set $X$ for which the pointwise stabilizer of $G$ has orbit lengths $1, q(q^{2m-2} - 1)/(q-1)$ and $q^{2m-1}$ under a relatively weak hypothesis about the pointwise stabilizer of a certain subset of $X$. A precise statement is

**Theorem.** Let $G$ be a transitive rank 3 group of permutations of a set $X$ such that the orbit lengths for the pointwise stabilizer are $1, q(q^{-2} - 1)/(q-1)$ and $q^{-1}$ for integers $q>1$ and $r>4$. Let $x^i$ denote the union of the orbits of length 1 and $q(q^{-2} - 1)/(q-1)$. Let $R(xy)$ denote $\{z^i : x, y \in z^i\}$. Assume $R(xy) \neq \{x, y\}$ for $y \in x^i - \{x\}$. Assume that the pointwise stabilizer of $x^i \cap y^j$ for $y \in x^i$ does not fix $R(xy)$ pointwise. Then $r$ is even, $q$ is a prime power and $G$ is isomorphic to either a group of symplectic collineations of projective $(r-1)$ space over $GF(q)$ containing $PSp(r, q)$ or a group of orthogonal collineations of projective $r$ space over $GF(q)$ containing $PΩ(r+1, q)$.

1. Introduction. The projective classical groups of symplectic type $PSp(2m, q)$ for $m \geq 2$ are transitive permutation groups of rank 3 when considered as groups of permutations of the absolute points of the corresponding projective space. Indeed the pointwise stabilizer of $PSp(2m, q)$ has 3 orbits of lengths $1, q(q^{2m-2} - 1)/(q-1)$ and $q^{2m-1}$. In a recent paper [7], the author characterized the symplectic groups $PSp(2m, q)$ for $m \geq 3$ as rank 3 permutation groups.

**Theorem A.** Let $G$ be a transitive rank 3 group of permutations of a set $X$ such that $G_x$, the stabilizer of a point $x \in X$, has orbit lengths $1, q(q^{-2} - 1)/(q-1)$ and $q^{-1}$ for integers $q \geq 2$ and $r \geq 5$. Let $x^i$ denote the union of the $G_x$-orbits of lengths 1 and $q(q^{-2} - 1)/(q-1)$. Let $R(xy)$ denote $\{z^i : x, y \in z^i\}$. Assume $R(xy) \neq \{x, y\}$. Assume that the pointwise stabilizer of $x^i$ is transitive on the points unequal to $x$ of $R(xy)$ for $y \in x^i$. Then $r$ is even, $q$ is a prime power and $G$ is isomorphic to a group of symplectic collineations of projective $(r-1)$ space over the field of $q$ elements, which contains $PSp(r, q)$.
We note that the orthogonal group $PΩ(2m + 1, q)$ for $m \geq 2$ acts on the singular points of the orthogonal geometry of a projective $2m$-space over the field of $q$ elements as a rank 3 permutation group in which its pointwise stabilizer has the same orbit lengths of 1, $q(q^{2m-2} - 1)/(q - 1)$ and $q^{2m-1}$ as $PSp(2m, q)$ in its action on the absolute points of the symplectic geometry. In the proof of Theorem A, the possibility that $G$ was an orthogonal group was eliminated because of the hypothesis that a hyperbolic line $R(xy)$ for $y \notin x^\perp$ carried at least 3 points. It seems reasonable to expect that with a change of hypothesis a characterization of the rank 3 groups $G$ in which the pointwise stabilizer has orbit lengths 1, $q(q^{r-2} - 1)/(q - 1)$ and $q^{2r-1}$ is possible and that these groups will be subgroups of the collineation groups of the symplectic geometry or of the orthogonal geometry. We establish a result of this nature in the following form.

**Theorem B.** Let $G$ be a transitive rank 3 group of permutations of a set $X$ such that the orbit lengths for $G_x$, the stabilizer of a point $x$ in $X$, are 1, $q(q^{r-2} - 1)/(q - 1)$ and $q^{r-1}$ for integers $q > 1$ and $r > 4$. Let $x^\perp$ denote the union of the $G_x$-orbits of length 1 and $q(q^{r-2} - 1)/(q - 1)$. Let $R(xy)$ denote $\bigcap\{z^\perp : x, y \in z^\perp\}$. Assume $R(xy)$ $\neq \{x, y\}$ for $y \in x^\perp - \{x\}$. Assume that the pointwise stabilizer of $x^\perp \cap y^\perp$ for $y \notin x^\perp$ does not fix $R(xy)$ pointwise. Then $r$ is even, $q$ is a prime power and $G \cong H$ where either $H$ is a group of symplectic collineations of projective $(r-1)$ space over $GF(q)$ such that $H \cong PSp(r, q)$ or $H$ is a group of orthogonal collineations of projective $r$ space over $GF(q)$ such that $H \cong PΩ(r + 1, q)$.

The proof of Theorem B actually yields the following corollary which distinguishes between the two cases.

**Corollary.** Assume the hypotheses of Theorem B.

(i) Assume that the pointwise stabilizer of $x^\perp$ is nontrivial. Then $r$ is even, $q$ is a prime power and $G \cong H$ where $H$ is a group of symplectic collineations of projective $(r-1)$ space over $GF(q)$ such that $H \cong PSp(r, q)$.

(ii) Assume that the pointwise stabilizer of $x^\perp$ is trivial and that the pointwise stabilizer of $x^\perp \cap y^\perp$ for $y \notin x^\perp$ does not fix $R(xy)$ pointwise. Then $r$ is even, $q$ is a prime power and $G \cong H$ where $H$ is a group of orthogonal collineations of projective $r$ space over $GF(q)$ such that $H \cong PΩ(r + 1, q)$.

Note that Corollary B(i) is a stronger result than Theorem A. We consider this paper a continuation of [7] and note that the
proof of Theorem B is similar to that of Theorem A. In § 2 we will prove Theorem B. At times we will refer the reader to [7] for the proofs of several statements. There are other characterizations of the rank 3 classical groups, due to D. Higman, W. Kantor and D. Perin [3, 4, 5].

2. The proof of Theorem B. In this section assume that $G$ is a rank 3 permutation group on $X$ which satisfies the hypotheses of Theorem B. Let $D(b)$ denote the $G_b$-orbit of length $q(q^r - 2)/ (q - 1)$ and let $C(b)$ denote the $G_b$-orbit of length $q^r - 1$. Let $v_r$ denote $(q^r - 1)/(q - 1)$.

**Lemma 2.1.** (i) $G$ is primitive of even order.
(ii) $\mu = \lambda + 2 = v_{r-2}$.
(iii) $a^b \cap b^b \neq R(ab)$ for $b \in D(a)$.

Note that 2.1 (iii) eliminates problems with generalized quadrangles.

**Lemma 2.2.** (i) $|a^b \cap C(b)| = q^{r-2}$ for $b \in D(a)$.
(ii) $G_{ab}$ is transitive on the points of $a^b \cap C(b)$ for $b \in D(a)$.

For the proofs, see Lemmas 3.1 and 3.2 of [7].

**Notation.** If $\{x_1, x_2, \ldots, x_i\}$ is a set of $i \geq 2$ distinct points of $X$, then let $R(x_1, x_2, \ldots, x_i)$ denote

$$\cap \{z^i: x_1, x_2, \ldots, x_i \in z^i \text{ for } z \in X\} = R(x_1, x_2, \ldots, x_i).$$

**Lemma 2.3.** (i) $y \in R(x_1, x_2, \ldots, x_i)$ if and only if $y^l = \cap \{x_j^i: 1 \leq j \leq i\}$.
(ii) $g(R(x_1, x_2, \ldots, x_i)) = R(g(x_1), g(x_2), \ldots, g(x_i))$ for $g \in G$.
(iii) $R(x_1, x_2, \ldots, x_i) = R(y_1, y_2, \ldots, y_i)$ if and only if

$$\cap \{x_j^i: 1 \leq j \leq i\} = \cap \{y_j^i: 1 \leq j \leq i\}.$$

**Remark.** This lemma is valid for any permutation group $G$ on $X$ and for any self-paired orbit $D(x)$ of $G_x$ where $x^i = \{x\} \cup D(x)$.

**Proof.** In the proof the intersections are taken from $j = 1$ to $i$.
(i) Assume $y \in R(x_1, x_2, \ldots, x_i)$. Let $w \in \cap x_j^i$. Then $x_1, x_2, \ldots, x_i \in w^i$ by Lemma 2.1 (vi) of [7]. Since $y \in R(x_1, x_2, \ldots, x_i)$ and $R(x_1, x_2, \ldots, x_i) \subseteq w^i$, it follows that $y \in w^i$ and $w \in y^i$.

Conversely assume $y^l = \cap x_j^i$. Let $x_1, x_2, \ldots, x_i \in w^i$. Then $w \in \cap x_j^i \subseteq y^i$. So $y \in w^i$ and $y \in R(x_1, x_2, \ldots, x_i)$. 
By (i) \( z \in R(g(x_i)g(x_2) \cdots g(x_i)) \) iff \( z^\perp \supseteq \cap g(x_j)^\perp \) iff \( (g^{-1}(z))^\perp \supseteq \cap j \) iff \( g^{-1}(z) \in R(x_i, x_2, \cdots x_i) \) iff \( z \in g(R(x_i, x_2, \cdots x_i)) \).

(iii) Assume \( R(x_i, x_2, \cdots x_i) = R(y_i, y_2, \cdots y_i) \). For \( 1 \leq j \leq i \), \( x_j \in R(y_i, y_2, \cdots y_i) \). By (i) \( x_j^\perp \supseteq \cap y_j^\perp \) for \( 1 \leq j \leq i \). So \( \cap x_j^\perp \supseteq \cap y_j^\perp \). It follows that \( \cap x_j^\perp = \cap y_j^\perp \).

Conversely assume \( \cap x_j^\perp = \cap y_j^\perp \). Then \( z \in R(x_i, x_2, \cdots x_i) \) iff \( x_j^\perp \supseteq \cap x_j^\perp = \cap y_j^\perp \) iff \( z \in R(y_i, y_2, \cdots y_i) \). This completes the proof of the lemma.

**Definition.** A \( 1 \)-clique is a set \( \{x\} \) for \( x \in X \).

For \( i \geq 2 \), an \( i \)-clique is a set \( \{x_1, x_2, \cdots, x_i\} \) of points of \( X \) such that \( \{x_1, x_2, \cdots, x_{i-1}\} \) is an \( (i-1) \)-clique, \( x_i \in D(x_j) \) for \( 1 \leq j \leq i-1 \) and \( x_i \in R(x_1, x_2, \cdots x_{i-1}) \) where \( R(x_1) = \{x_1\} \).

If \( \{x_1, x_2, \cdots, x_i\} \) is an \( i \)-clique, then we will call \( R(x_1, x_2, \cdots x_i) \) an “\( i \)-space.”

Note that a “2-space” is a totally singular line of [2] and a “3-space” is a “plane” of [7]. Eventually an “\( i \)-space” will correspond to a totally singular subspace generated by \( i \) linearly independent singular points of a classical geometry.

**Notation.** Let \( T(xy) \) denote the pointwise stabilizer in \( G \) of \( x^\perp \cap y^\perp \) for \( y \in C(x) \). Thus

\[
T(xy) = \cap \{G_z: z \in x^\perp \cap y^\perp\}.
\]

**Proposition 2.4.** \( T(xy) \subseteq G_{R(xy)} \) and \( T(xy) \) is transitive on the points of \( R(xy) \) for \( y \in x^\perp \).

**Proof.** First we prove that \( G_{R(xy)} \) is primitive on the points of \( R(xy) \). Indeed if \( |R(xy)| > 2 \), then \( G_{R(xy)} \) is 2-transitive on the points of \( R(xy) \) by a lemma in [2]. If \( R(xy) = \{x, y\} \), then \( |G: G_{R(xy)}| = nl/2 \) if \( y \in x^\perp \) and \( |G: G_{xy}| = nl \). Therefore \( |G_{R(xy)}: G_{R(xy)}| = 2 \) because \( G_{R(xy)} = G_{xy} \).

If \( g \in G_{R(xy)} \), then

\[
g(R(xy)) = R(g(x)g(y)) = R(xy)
\]

and

\[
g(x)^\perp \cap g(y)^\perp = x^\perp \cap y^\perp
\]

by Lemma 2.3. But

\[
T(xy)^g = \cap \{G_z: z \in x^\perp \cap y^\perp\} = T(g(x)g(y))
\]
and so $T(xy)^g = T(xy)$. Therefore $T(xy)$ is a normal subgroup of the primitive group $G_{R(xy)}$. Since $T(xy)$ does not fix $R(xy)$ pointwise by hypothesis of the theorem, it follows that $T(xy)$ is transitive on the points of $R(xy)$.

Note that $G_{R(xy)}$ is a doubly transitive group on the points of $R(xy)$ and has a normal subgroup $I(xy)$. By familiar classification theorems not needed here, $|R(xy)| - 1$ is usually a prime power.

Note that if $T(x)$, the pointwise stabilizer of $x^+$, is nontrivial, then $T(xy)$ does not fix $R(xy)$ pointwise for $y \in x^+$ because $T(x)$ is semiregular off $x^+$ by a lemma in [2] and $T(x) \leq T(xy)$.

Denote the group generated by $T(xy)$ for all $x, y \in X$ with $y \in C(x)$ simply as $K$. Thus

$$K = \langle T(xy) : x, y \in X, y \in C(x) \rangle .$$

**Proposition 2.5.** (i) If $\{x_1, x_2, \ldots, x_i\}$ is a set of $i$ distinct points of $X$, then $K_{x_1 x_2 \ldots x_i}$ is transitive on the points of $\cap \{x_j^+ : 1 \leq j \leq i\} - R(x_1 x_2 \ldots x_i)$.

(ii) $K$ is transitive on $i$-cliques.

**Proof.** (i) In the proof the intersections are taken from $j = 1$ to $i$. Let $c$ and $h$ be distinct points of $\cap x_j^+ - R(x_1 x_2 \ldots x_i)$. Either $c \in C(h)$ or $c \in D(h)$. If $c \in C(h)$, then $R(ch)$ is a hyperbolic line in $\cap x_j^+$. Since $|G|$ is even, $x_1, x_2, \ldots, x_i \in c^+ \cap h^+$ and so $T(ch)$ fixes $x_1, x_2, \ldots, x_i$. By Proposition 2.4, there exists $t \in T(ch) \leq K_{x_1 x_2 \ldots x_i}$ such that $t(c) = h$.

Assume now that $c \in D(h)$. Since $c, h \in R(x_1 x_2 \ldots x_i)$, there exists by Lemma 2.3 (i) $u \in \cap x_j^+ \cap C(c)$ and $v \in \cap x_j^+ \cap C(h)$. There are 3 possible cases to consider:

1. $u \in C(h), (2) v \in C(c) \cap C(h)$ and $v \in D(c)$.

(1) If $u \in \cap x_j^+ \cap C(c) \cap C(h)$, then $R(cu)$ is a hyperbolic line in $\cap x_j^+$. By Proposition 2.4, there exists $t \in T(cu) \leq K_{x_1 x_2 \ldots x_i}$ such that $t(c) = u$. The line $R(uh)$ is hyperbolic and lies in $\cap x_j^+$. By Proposition 2.4, there exists $s \in T(uh) \leq K_{x_1 x_2 \ldots x_i}$ such that $s(u) = h$. Thus $st(c) = h$ and $st \in K_{x_1 x_2 \ldots x_i}$.

(2) If $v \in \cap x_j^+ \cap C(c) \cap C(h)$, then a proof similar to that of case (1) yields the desired result.

(3) $u \in \cap x_j^+ \cap C(c) \cap D(h)$ and $v \in \cap x_j^+ \cap D(c) \cap C(h)$. Since $c \in D(h)$, there exists $w \in R(ch) - \{c, h\}$ because by hypothesis $|R(ch)| > 2$. Note $w \in C(u)$, for if $w \in u^+$, then $c \in R(ch) = R(wh) \subseteq u^+$, a contradiction in case (3). Now $w \in R(ch) \subseteq x_j^+$. But $w \in R(x_1 x_2 \ldots x_i)$ because $u \in \cap x_j^+ \cap C(u)$. So $u \in \cap x_j^+ \cap C(c) \cap C(w)$. By case (1) there exists $t \in K_{x_1 x_2 \ldots x_i}$ such that $t(c) = w$. Note $w \in C(v)$, for if $w \in v^+$, then $h \in R(ch) = R(wh) \subseteq v^+$, a contradiction. Now $v \in \cap x_j^+$
\( C(w) \cap C(h) \). By case (1) there exists \( s \in K_{x_1 x_2 \cdots x_i} \) such that \( s(w) = h \).

So \( st(c) = h \) and \( st \in K_{x_1 x_2 \cdots x_i} \).

(ii) Let \( \{x_1, x_2, \cdots, x_i\} \) and \( \{y_1, y_2, \cdots, y_i\} \) be 2 \( i \)-cliques. The proof is by induction on \( i \). First note that \( K \) is transitive on \( X \) because \( K \) is a normal subgroup of the primitive group \( G \). If \( i = 1 \), then there exists \( k \in K \) such that \( k(x_1) = y_1 \). Assume \( i > 1 \). By the induction assumption there exists \( g \in K \) such that \( g(x_j) = y_j \) for \( j = 1, 2, \cdots, i - 1 \). From Lemma 2.3 (ii) and the definition of \( i \)-clique, it follows that \( \{y_1, y_2, \cdots, y_{i-1}, g(x_i)\} \) is an \( i \)-clique because \( \{x_1, x_2, \cdots, x_{i-1}, x_i\} \) is an \( i \)-clique. Since

\[ g(x_i), y_i \in \cap \{y_j: 1 \leq j \leq i - 1\} = R(y_1 y_2 \cdots y_{i-1}) , \]

by (i) there is \( h \in K_{x_1 y_2 \cdots y_{i-1}} \) such that \( h(g(x_i)) = y_i \). Thus \( hg(x_j) = y_j \) for \( j = 1, 2, \cdots, i \). This completes the proof of the proposition.

Note that 3-cliques exist by Lemma 2.1 (iii).

**Proposition 2.6.** \( G_a \) is a rank 3 permutation group on the set of totally singular lines through \( a \). For \( b \in D(a) \), \( G_{a \cap R(ab)} \) has nontrivial orbits

\[ \{R(ac): c \in a \perp \cap b^\perp = R(ab)\} \]

and

\[ \{R(ac): c \in a \perp \cap C(b)\} \].

The proof is similar to that of Proposition 3.4 of [7]. This proposition follows from Lemmas 2.2 and 2.3 and Proposition 2.5 (i) for \( i = 2 \) just as Proposition 3.4 of [7] follows from Lemmas 3.2 and 2.2 and Proposition 3.3 of [7].

**Proposition 2.7.** Totally singular lines carry \( q + 1 \) points.

**Proposition 2.8.** If \( b \in D(a) \), the \( X = \cup \{c^\perp: c \in R(ab)\} \).

**Proposition 2.9.** \( X \) together with its totally singular lines forms a nondegenerate Shult space of finite rank \( \geq 3 \) in which lines carry \( q + 1 \) points.

The proofs of the above three statements are identical to the proofs of Propositions 3.5, 3.6, and 3.7 of [7].

**Lemma 2.10.** If \( \{x_1, x_2, \cdots, x_i\} \) is an \( i \)-clique, then \( R(x_1 x_2 \cdots x_i) \) is a Shult subspace of \( X \).
Proof. In the proof the intersections are taken from \( j = 1 \) to \( i \).

Let \( d, e \in R(x, x_2, \ldots, x_i) \). By definition of \( i \)-clique, \( x_k \in \cap x_j^j \) for \( 1 \leq k \leq j \) and so by definition of "\( i \)-space" and by Lemma 2.3 (i) it follows that

\[
d \in R(x, x_2, \ldots, x_i) \subseteq \cap x_j^j \subseteq e^i.
\]

Thus any two points of \( R(x, x_2, \ldots, x_i) \) are adjacent. Let the line \( R(xy) \) meet \( R(x_1, x_2, \ldots, x_i) \) in \( \{u, v\} \). Then \( R(xy) = R(uv) \) and \( x^j \cap y^j = u^j \cap v^j \). If \( z \in R(xy) \), then

\[
z^j \supseteq x^j \cap y^j = u^j \cap v^j \supseteq \cap x_j^j
\]

since \( u, v \in R(x_1, x_2, \ldots, x_i) \) by Lemma 2.3. Thus \( z \in R(x_1, x_2, \ldots, x_i) \) and \( R(xy) \subseteq R(x_1, x_2, \ldots, x_i) \). Therefore \( R(x_1, x_2, \ldots, x_i) \) is a Shult subspace of \( X \), as desired.

**Proposition 2.11.** (i) \( q \) is a prime power and \( r \) is even.

(ii) Either \( X \) is isomorphic to the polar space \( S \) associated with an alternating form \( f \) defined on a protective space \( P \) of dimension \( r - 1 \) over \( GF(q) \) or \( X \) is isomorphic to the polar space \( S \) associated with a symmetric form \( f \) defined on a protective space \( P \) of dimension \( r \) over \( GF(q) \) for \( q \) odd.

For the proof see Proposition 3.9 of [7].

Since \( r \) is even and \( r \geq 5 \), there exists a natural number \( m \geq 3 \) such that \( r = 2m \).

**Proposition 2.12.** (i) \( G \) is isomorphic to a subgroup of \( P\Gamma U(f) \), the group of collineations of \( P \) which preserve the form \( f \).

(ii) For \( x \in X \), \( \phi(x^j) = \{w \in P : f(w, w) = 0, f(w, \phi(x_j)) = 0\} \) where \( \phi : X \to S \) is a polar space isomorphism.

(iii) For an \( i \)-clique, \( |R(x_1, x_2, \ldots, x_i)| = v_i \) and \( \cap \{x_j^j : 1 \leq j \leq i\} = v_{r-i} \).

(iv) \( X \) contains \( m \)-cliques but does not contain \((m + 1)\)-cliques.

**Proof.** For (i) and (ii) see Proposition 3.10 (i) and (ii) of [7].

(iii) From (ii) it follows that

\[
\phi(R(x_1, x_2, \ldots, x_i)) = \cap \{\phi(z^j) : \phi(x_1), \phi(x_2), \ldots, \phi(x_i) \in \phi(z^j)\}
\]

which equals the set of singular points in the intersection of all the hyperplanes containing \( \phi(x_1), \phi(x_2), \ldots, \phi(x_i) \). But this set is the projective subspace generated by \( \phi(x_1), \phi(x_2), \ldots, \phi(x_i) \) since \( \phi(x_k) \perp \phi(x_j) \) for all \( k, j \). Thus \( |R(x_1, x_2, \ldots, x_i)| = v_i \).

From (ii) \( \cap \{x_j^j : 1 \leq j \leq i\} = v_{r-i} \).
(iv) Since \( r = 2m \), (iv) follows from (iii).

Now let \( \{x_1, x_2, \ldots, x_m\} \) be a fixed \( m \)-clique of \( X \). Then
\[
x_1 \subset R(x_1 x_2) \subset R(x_1 x_2 x_3) \subset \cdots \subset R(x_1 x_2 \cdots x_m)
\]
is a chain of Shult subspaces of \( X \) of length \( m \geq 3 \). Define sub-
groups \( K_i \) of \( K \) as follows:
\[
K_1 = K \quad K_i = K_{i-1} \cap K_{R(x_1 x_2 \cdots x_{i-1})} \quad \text{for} \quad 2 \leq i \leq m+1.
\]
Note the choice of the fixed \( i \)-clique is arbitrary since \( K \) is transi-
tive on \( i \)-cliques.

**Proposition 2.13.** (i) \( K_i \) is transitive on the set of “\( i \)-spaces” containing \( R(x_1 x_2 \cdots x_{i-1}) \), for \( 2 \leq i \leq m \).

(ii) \( |K_i| = \prod_{j=1}^{m} v_{x^j} \).

*Proof.* (i) Let \( R(x_1 x_2 \cdots x_{i-1} d) \) and \( R(x_1 x_2 \cdots x_{i-1} e) \) be “\( i \)-spaces” containing \( R(x_1 x_2 \cdots x_{i-1}) \). Then
\[
d, e \in \bigcap_{j=1}^{i-1} x_j^j - R(x_1 x_2 \cdots x_{i-1}),
\]
a set on which \( K_{x_1 x_2 \cdots x_{i-1}} \) is transitive by Proposition 2.5. There
exists \( k \in K_{x_1 x_2 \cdots x_{i-1}} \) such that \( k(d) = e \). By Lemma 2.3 (iii), it follows
that
\[
k(R(x_1 x_2 \cdots x_{i-1} d)) = R(x_1 x_2 \cdots x_{i-1} e)
\]
and that \( k \in K_i \).

(ii) For \( 2 \leq i \leq m \) the number of “\( i \)-spaces” containing \( R(x_1 x_2 \cdots x_{i-1}) \) is
\[
\left| \bigcap_{j=1}^{i-1} x_j^j \right| - |R(x_1 x_2 \cdots x_{i-1})| \left| |R(x_1 x_2 \cdots x_{i-1})| - |R(x_1 x_2 \cdots x_{i-1})| \right|
\]
\[
= (v_{2m-(i-1)} - v_{i-1})(v_i - v_{i-1}) = v_{2m-(i-1)}.
\]
So \( |K_i : K_{i+1}| = v_{2m-(i-1)} \) by (i). Since \( K \) is a normal subgroup of
the primitive group \( G \), \( K \) is transitive and \( |K_i : K_2| = v_{2m} \). Now (ii)
follows.

**Proposition 2.14.** (i) \( \psi(K) \) is a flag-transitive subgroup of \( \text{PGU}(f) \), the group of projective transformations of \( P \) which pre-
serve \( f \).

(ii) If \( X \) is symplectic, then \( \psi(K) \geq \text{PSp}(2m, q) \).
(iii) If $X$ is orthogonal, then $\psi(K) \cong P\Omega(2m + 1, q)$.

Proof. Let $x, y \in X$ with $y \in C(x)$. Since $T(xy)$ is the pointwise stabilizer in $G$ of $x^\perp \cap y^\perp$, it follows that $\psi(T(xy))$ is the pointwise stabilizer in $\psi(G)$ of $\phi(x)^\perp \cap \phi(y)^\perp$. If $t$ is a nontrivial element of $T(xy)$, then $\psi(t) \in P\Gamma U(f)$ and fixes $\phi(x)^\perp \cap \phi(y)^\perp$ pointwise. This implies that $\psi(t) \in PGU(f)$ and so $\psi(K) \leq PGU(f)$.

Now $\psi(K_{m+1})$ fixes the flag
$$\{\phi(x_1), \langle\phi(x_1), \phi(x_2)\rangle, \ldots, \langle\phi(x_1), \phi(x_2), \ldots, \phi(x_m)\rangle\}.$$ If $B$ is the subgroup of $PGU(f)$ which fixes the above flag, then $B$ is a Borel subgroup of $PGU(f)$ and $B \cap \psi(K) = \psi(K_{m+1})$. Therefore by Proposition 2.13 (ii)
$$|B\psi(K)| = |B| \cdot |\psi(K) : \psi(K_{m+1})|$$
$$= q^{m^2}(q - 1)^m \cdot \prod_{i=1}^{m} v_{2i} = |PGU(f)|.$$

Thus $B\psi(K) = PGU(f)$ and $\psi(K)$ is a flag-transitive subgroup of $PGU(f)$. By a theorem of Seitz [6], it follows that
$$\psi(K) \cong PSp(2m, q)$$
if $X$ is symplectic and
$$\psi(K) \cong P\Omega(2m + 1, q)$$
if $X$ is orthogonal, as desired.

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Received July 17, 1975 and in revised form July 9, 1976.

Southern Illinois University
Carbondale, IL 62901
Kazuo Anzai and Shiro Ishikawa, *On common fixed points for several continuous affine mappings* .................................................. 1

Bruce Alan Barnes, *When is a representation of a Banach ∗-algebra Naimark-related to a ∗-representation* ........................................... 5


Donald S. Coram and Paul Frazier Duvall, Jr., *Approximate fibrations and a movability condition for maps* ........................................... 41

Kenneth R. Davidson and Che-Kao Fong, *An operator algebra which is not closed in the Calkin algebra* ........................................... 57

Garret J. Etgen and James Pawlowski, *A comparison theorem and oscillation criteria for second order differential systems* ............... 59

Philip Palmer Green, *C*-algebras of transformation groups with smooth orbit space ................................................................. 71

Charles Allen Jones and Charles Dwight Lahr, *Weak and norm approximate identities are different* .................................................... 99

G. K. Kalisch, *On integral representations of piecewise holomorphic functions* ................................................................. 105

Y. Kodama, *On product of shape and a question of Sher* ...................... 115

Heinz K. Langer and B. Textorius, *On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space* ................................................................. 135

Albert Edward Livingston, *On the integral means of univalent, meromorphic functions* ................................................................. 167

Wallace Smith Martindale, III and Susan Montgomery, *Fixed elements of Jordan automorphisms of associative rings* ....................... 181

R. Kent Nagle, *Monotonicity and alternative methods for nonlinear boundary value problems* ................................................................. 197

Richard John O’Malley, *Approximately differentiable functions: the r topology* ................................................................. 207

Mangesh Bhalchandra Rege and Kalathoor Varadarajan, *Chain conditions and pure-exactness* ................................................................. 223

Christine Ann Shannon, *The second dual of C(X)* ................................. 237

Sin-ei Takahasi, *A characterization for compact central double centralizers of C*-algebras* ................................................................. 255

Theresa Phillips Vaughan, *A note on the Jacobi-Perron algorithm* ............ 261

Arthur Anthony Yanushka, *A characterization of PSp(2m, q) and PΩ(2m + 1, q) as rank 3 permutation groups* ................................................................. 273