A DECOMPOSITION OF ADDITIVE SET FUNCTIONS

WAYNE C. BELL
A DECOMPOSITION OF ADDITIVE SET FUNCTIONS

WAYNE C. BELL

In this paper it is demonstrated that if $\mu$ is an additive function from a field $F$ into the nonnegative reals, then $\mu$ can be separated into two mutually singular parts, $\mu_1$ and $\mu_2$, where $\mu_1$ is representable in the sense that its Lebesgue decomposition projection operator has a refinement integral representation and $\mu_2$ is such that for each $E \in F$ the contraction of $\mu_2$ to $E$ is representable iff $\mu_2(E) = 0$. If $\mu_2$ is maximal, then the decomposition is unique.

1. Introduction. Suppose $S$ is a set, $F$ a field of subsets of $S$, $b(F)$ the set of bounded functions from $F$ into $R$, and $ba(F)$ the set of functions in $b(F)$ which are additive on disjoint elements of $F$. For $H \subseteq ba(F)$ denote by $H^+$ the set of nonnegative valued elements of $H$ and let $\mu$ be in $ba(F)^+$. For $\lambda \in ba(F)^+$ denote by $A_\lambda$ the set of elements in $ba(F)$ which are absolutely continuous with respect to $\lambda$ and by $\alpha_\lambda$ the Lebesgue decomposition projection operator for $\lambda$, i.e., for $\eta \in ba(F)$, $\alpha_\lambda(\eta)$ is that part of $\eta$ which is absolutely continuous with respect to $\lambda$ [5]. For $\lambda \in ba(F)^+$ we say that $\lambda$ is representable if there exists a $g: F \to R$ such that $\alpha_\lambda(\eta) = \int g \eta$ for each $\eta \in ba(F)$ in which case $g$ will be said to represent $\lambda$.

2. Preliminary theorems. All integrals in this paper are refinement limits of sums over finite subdivision of $S$ by elements of $F$. If $\beta: F \to R$ and $\int_S \beta(I) \exists$ we will denote by $\int \beta$ the function $\{(v, \int_v \beta(I)) \mid v \in F\}$. For further details concerning the integral and 2. K. 1 and 2. K. 2 below see [1].

THEOREM 2. K. 1. If $\alpha: F \to R$ and $\int_S \alpha(I) \exists$, then
\[
\int_S |\alpha(v) - \int_v \alpha(I)|
\]
exists and is zero. Consequently, if $\beta \in b(F)$ and $v \in F$, then $\int_v \beta(I) \int_I \alpha(J) \exists$ iff $\int_v \beta(I) \alpha(I) \exists$ in which case they are equal.

Proof. [9].

COROLLARY 2. K. 2. If $\alpha: F \to R$ and $\beta: F \to R$ and each of
\[ \int_s \alpha(I) \text{ and } \int_s \beta(I) \text{ exists and } M \text{ is either max or min then } \int M\{\alpha, \beta\} \text{ exists iff } \int M \{\int \alpha, \int \beta\} \text{ exists in which case they are equal.} \]

**Proof.** [1].

Notice that if \( h \) represents \( \mu \), then for \( \lambda \in ba(F)^+ \) we have \( 0 \leq \alpha(\lambda) = \int h\lambda \leq \lambda \) so that \( \int h\lambda = \int \max\{0, \min\{h, 1\}\} \lambda \) and therefore \( h \) can be replaced by a bounded function. Also any representation for \( \mu \) which is valid for \( ba(F)^+ \) is valid for \( \eta \in ba(F) \) since \( \alpha(\eta) = \alpha(\eta^+) - \alpha(\eta^-) \) [5] where \( \eta^+ \) and \( \eta^- \) are the positive and negative parts of \( \eta \), respectively. Consequently we will restrict our attention to \( ba(F)^+ \).

We will also have need of the following theorem due to Appling.

**Theorem 2.A.** If \( \mu \in ba(F)^+ \), \( \eta \in A_\mu \), \( \beta \in b(F) \) and \( \int \beta\mu \) exists, then \( \int \beta\eta \) exists.

**Proof.** [3].

If in subsequent statements the existence of a given integral or its equivalence to a given integral is an immediate consequence of the statements of this section, the integral will often only be written and the proof of existence or equivalence will be left to the reader.

3. Two lemmas. By the remarks of the previous section if \( \mu \) has a representing function, then it has a bounded representing function which, by the following lemma we may assume to be the characteristic function of some subset of \( F \).

**Lemma 3.1.** Suppose \( h \in b(F) \) and for each \( \lambda \in ba(F)^+ \) we have \( \int h\lambda \) exists and is equal to \( \int h \int h\lambda \). Then there exists a \( g: F \to \{0, 1\} \) such that for each \( \lambda \in ba(F)^+ \) we have \( \int g\lambda \) exists and is equal to \( \int h\lambda \).

**Proof.** Let \( \alpha = h^2 \), \( \beta = \min\{\alpha, 1\} \) and suppose \( \lambda \in ba(F)^+ \). It is an easy consequence of 2.K.1 and 2.K.2 that

\[
\int \alpha\lambda = \int \alpha^2\lambda = \int \alpha \int \alpha\lambda = \int h\lambda \quad \text{and} \quad \int \beta\lambda = \int \beta \int \beta\lambda = \int \beta^2\lambda.
\]

Also
\[ \int \alpha \lambda \leq \int \max \{\alpha, 1\} \lambda - \lambda + \lambda \leq \int \max \{\alpha, 1\}(\max \{\alpha, 1\} - 1) \lambda + \lambda = \lambda \]

hence \( \int \beta \lambda = \int \min \{\alpha, 1\} \lambda = \int \alpha \lambda = \int h \lambda. \) Now

\[ 0 \leq \int \min \{\beta, 1 - \beta\} \lambda = \int (1 - \beta) \min \{\beta, 1 - \beta\} \lambda + \int \beta \min \{\beta, 1 - \beta\} \lambda \]

\[ = \int \min \{\beta - \beta^2, (1 - \beta)^2\} \lambda + \int \min \left\{ \beta^2 \lambda, \int \beta \lambda - \int \beta^2 \lambda \right\} \]

\[ = \int \min \left\{ \int \beta \lambda - \int \beta^2 \lambda, \int (1 - \beta)^2 \lambda \right\} + 0 = 0. \]

For each \( v \in F \) let \( \ell(v) = \begin{cases} \beta(v) & \text{if } \beta(v) \leq 1/2 \\ 0 & \text{otherwise.} \end{cases} \) Then \( 0 \leq \ell \leq \min \{\beta, 1 - \beta\} \) so that \( \int \ell \lambda \) exists and is zero. For each \( v \in F \) let

\[ g(v) = \begin{cases} 1 & \text{if } \beta(v) > 1/2 \\ 0 & \text{if } \beta(v) \leq 1/2 \end{cases} = \min \{2(\beta(v) - \ell(v)), 1\}. \]

Now by 2.K.2. \( \int g \lambda \) exists and we have

\[ \int g \lambda = \int \min \{2(\beta - \ell), 1\} \lambda = \int \min \left\{ 2 \int \beta \lambda - 2 \int \ell \lambda, \lambda \right\} \]

\[ = \int \min \left\{ 2 \int \beta \lambda, \lambda \right\} = \int \beta \lambda + \int \min \left\{ \int \beta \lambda, \lambda - \int \beta \lambda \right\} \]

\[ = \int \beta \lambda - \int \min \{\beta, 1 - \beta\} \lambda = \int \beta \lambda. \]

If \( D \) is a subdivision of \( S \), i.e., a finite disjoint subset of \( F \) whose union is \( S \), then \( H \) is a refinement of \( D, H \ll D \), means that \( H \) is a subdivision of \( S \) and for each \( v \in D \) there exists a subset \( H_v \) of \( H \) whose union is \( v \).

**Lemma 2.** Suppose \( \lambda \in ba(F)^+ \), \( (E_i) \) is a disjoint sequence in \( F \), \( B > 0 \) and for each \( i \in N \) we have \( g_i : F \to [0, B] \) and \( \int g_i \lambda \) exists. Suppose also that if \( i \in N \), \( I \in F \) and \( g_i(I) \neq 0 \), then \( I \subseteq E_i \). Then for each \( v \in F \), \( \int g(I) \lambda(I) \) exists and is \( \sum_{i=1}^{\infty} g_i(I) \lambda(I) \), where \( g(I) = \sum_{i=1}^{\infty} g_i(I) \) for each \( I \in F \).

**Proof.** Let \( v \in F \) and \( c > 0 \). Let \( n \) be such that \( \sum_{i=1}^{\infty} \lambda(E_i \cap v) < c/4B \). For each \( i \leq n \) let \( D_i \ll \{E_i \cap v\} \) be such that if \( K \ll D_i \), then

\[ \left| \sum_K g_i(I) \lambda(I) - \int_{v \cap E_i} g_i(I) \lambda(I) \right| < c/2n. \]

Let
and suppose \( H \ll D \). Let \( H_i = \{ I \in H | I \subseteq E_i \} \) for each \( i \) and \( H' = H \sim \bigcup_{i=1}^{n} H_i \). Note that if \( I \in H_i \), then \( g_i(I) = g(I) \). Now

\[
\left| \sum_H g(I) \lambda(I) - \sum_1^\infty \int g_i(I) \lambda(I) \right| \\
\leq \left| \sum_{i=1}^n \sum_{H_i} g(I) \lambda(I) - \sum_1^\infty \int_{v \cap E_i} g_i(I) \lambda(I) \right| \\
+ \left| \sum_{H'} g(I) \lambda(I) \right| + \left| \sum_{j=n+1}^\infty \int_{E_j \cap v} g_i(I) \lambda(I) \right| \\
\leq \sum_{i=1}^n \left| \sum_{H_i} g_i(I) \lambda(I) - \int_{v \cap E_i} g_i(I) \lambda(I) \right| \\
+ \sum \{ g(I) \lambda(I) | I \in H', I \subseteq E_j \cap v \text{ and } j > n \} \\
+ \sum_{j=n+1}^\infty B \lambda(E_j \cap v) \\
< \sum_{i=1}^n c/2n + \sum_{j=n+1}^\infty B \lambda(E_j \cap v) + B \cdot c/4B \\
\leq c/2 + B \cdot c/4B + c/4 = c.
\]

For \( v \in F \) denote by \( x_v \) the characteristic function of \( \{ I \in F | I \subseteq v \} \) and by \( c_v(\mu) \) the contraction of \( \mu \) to \( v \), i.e., \( c_v(\mu) = \int x_v \mu \).

A linear transformation, \( T \), from \( ba(F) \) into \( ba(F) \) is in the class \( \mathcal{G} \) [2] iff there exists a \( K > 0 \) such that for each \( v \in F \) and \( \xi \) in \( ba(F) \) we have

\[
\int_v | T(\xi)(I) | \leq K \int_v | \xi(I) | .
\]

**Theorem 3.A.** If \( T \in \mathcal{G} \), \( \eta \in ba(F)^+ \) and \( \delta \in A_v \), then \( T(\delta) = \int (T(\eta)/\eta) \delta \).

**Proof.** [2].

In [4] it was shown that the elements of \( \mathcal{G} \) commute. Now, if \( v \in F \) and \( \lambda \in A^+_v \), then \( c_v, \alpha_\mu \) and \( \alpha_\lambda \) are clearly in \( \mathcal{G} \). Therefore for \( \xi \in ba(F) \) we have \( \alpha_\lambda(\xi) \in A_\mu \), so that

\[
3.c.1. \quad \alpha_\lambda(\xi) = \alpha_\mu(\alpha_\lambda(\xi)) = \alpha_\lambda(\alpha_\mu(\xi)) = \int (\alpha_\lambda(\mu)/\mu) \alpha_\mu(\xi) ,
\]

consequently if \( \mu \) is representable, then \( \lambda \) is also. If we replace \( \lambda \), in 3.c.1, by \( c_v(\mu) \) we have:

\[
3.c.2. \quad \alpha_{c_v(\mu)}(\xi) = (c_v \circ \alpha_\mu)(\xi) ,
\]
hence we may say that if $g$ represents $c_v(\mu)$ and $I \in F$ is such that $I \subseteq v$, then $g \cdot x_I$ represents $c_v(\mu)$.

4. The decomposition. Suppose $R \subseteq F$ is a ring of subsets of $S$ such that $I \in F$ and $I \subseteq v \in R$ implies that $I \in R$, then if $f$ is the characteristic function of $R$ and $\lambda \in \text{ba}(F)^+$ the expression $\sum I f(I) \lambda(I)$ is nondecreasing for successive refinements and bounded by $\lambda(S)$ so that $\int f \lambda$ exists.

**Theorem 1.** Suppose $R \subseteq F$ is a ring of subsets of $S$ for which $I \in F$ and $I \subseteq v \in R$ imply $I \in R$. Suppose further that $c_v(\mu)$ is representable for each $v \in R$ and $\int f \mu = \mu$ where $f$ is the characteristic function of $R$. Then $\mu$ is representable.

**Proof.** Since $\mu = \int f \mu$ we have for each $n$ there exists $D_n \ll \{S\}$ such that if $E \ll D_n$, then $\mu(S) - \sum I f(I) \mu(I) < 1/n$ and $D_n$ can be chosen so that $D_n \ll D_{n-1}$. Therefore if $v_n = \bigcup \{ I \in D_n \mid f(I) = 1 \}$, then $v_n \subseteq v_{n+1}$ and $\mu(S \sim v_n) < 1/n$ for each $n$. Let $E_i = v_i$ and $E_i = v_i \sim v_{i-1}$ for $i > 1$. For each $i$ let $\mu_i = c_{E_i}(\mu)$ and $g_i : F \to \{0, 1\}$ be such that $g_i \cdot x_{E_i} = g_i$ and $\alpha_{E_i}(\lambda) = \int g_i \lambda$ for each $\lambda \in \text{ba}(F)^+$. Let $g = \sum_i g_i$ and suppose $\lambda \in \text{ba}(F)^+$. By Lemma 2, $\int g \lambda$ exists and is $\sum_i \int g_i \lambda$ and for each $i$ we have $\alpha_{E_i}(\lambda) = \int g_i \lambda \in A_{\mu_i} \subseteq A_\mu$ and therefore $\int g \lambda \in A_\mu$.

Thus, if $\lambda = \int g \lambda$, then $\lambda \in A_\mu$.

Now suppose $\lambda \in A_\mu^+$. Let $c > 0$ and $n$ be such that $\mu(I) < 1/n$ implies that $\lambda(I) < c$. Then

\[
0 \leq \lambda(S) - \int_S g(I) \lambda(I) \leq \lambda(S) - \sum_1^n \int_S g_i(I) \lambda(I) = \lambda(S) - \sum_1^n \alpha_{E_i}(\lambda)(S)
\]

\[
= \lambda(S) - \sum_1^n c_{E_i} \cdot \alpha_{E_i}(\lambda)(S) = \lambda(S) - \sum_1^n \alpha_{E_i}(\lambda)(E_i)
\]

\[
= \lambda(S) - \sum_1^n \lambda(E_i) = \lambda(S) - \lambda(v_n) = \lambda(S \sim v_n) < c .
\]

Therefore $\lambda \in A_\mu$ iff $\lambda = \int g \lambda$.

Now, as previously established, $\int g \lambda \in A_\mu$. Since $\int g \lambda \leq \lambda$ it follows that $\int g \lambda \leq \alpha_\mu(\lambda) = \int g \alpha_\mu(\lambda) \leq \int g \lambda$, hence $g$ represents $\mu$.

If $\gamma$ and $\delta$ are in $\text{ba}(F)^+$ we will say that they are mutually singular if whenever $\lambda \in \text{ba}(F)^+$ and $\lambda \leq \gamma$ and $\lambda \leq \delta$, then $\lambda = 0$. This is the notion of singularity used in [5] and [10] which is equivalent to that of [6]. It is also equivalent to $\int \min \{\gamma, \delta\} = 0$. 


Since \( \gamma \) and \( \delta \) are only finitely additive we cannot, necessarily, obtain disjoint sets \( s_1 \) and \( s_2 \) such that \( \gamma(s_1) = \delta(s_2) = 0 \) with \( s_1 \cup s_2 = t \).

**Theorem 2.** There exist \( \mu_1 \) and \( \mu_2 \) in \( ba(F)^+ \) such that:
1. \( \mu_1 \) and \( \mu_2 \) are mutually singular and \( \mu = \mu_1 + \mu_2 \).
2. \( \mu_1 \) is representable.
3. For each \( v \in F \) we have \( c_v(\mu_2) \) is representable iff \( \mu_1(v) = 0 \).
4. If \( \mu_2 \) is in \( ba(F)^+ \), \( \mu_2 \leq \mu \) and for each \( v \in F \) we have \( c_v(\mu_2) \) is representable iff \( \mu_2(v) = 0 \), then \( \mu_2 = \mu_3 \).

**Proof.** If \( I, v \in F, I \subseteq v \) and \( h \) represents \( c_v(\mu) \), then by 3.c.2. \( x_I \cdot h \) represents \( c_I(\mu) \). Consequently \( R = \{ v \in F \mid c_v(\mu) \text{ is representable} \} \) is a ring satisfying the conditions of Theorem 1 since for \( I \) and \( v \) in \( R \) with \( h, k \) representing \( c_I(\mu) \) and \( c_v(\mu) \) respectively we have \( h + x_{-I} \cdot k \) represents \( c_{I \cup v}(\mu) \). Let \( f \) be the characteristic function of \( R \) and \( \mu_1 = \int f \mu \). Then for each \( v \in R \) we have

\[
c_v(\mu_1) = \int x_v \mu_1 = \int x_v \int f \mu = \int x_v f \mu = \int x_v \mu = c_v(\mu)
\]

so that \( c_v(\mu_1) \) is representable. Also

\[
\int f \mu_1 = \int f \int f \mu = \int f^2 \mu = \int f \mu = \mu_1
\]

and thus, by Theorem 1, \( \mu_1 \) is representable. Let \( \mu_2 = \mu - \mu_1 = \int (1 - f) \mu \) and note that \( \mu_1 \) and \( \mu_2 \) are mutually singular since \( \min \{f, 1 - f\} = 0 \). Therefore for \( \lambda \in ba(F)^+ \) we have \( \alpha_{\mu_1}(\lambda) \) and \( \alpha_{\mu_2}(\lambda) \) are mutually singular hence

\[
\alpha_{\mu_1}(\lambda) + \alpha_{\mu_2}(\lambda) = \int \max \{\alpha_{\mu_1}(\lambda), \alpha_{\mu_2}(\lambda)\} \leq \alpha_\mu(\lambda) \leq \alpha_{\mu_1}(\lambda) + \alpha_{\mu_2}(\lambda),
\]

i.e., \( \alpha_{\mu_1} + \alpha_{\mu_2} = \alpha_\mu \). Now suppose \( v \in F \) and \( c_v(\mu_2) \) is representable, then \( c_v(\mu) = c_v(\mu_1) + c_v(\mu_2) \) is representable so that \( v \in R \). Therefore \( f(I) = 1 \) for each \( I \in F \) for which \( I \subseteq v \). Hence

\[
\mu_2(v) = \int_v (1 - f(I)) \mu(I) = 0.
\]

Finally suppose \( \mu_3 \in ba(F)^+ \) and \( \mu_2 \leq \mu_3 \leq \mu \) and \( c_v(\mu_3) \) is representable iff \( \mu_3(v) = 0 \). For each \( v \in R \) we have \( c_v(\mu_3) \) is representable by 3.c.1. so that \( \mu_3(v) = 0 \). Therefore

\[
\mu_3 = \int f \mu_3 + \int (1 - f) \mu_3 \leq 0 + \int (1 - f) \mu = \mu_2.
\]

This decomposition differs from those of [6], [7] and [10] in that
it does not give rise to a normal subspace [5]. To see that this is true suppose that the set $R$ of those elements of $ba(F)$ whose total variations are representable is a normal subspace and note that if $a \in ba(F)^+$ and for each $v \in F$ we have $a(v) \in \{0, a(S)\}$, then $a \in R$. Therefore for any summable sequence, $(a_n)$, of such elements we have $\lambda = \sum_{n=1}^{\infty} a_n \in R$. Consequently $\alpha_i$ has an integral representation. However by [4] this is true iff the linear functional $\eta \mapsto \alpha_i(\eta)(S)$ has an integral representation and in [8] it was shown that this is not always true.

REFERENCES


Received October 29, 1976 and in revised form April 25, 1977.
George E. Andrews, *Plane partitions. II. The equivalence of the Bender-Knuth and MacMahon conjectures* ........................................ 283
Lee Wilson Badger, *An Ehrenfeucht game for the multivariable quantifiers of Malitz and some applications* ........................................ 293
Wayne C. Bell, *A decomposition of additive set functions* ......................... 305
Bruce Blackadar, *Infinite tensor products of C*-algebras* .......................... 313
Arne Brøndsted, *The inner aperture of a convex set* .................................. 335
N. Burgoyne, *Finite groups with Chevalley-type components* ................... 341
Richard Dowell Byrd, Justin Thomas Lloyd and Roberto A. Mena, *On the retractability of some one-relator groups* ......................... 351
Paul Robert Chernoff, *Schrödinger and Dirac operators with singular potentials and hyperbolic equations* .............................. 361
John J. F. Fournier, *Sharpness in Young’s inequality for convolution* ........... 383
Stanley Phillip Franklin and Barbara V. Smith Thomas, *On the metrizability of kω-spaces* .................................................. 399
David Andrew Gay, Andrew McDaniel and William Yslas Vélez, *Partially normal radical extensions of the rationals* ...................... 403
Jean-Jacques Gervais, *Sufficiency of jets* .................................................. 419
Kenneth R. Goodearl, *Completions of regular rings. II* ................................ 423
Sarah J. Gottlieb, *Algebraic automorphisms of algebraic groups with stable maximal tori* .................................................. 461
Donald Gordon James, *Invariant submodules of unimodular Hermitian forms* ................................................................. 471
J. Kyle, *Wδ(T) is convex* ............................................................................ 483
Ernest A. Michael and Mary Ellen Rudin, *A note on Eberlein compacts* ...... 487
Ernest A. Michael and Mary Ellen Rudin, *Another note on Eberlein compacts* ................................................................. 497
Thomas Bourque Muenzenberger and Raymond Earl Smithson, *Fixed point theorems for acyclic and dendritic spaces* .................. 501
Budh Singh Nashier and A. R. Rajwade, *Determination of a unique solution of the quadratic partition for primes p ≡ 1 (mod 7)* .......... 513
Frederick J. Scott, *New partial asymptotic stability results for nonlinear ordinary differential equations* ...................................... 523
Frank Servedio, *Affine open orbits, reductive isotropy groups, and dominant gradient morphisms; a theorem of Mikio Sato* ............... 537
D. Suryanarayana, *On the distribution of some generalized square-full integers* ................................................................. 547
Wolf von Wahl, *Instationary Navier-Stokes equations and parabolic systems* ................................................................. 557