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# A CHARACTERIZATION OF $PSp(2m, q)$ AND $P\Omega(2m+1, q)$ AS RANK 3 PERMUTATION GROUPS

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**This paper characterizes the projective symplectic groups $PSp(2m, q)$ and the projective orthogonal groups $P\Omega(2m+1, q)$ as the only transitive rank 3 permutation groups $G$ of a set $X$ for which the pointwise stabilizer of $G$ has orbit lengths 1, $q(q^{2m-2}-1)/(q-1)$ and $q^{2m-1}$ under a relatively weak hypothesis about the pointwise stabilizer of a certain subset of $X$. A precise statement is**

THEOREM. **Let $G$ be a transitive rank 3 group of permutations of a set $X$ such that the orbit lengths for the pointwise stabilizer are 1, $q(q^{r-2}-1)/(q-1)$ and $q^{r-1}$ for integers $q>1$ and $r>4$. Let $x^\perp$ denote the union of the orbits of length 1 and $q(q^{r-2}-1)/(q-1)$. Let $R(xy)$ denote $\cap\{z^\perp: x, y \in z^\perp\}$. Assume $R(xy) \neq \{x, y\}$ for $y \in x^\perp - \{x\}$. Assume that the pointwise stabilizer of $x^\perp \cap y^\perp$ for $y \notin x^\perp$ does not fix $R(xy)$ pointwise. Then $r$ is even, $q$ is a prime power and $G$ is isomorphic to either a group of symplectic collineations of projective $(r-1)$ space over $GF(q)$ containing $PSp(r, q)$ or a group of orthogonal collineations of projective $r$ space over $GF(q)$ containing $P\Omega(r+1, q)$.**

## 1. Introduction.

The projective classical groups of symplectic type $PSp(2m, q)$ for $m \geqq 2$ are transitive permutation groups of rank 3 when considered as groups of permutations of the absolute points of the corresponding projective space. Indeed the pointwise stabilizer of $PSp(2m, q)$ has 3 orbits of lengths 1, $q(q^{2m-2} - 1)/(q - 1)$ and $q^{2m-1}$. In a recent paper [7], the author characterized the symplectic groups $PSp(2m, q)$ for $m \geqq 3$ as rank 3 permutation groups.

THEOREM A. *Let $G$ be a transitive rank 3 group of permutations of a set $X$ such that $G_x$, the stabilizer of a point $x \in X$, has orbit lengths 1, $q(q^{r-2} - 1)/(q - 1)$ and $q^{r-1}$ for integers $q \geqq 2$ and $r \geqq 5$. Let $x^\perp$ denote the union of the $G_x$-orbits of lengths 1 and $q(q^{r-2} - 1)/(q - 1)$. Let $R(xy)$ denote $\cap\{z^\perp: x, y \in z^\perp\}$. Assume $R(xy) \neq \{x, y\}$. Assume that the pointwise stabilizer of $x^\perp$ is transitive on the points unequal to $x$ of $R(xy)$ for $y \notin x^\perp$. Then $r$ is even, $q$ is a prime power and $G$ is isomorphic to a group of symplectic collineations of projective $(r - 1)$ space over the field of $q$ elements, which contains $PSp(r, q)$.*

We note that the orthogonal group $P\Omega(2m+1, q)$ for $m \geqq 2$ acts on the singular points of the orthogonal geometry of a projective $2m$-space over the field of $q$ elements as a rank 3 permutation group in which its pointwise stabilizer has the same orbit lengths of 1, $q(q^{2m-2}-1)/(q-1)$ and $q^{2m-1}$ as $PSp(2m, q)$ in its action on the absolute points of the symplectic geometry. In the proof of Theorem A, the possibility that $G$ was an orthogonal group was eliminated because of the hypothesis that a hyperbolic line $R(xy)$ for $y \notin x^{\perp}$ carried at least 3 points. It seems reasonable to expect that with a change of hypothesis a characterization of the rank 3 groups $G$ in which the pointwise stabilizer has orbit lengths 1, $q(q^{r-2}-1)/(q-1)$ and $q^{2r-1}$ is possible and that these groups will be subgroups of the collineation groups of the symplectic geometry or of the orthogonal geometry. We establish a result of this nature in the following form.

THEOREM B. *Let $G$ be a transitive rank 3 group of permutations of a set $X$ such that the orbit lengths for $G_x$, the stabilizer of a point $x$ in $X$, are 1, $q(q^{r-2}-1)/(q-1)$ and $q^{r-1}$ for integers $q>1$ and $r>4$. Let $x^{\perp}$ denote the union of the $G_x$-orbits of length 1 and $q(q^{r-2}-1)/(q-1)$. Let $R(xy)$ denote $\cap\{z^{\perp}: x, y \in z^{\perp}\}$. Assume $R(xy) \neq \{x, y\}$ for $y \in x^{\perp}-\{x\}$. Assume that the pointwise stabilizer of $x^{\perp} \cap y^{\perp}$ for $y \notin x^{\perp}$ does not fix $R(xy)$ pointwise. Then $r$ is even, $q$ is a prime power and $G \cong H$ where either $H$ is a group of symplectic collineations of projective $(r-1)$ space over $GF(q)$ such that $H \trianglerighteq PSp(r, q)$ or $H$ is a group of orthogonal collineations of projective $r$ space over $GF(q)$ such that $H \trianglerighteq P\Omega(r+1, q)$.*

The proof of Theorem B actually yields the following corollary which distinguishes between the two cases.

COROLLARY. *Assume the hypotheses of Theorem* B.

(i) *Assume that the pointwise stabilizer of $x^{\perp}$ is nontrivial. Then $r$ is even, $q$ is a prime power and $G \cong H$ where $H$ is a group of symplectic collineations of projective $(r-1)$ space over $GF(q)$ such that $H \trianglerighteq PSp(r, q)$.*

(ii) *Assume that the pointwise stabilizer of $x^{\perp}$ is trivial and that the pointwise stabilizer of $x^{\perp} \cap y^{\perp}$ for $y \notin x^{\perp}$ does not fix $R(xy)$ pointwise. Then $r$ is even, $q$ is a prime power and $G \cong H$ where $H$ is a group of orthogonal collineations of projective $r$ space over $GF(q)$ such that $H \trianglerighteq P\Omega(r+1, q)$.*

Note that Corollary B(i) is a stronger result than Theorem A. We consider this paper a continuation of [7] and note that the proof of Theorem B is similar to that of Theorem A. In § 2 we will prove Theorem B. At times we will refer the reader to [7] for the proofs of several statements. There are other characterizations of the rank 3 classical groups, due to D. Higman, W. Kantor and D. Perin [3, 4, 5].

**2. The proof of Theorem B.** In this section assume that $G$ is a rank 3 permutation group on $X$ which satisfies the hypotheses of Theorem B. Let $D(b)$ denote the $G_b$-orbit of length $q(q^{r-2}-1)/(q-1)$ and let $C(b)$ denote the $G_b$-orbit of length $q^{r-1}$. Let $v_r$ denote $(q^r-1)/(q-1)$.

LEMMA 2.1. (i) *$G$ is primitive of even order.*
(ii) $\mu = \lambda + 2 = v_{r-2}$.
(iii) $a^\perp \cap b^\perp \neq R(ab)$ *for* $b \in D(a)$.

Note that 2.1 (iii) eliminates problems with generalized quadrangles.

LEMMA 2.2. (i) $|a^\perp \cap C(b)| = q^{r-2}$ *for* $b \in D(a)$.
(ii) $G_{ab}$ *is transitive on the points of* $a^\perp \cap C(b)$ *for* $b \in D(a)$.

For the proofs, see Lemmas 3.1 and 3.2 of [7].

NOTATION. If $\{x_1, x_2, \cdots, x_i\}$ is a set of $i \geqq 2$ distinct points of $X$, then let $R(x_1, x_2, \cdots, x_i)$ denote

$$\cap \{z^\perp : x_1, x_2, \cdots, x_i \in z^\perp \quad \text{for} \quad z \in X\} = R(x_1, x_2, \cdots, x_i) .$$

LEMMA 2.3. (i) $y \in R(x_1x_2\cdots x_i)$ *if and only if* $y^\perp \supseteqq \cap \{x_j^\perp : 1 \leqq j \leqq i\}$.
(ii) $g(R(x_1x_2\cdots x_i)) = R(g(x_1)g(x_2)\cdots g(x_i))$ *for* $g \in G$.
(iii) $R(x_1x_2\cdots x_i) = R(y_1y_2\cdots y_i)$ *if and only if*

$$\cap \{x_j^\perp : 1 \leqq j \leqq i\} = \cap \{y_j^\perp : 1 \leqq j \leqq i\} .$$

REMARK. This lemma is valid for any permutation group $G$ on $X$ and for any self-paired orbit $D(x)$ of $G_x$ where $x^\perp = \{x\} \cup D(x)$.

*Proof.* In the proof the intersections are taken from $j=1$ to $i$.
(i) Assume $y \in R(x_1x_2\cdots x_i)$. Let $w \in \cap x_j^\perp$. Then $x_1, x_2, \cdots, x_i \in w^\perp$ by Lemma 2.1 (vi) of [7]. Since $y \in R(x_1x_2\cdots x_i)$ and $R(x_1x_2\cdots x_i) \subseteqq w^\perp$, it follows that $y \in w^\perp$ and $w \in y^\perp$.

Conversely assume $y^\perp \supseteqq \cap x_j^\perp$. Let $x_1, x_2, \cdots, x_i \in w^\perp$. Then $w \in \cap x_j^\perp \subseteqq y^\perp$. So $y \in w^\perp$ and $y \in R(x_1x_2\cdots x_i)$.

(ii) By (i) $z \in R(g(x_1)g(x_2)\cdots g(x_i))$ iff $z^\perp \supseteqq \cap\, g(x_j)^\perp$ iff $(g^{-1}(z))^\perp \supseteqq \cap\, x_j^\perp$ iff $g^{-1}(z) \in R(x_1x_2\cdots x_i)$ iff $z \in g(R(x_1x_2\cdots x_i))$.

(iii) Assume $R(x_1x_2\cdots x_i) = R(y_1y_2\cdots y_i)$. For $1 \leqq j \leqq i$, $x_j \in R(y_1y_2\cdots y_i)$. By (i) $x_j^\perp \supseteqq \cap\, y_k^\perp$ for $1 \leqq j \leqq i$. So $\cap\, x_j^\perp \supseteqq \cap\, y_k^\perp$. It follows that $\cap\, x_j^\perp = \cap\, y_k^\perp$.

Conversely assume $\cap\, x_j^\perp = \cap\, y_j^\perp$. Then $z \in R(x_1x_2\cdots x_i)$ iff $z^\perp \supseteqq \cap\, x_j^\perp = \cap\, y_j^\perp$ iff $z \in R(y_1y_2\cdots y_i)$. This completes the proof of the lemma.

DEFINITION. A *l-clique* is a set $\{x\}$ for $x \in X$.

For $i \geqq 2$, an *i-clique* is a set $\{x_1, x_2, \cdots, x_i\}$ of points of $X$ such that $\{x_1, x_2, \cdots, x_{i-1}\}$ is an $(i-1)$-clique, $x_i \in D(x_j)$ for $1 \leqq j \leqq i-1$ and $x_i \notin R(x_1x_2\cdots x_{i-1})$ where $R(x_1) = \{x_1\}$.

If $\{x_1, x_2, \cdots, x_i\}$ is an $i$-clique, then we will call $R(x_1x_2\cdots x_i)$ an "*i-space.*"

Note that a "2-space" is a totally singular line of [2] and a "3-space" is a "plane" of [7]. Eventually an "$i$-space" will correspond to a totally singular subspace generated by $i$ linearly independent singular points of a classical geometry.

NOTATION. Let $T(xy)$ denote the pointwise stabilizer in $G$ of $x^\perp \cap y^\perp$ for $y \in C(x)$. Thus

$$T(xy) = \cap \{G_z\colon z \in x^\perp \cap y^\perp\} .$$

PROPOSITION 2.4. *$T(xy) \trianglelefteq G_{R(xy)}$ and $T(xy)$ is transitive on the points of $R(xy)$ for $y \notin x^\perp$.*

*Proof.* First we prove that $G_{R(xy)}$ is primitive on the points of $R(xy)$. Indeed if $|R(xy)| > 2$, then $G_{R(xy)}$ is 2-transitive on the points of $R(xy)$ by a lemma in [2]. If $R(xy) = \{x, y\}$, then $|G\colon G_{R(xy)}| = nl/2$ if $y \notin x^\perp$ and $|G\colon G_{xy}| = nl$. Therefore $|G_{R(xy)}\colon G_{R(xy)x}| = 2$ because $G_{R(xy)x} = G_{xy}$.

If $g \in G_{R(xy)}$, then

$$g(R(xy)) = R(g(x)g(y)) = R(xy)$$

and

$$g(x)^\perp \cap g(y)^\perp = x^\perp \cap y^\perp$$

by Lemma 2.3. But

$$T(xy)^g = \cap \{G_{g(z)}\colon z \in x^\perp \cap y^\perp\} = T(g(x)g(y))$$

and so $T(xy)^g = T(xy)$. Therefore $T(xy)$ is a normal subgroup of the primitive group $G_{R(xy)}$. Since $T(xy)$ does not fix $R(xy)$ pointwise by hypothesis of the theorem, it follows that $T(xy)$ is transitive on the points of $R(xy)$.

Note that $G_{R(xy)}$ is a doubly transitive group on the points of $R(xy)$ and has a normal subgroup $I(xy)$. By familiar classification theorems not needed here, $|R(xy)| - 1$ is usually a prime power.

Note that if $T(x)$, the pointwise stabilizer of $x^\perp$, is nontrivial, then $T(xy)$ does not fix $R(xy)$ pointwise for $y \notin x^\perp$ because $T(x)$ is semiregular off $x^\perp$ by a lemma in [2] and $T(x) \leqq T(xy)$.

Denote the group generated by $T(xy)$ for all $x, y \in X$ with $y \in C(x)$ simply as $K$. Thus

$$K = \langle T(xy) \colon x, y \in X, y \in C(x) \rangle .$$

PROPOSITION 2.5. (i) *If* $\{x_1, x_2, \cdots, x_i\}$ *is a set of* $i$ *distinct points of* $X$, *then* $K_{x_1x_2\cdots x_i}$ *is transitive on the points of* $\cap \{x_j^\perp \colon 1 \leqq j \leqq i\} - R(x_1x_2\cdots x_i)$.

(ii) $K$ *is transitive on* $i$*-cliques.*

*Proof.* (i) In the proof the intersections are taken from $j = 1$ to $i$. Let $c$ and $h$ be distinct points of $\cap x_j^\perp - R(x_1x_2\cdots x_i)$. Either $c \in C(h)$ or $c \in D(h)$. If $c \in C(h)$, then $R(ch)$ is a hyperbolic line in $\cap x_j^\perp$. Since $|G|$ is even, $x_1, x_2, \cdots, x_i \in c^\perp \cap h^\perp$ and so $T(ch)$ fixes $x_1, x_2, \cdots, x_i$. By Proposition 2.4, there exists $t \in T(ch) \leqq K_{x_1x_2\cdots x_i}$ such that $t(c) = h$.

Assume now that $c \in D(h)$. Since $c, h \notin R(x_1x_2\cdots x_i)$, there exists by Lemma 2.3 (i) $u \in \cap x_j^\perp \cap C(c)$ and $v \in \cap x_j^\perp \cap C(h)$. There are 3 possible cases to consider:

(1) $u \in C(h)$, (2) $v \in C(c)$ and (3) $u \in D(h)$ and $v \in D(c)$.

(1) If $u \in \cap x_j^\perp \cap C(c) \cap C(h)$, then $R(cu)$ is a hyperbolic line in $\cap x_j^\perp$. By Proposition 2.4, there exists $t \in T(cu) \leqq K_{x_1x_2\cdots x_i}$ such that $t(c) = u$. The line $R(uh)$ is hyperbolic and lies in $\cap x_j^\perp$. By Proposition 2.4, there exists $s \in T(uh) \leqq K_{x_1x_2\cdots x_i}$ such that $s(u) = h$. Thus $st(c) = h$ and $st \in K_{x_1x_2\cdots x_i}$.

(2) If $v \in \cap x_j^\perp \cap C(c) \cap C(h)$, then a proof similar to that of case (1) yields the desired result.

(3) $u \in \cap x_j^\perp \cap C(c) \cap D(h)$ and $v \in \cap x_j^\perp \cap D(c) \cap C(h)$. Since $c \in D(h)$, there exists $w \in R(ch) - \{c, h\}$ because by hypothesis $|R(ch)| > 2$. Note $w \in C(u)$, for if $w \in u^\perp$, then $c \in R(ch) = R(wh) \subseteqq u^\perp$, a contradiction in case (3). Now $w \in R(ch) \subseteqq \cap x_j^\perp$. But $w \notin R(x_1x_2\cdots x_i)$ because $u \in \cap x_j^\perp \cap C(w)$. So $u \in \cap x_j^\perp \cap C(c) \cap C(w)$. By case (1) there exists $t \in K_{x_1x_2\cdots x_i}$ such that $t(c) = w$. Note $w \in C(v)$, for if $w \in v^\perp$, then $h \in R(ch) = R(wh) \subseteqq v^\perp$, a contradiction. Now $v \in \cap x_j^\perp \cap$

$C(w) \cap C(h)$. By case (1) there exists $s \in K_{x_1 x_2 \cdots x_i}$ such that $s(w) = h$. So $st(c) = h$ and $st \in K_{x_1 x_2 \cdots x_i}$.

(ii) Let $\{x_1, x_2, \cdots, x_i\}$ and $\{y_1, y_2, \cdots, y_i\}$ be 2 $i$-cliques. The proof is by induction on $i$. First note that $K$ is transitive on $X$ because $K$ is a normal subgroup of the primitive group $G$. If $i=1$, then there exists $k \in K$ such that $k(x_1) = y_1$. Assume $i > 1$. By the induction assumption there exists $g \in K$ such that $g(x_j) = y_j$ for $j = 1, 2, \cdots, i-1$. From Lemma 2.3 (ii) and the definition of $i$-clique, it follows that $\{y_1, y_2, \cdots, y_{i-1}, g(x_i)\}$ is an $i$-clique because $\{x_1, x_2, \cdots, x_{i-1}, x_i\}$ is an $i$-clique. Since

$$g(x_i),\ y_i \in \bigcap \{y_j^\perp : 1 \leqq j \leqq i-1\} - R(y_1 y_2 \cdots y_{i-1}) ,$$

by (i) there is $h \in K_{y_1 y_2 \cdots y_{i-1}}$ such that $h(g(x_i)) = y_i$. Thus $hg(x_j) = y_j$ for $j = 1, 2, \cdots, i$. This completes the proof of the proposition.

Note that 3-cliques exist by Lemma 2.1 (iii).

PROPOSITION 2.6. *$G_a$ is a rank 3 permutation group on the set of totally singular lines through $a$. For $b \in D(a)$, $G_{aR(ab)}$ has nontrivial orbits*

$$\{R(ac) : c \in a^\perp \cap b^\perp = R(ab)\}$$

*and*

$$\{R(ac) : c \in a^\perp \cap C(b)\} .$$

The proof is similar to that of Proposition 3.4 of [7]. This proposition follows from Lemmas 2.2 and 2.3 and Proposition 2.5 (i) for $i = 2$ just as Proposition 3.4 of [7] follows from Lemmas 3.2 and 2.2 and Proposition 3.3 of [7].

PROPOSITION 2.7. *Totally singular lines carry $q + 1$ points.*

PROPOSITION 2.8. *If $b \in D(a)$, the $X = \bigcup \{c^\perp : c \in R(ab)\}$.*

PROPOSITION 2.9. *$X$ together with its totally singular lines forms a nondegenerate Shult space of finite rank $\geqq 3$ in which lines carry $q + 1$ points.*

The proofs of the above three statements are identical to the proofs of Propositions 3.5, 3.6, and 3.7 of [7].

LEMMA 2.10. *If $\{x_1, x_2, \cdots, x_i\}$ is an $i$-clique, then $R(x_1 x_2 \cdots x_i)$ is a Shult subspace of $X$.*

*Proof.* In the proof the intersections are taken from $j=1$ to $i$.

Let $d, e \in R(x_1x_2\cdots x_i)$. By definition of $i$-clique, $x_k \in \cap x_j^\perp$ for $1 \leqq k \leqq j$ and so by definition of "$i$-space" and by Lemma 2.3 (i) it follows that

$$d \in R(x_1x_2\cdots x_i) \subseteqq \cap x_j^\perp \subseteqq e^\perp .$$

Thus any two points of $R(x_1x_2\cdots x_i)$ are adjacent. Let the line $R(xy)$ meet $R(x_1x_2\cdots x_i)$ in $\{u, v\}$. Then $R(xy) = R(uv)$ and $x^\perp \cap y^\perp = u^\perp \cap v^\perp$. If $z \in R(xy)$, then

$$z^\perp \supseteqq x^\perp \cap y^\perp = u^\perp \cap v^\perp \supseteqq \cap x_j^\perp$$

since $u, v \in R(x_1x_2\cdots x_i)$ by Lemma 2.3. Thus $z \in R(x_1x_2\cdots x_i)$ and $R(xy) \subseteqq R(x_1x_2\cdots x_i)$. Therefore $R(x_1x_2\cdots x_i)$ is a Shult subspace of $X$, as desired.

PROPOSITION 2.11. (i) *$q$ is a prime power and $r$ is even.*

(ii) *Either $X$ is isomorphic to the polar space $S$ associated with an alternating form $f$ defined on a projective space $P$ of dimension $r-1$ over $GF(q)$ or $X$ is isomorphic to the polar space $S$ associated with a symmetric form $f$ defined on a projective space $P$ of dimension $r$ over $GF(q)$ for $q$ odd.*

For the proof see Proposition 3.9 of [7].

Since $r$ is even and $r \geqq 5$, there exists a natural number $m \geqq 3$ such that $r = 2m$.

PROPOSITION 2.12. (i) *$G$ is isomorphic to a subgroup of $P\Gamma U(f)$, the group of collineations of $P$ which preserve the form $f$.*

(ii) *For $x \in X$, $\phi(x^\perp) = \{w \in P\colon f(w, w) = 0, f(w, \phi(x)) = 0\}$ where $\phi\colon X \to S$ is a polar space isomorphism.*

(iii) *For an $i$-clique, $|R(x_1x_2\cdots x_i)| = v_i$ and $|\cap \{x_j^\perp\colon 1 \leqq j \leqq i\}| = v_{r-i}$.*

(iv) *$X$ contains $m$-cliques but does not contain $(m+1)$-cliques.*

*Proof.* For (i) and (ii) see Proposition 3.10 (i) and (ii) of [7].

(iii) From (ii) it follows that

$$\phi(R(x_1x_2\cdots x_i)) = \cap \{\phi(z)^\perp\colon \phi(x_1), \phi(x_2), \cdots, \phi(x_i) \in \phi(z^\perp)\}$$

which equals the set of singular points in the intersection of all the hyperplanes containing $\phi(x_1), \phi(x_2), \cdots, \phi(x_i)$. But this set is the projective subspace generated by $\phi(x_1), \phi(x_2), \cdots, \phi(x_i)$ since $\phi(x_k) \perp \phi(x_j)$ for all $k, j$. Thus $|R(x_1x_2\cdots x_i)| = v_i$.

From (ii) $|\cap \{x_j^\perp\colon 1 \leqq j \leqq i\}| = v_{r-i}$.

(iv) Since $r = 2m$, (iv) follows from (iii).

Now let $\{x_1, x_2, \cdots, x_m\}$ be a fixed $m$-clique of $X$. Then

$$x_1 \subset R(x_1x_2) \subset R(x_1x_2x_3) \subset \cdots \subset R(x_1x_2\cdots x_m)$$

is a chain of Shult subspaces of $X$ of length $m \geqq 3$. Define subgroups $K_i$ of $K$ as follows:

$$K_1 = K$$
$$K_i = K_{i-1} \cap K_{R(x_1x_2\cdots x_{i-1})} \qquad \text{for} \quad 2 \leqq i \leqq m+1\,.$$

Note the choice of the fixed $i$-clique is arbitrary since $K$ is transitive on $i$-cliques.

PROPOSITION 2.13. (i) *$K_i$ is transitive on the set of "$i$-spaces" containing $R(x_1x_2\cdots x_{i-1})$, for $2 \leqq i \leqq m$.*

(ii) $|K\colon K_{m+1}| = \prod_{j=1}^{m} v_{2j}$.

*Proof.* (i) Let $R(x_1x_2\cdots x_{i-1}d)$ and $R(x_1x_2\cdots x_{i-1}e)$ be "$i$-spaces" containing $R(x_1x_2\cdots x_{i-1})$. Then

$$d, e \in \bigcap_{j=1}^{i-1} x_j^{\perp} - R(x_1x_3\cdots x_{i-1})\,,$$

a set on which $K_{x_1x_2\cdots x_{i-1}}$ is transitive by Proposition 2.5. There exists $k \in K_{x_1x_2\cdots x_{i-1}}$ such that $k(d) = e$. By Lemma 2.3 (iii), it follows that

$$k(R(x_1x_2\cdots x_{i-1}d)) = R(x_1x_2\cdots x_{1-i}e)$$

and that $k \in K_i$.

(ii) For $2 \leqq i \leqq m$ the number of "$i$-spaces" containing $R(x_1x_2\cdots x_{i-1})$ is

$$\left(\left|\bigcap_{j=1}^{i-1} x_j^{\perp}\right| - |R(x_1x_2\cdots x_{i-1})|\right)\Big/(|R(x_1x_2\cdots x_i)| - |R(x_1x_2\cdots x_{i-1})|)$$
$$= (v_{2m-(i-1)} - v_{i-1})/(v_i - v_{i-1}) = v_{2(m-(i-1))}\,.$$

So $|K_i\colon K_{i+1}| = v_{2(m-(i-1))}$ by (i). Since $K$ is a normal subgroup of the primitive group $G$, $K$ is transitive and $|K_1\colon K_2| = v_{2m}$. Now (ii) follows.

PROPOSITION 2.14. (i) *$\psi(K)$ is a flag-transitive subgroup of $PGU(f)$, the group of projective transformations of $P$ which preserve $f$.*

(ii) *If $X$ is symplectic, then $\psi(K) \geqq PSp(2m, q)$.*

(iii) *If $X$ is orthogonal, then* $\psi(K) \geqq P\Omega(2m+1, q)$.

*Proof.* Let $x, y \in X$ with $y \in C(x)$. Since $T(xy)$ is the pointwise stabilizer in $G$ of $x^\perp \cap y^\perp$, it follows that $\psi(T(xy))$ is the pointwise stabilizer in $\psi(G)$ of $\phi(x)^\perp \cap \phi(y)^\perp$. If $t$ is a nontrivial element of $T(xy)$, then $\psi(t) \in P\Gamma U(f)$ and fixes $\phi(x)^\perp \cap \phi(y)^\perp$ pointwise. This implies that $\psi(t) \in PGU(f)$ and so $\psi(K) \leqq PGU(f)$.

Now $\psi(K_{m+1})$ fixes the flag

$$\{\phi(x_1), \langle \phi(x_1), \phi(x_2)\rangle, \cdots, \langle \phi(x_1), \phi(x_2), \cdots, \phi(x_m)\rangle\} .$$

If $B$ is the subgroup of $PGU(f)$ which fixes the above flag, then $B$ is a Borel subgroup of $PGU(f)$ and $B \cap \psi(K) = \psi(K_{m+1})$. Therefore by Proposition 2.13 (ii)

$$\begin{aligned}|B\psi(K)| &= |B|\cdot|\psi(K): \psi(K_{m+1})| \\ &= q^{m^2}(q-1)^m \cdot \prod_{i=1}^{m} v_{2i} = |PGU(f)| .\end{aligned}$$

Thus $B\psi(K) = PGU(f)$ and $\psi(K)$ is a flag-transitive subgroup of $PGU(f)$. By a theorem of Seitz [6], it follows that

$$\psi(K) \geqq PSp(2m, q)$$

if $X$ is symplectic and

$$\psi(K) \geqq P\Omega(2m+1, q)$$

if $X$ is orthogonal, as desired.

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