SUFFICIENCY OF JETS

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We give a necessary and sufficient condition for the \( C^\infty \)-sufficiency of a jet; this generalizes and improves some results of J. N. Mather and J. C. Tougeron. Our result, given in terms of \( G \)-sufficiency which is a generalization of the ordinary sufficiency, can be applied to many cases.

NOTATIONS. Let \( G \) be a \( q \)-dimensional Lie subgroup of \( \text{Gl}_p(R) \). Let \( G(n) = C^\infty_0(\mathbb{R}_n, G) \) be the group of germs at 0 of smooth mappings \( g \) from \( \mathbb{R}^n \) to \( G \) such that \( g(0) = e \) (where \( e \) is the identity of \( G \)) and \( \text{Diff}(n) \) the group of germs at 0 of smooth diffeomorphisms \( \tau \) from a neighborhood of 0 in \( \mathbb{R}^n \) on a neighborhood of 0 in \( \mathbb{R}^n \) such that \( \tau(0) = 0 \). Let \( \mathcal{G} \) be the ring of germs at 0 of smooth functions from \( \mathbb{R}^n \) to \( R \) and \( m \) its maximal ideal. For \( f \in \bigoplus_p m \), \( j^r(f) \) will denote the \( r \)-jet of \( f \) at 0. The set \( \mathcal{G}(n) = G(n) \times \text{Diff}(n) \) is a group with the following multiplication: \( (g_1, \tau_1) \cdot (g_2, \tau_2) = (g_1 \cdot (g_2 \circ \tau_1^{-1}), \tau_1 \circ \tau_2) \). Then we may define an action of \( \mathcal{G}(n) \) on \( \bigoplus_p m \) by the formula: for \( (g, \tau) \in \mathcal{G}(n) \) and \( f \in \bigoplus_p m \), \( (g, \tau) \cdot f \) is the germ at 0 of the mapping \( x \mapsto \tilde{g}(x) \cdot (\tilde{f} \circ \tilde{\tau}^{-1}(X)) \) where \( \tilde{g}, \tilde{f}, \) and \( \tilde{\tau} \) are representatives of \( g, f, \) and \( \tau \) respectively.

DEFINITION 1. An \( r \)-jet \( z \) of an element of \( \bigoplus_p m \) is \( G \)-sufficient if for any \( f \in \bigoplus_p m \) such that \( j^r(f) = z \) there exists \( (g, \tau) \in \mathcal{G}(n) \) such that \( (g, \tau) \cdot f = z \).

REMARK. When \( G = \{ e \} \) and \( p = 1 \) the \( G \)-sufficiency is the ordinary \( C^\infty \)-sufficiency of jets.

We will use the well known:

NAKAYAMA'S LEMMA. Let \( A \) be a commutative ring with identity and let \( I \) be an ideal in \( A \) such that \( 1 + a \) is invertible for any \( a \in I \). Let \( M \) and \( N \) be submodules of an \( A \)-module \( P \) such that \( M \) is finitely generated and \( M \subset N + I \). \( M \). Then \( M \subset N \).

JETS \( G \)-sufficient. Let \( \{ A_1, \cdots, A_q \} \) be a base over \( R \) of the Lie algebra \( T_0 G \) of \( G \). For every \( g \in G(n) \) there exists \( u = (u_1, \cdots, u_q) \in \bigoplus_q m \) such that

\[
g(x) = \sum_{i=1}^q u_i(x).A_i.
\]
Hence we may identify $G(n)$ with $\bigoplus m$.

Let $G^r$ be the analytic Lie group of the $r$-jets of the elements of $G(n)$ and let $X^r$ be the space of $r$-jets of the elements of $\bigoplus m$. The group action of $G(n)$ on $\bigoplus m$ induces, for each $r$, a well defined group action of $G^r$ on $X^r$. One easily sees that this group action is analytic for each $r$.

For $f \in \bigoplus m$, let $M_f$ be the $\mathcal{C}$-linear mapping:

$$M_f: \mathcal{C}_n^{p+q} \to \mathcal{C}_n^p,$$

where $M_f$ is given by the $p \times (q + n)$-matrix with $A_1 f, \ldots, A_q f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n$ as columns. It is easily seen that for $f \in \bigoplus m$ the mapping

$$\tilde{M}_f: \bigoplus_{q+n} \left( \frac{m}{m^{r+1}} \right) \to \bigoplus_p \left( \frac{m}{m^{r+1}} \right),$$

derived from $M_f$, is the tangent mapping at the identity of the mapping

$$G^r \ni \gamma \to \gamma \cdot j^r(f) \in X^r.$$

**Theorem 1.** Let $z \in X^r$. The following statements are equivalent:

(i) $z$ is $G$-sufficient.

(ii) For any homogeneous jet $w$ of degree $r + 1$ we have $m \cdot \text{Im} M_{z+w} \supseteq m^{r+1} \cdot \mathcal{C}_n^p$ (where $\text{Im} M_{z+w}$ is the range of $M_{z+w}$).

**Proof.**

(i) $\implies$ (ii) Let $w$ and $w'$ be two homogeneous jets of degree $r + 1$. Since $z$ is $G$-sufficient, there exist $(g, \tau)$ and $(g', \tau') \in G(n)$ such that $(g, \tau) \cdot z = z + w$ and $(g', \tau') \cdot z = z + w'$; hence $(g', \tau') \cdot (g, \tau)^{-1} \cdot (z + w) = z + w'$.

Consequently, if we put $\gamma = j^{r+1}((g', \tau') \cdot (g, \tau)^{-1})$, we have $\gamma \cdot (z + w) = z + w'$. We have thus shown that for any homogeneous jet $w$ of degree $r + 1$ the $\mathcal{C}^{r+1}$-orbit of $z + w$ in $X^{r+1}$ contains $\{z + w' \mid w' \text{ is a homogeneous jet of degree } r + 1\}$. Since the tangent mapping at the identity of the mapping $\mathcal{C}^{r+1} \ni \gamma \to \gamma \cdot (z + w) \in X^{r+1}$ is

$$\tilde{M}_{z+w} \bigoplus_{q+n} \left( \frac{m}{m^{r+2}} \right) \to \bigoplus_p \left( \frac{m}{m^{r+2}} \right),$$

derived from $M_{z+w}$, we have $\text{Im} \tilde{M}_{z+w} \supseteq \bigoplus_p \left( m^{r+1}/m^{r+2} \right)$, i.e., $m \cdot \text{Im} M_{z+w} + m^{r+2} \cdot \mathcal{C}_n^p \supseteq m^{r+1} \cdot \mathcal{C}_n^p$. From the Nakayama's lemma, we
conclude that $m \cdot \text{Im } M_{z+w} \supset m^{r+1} \cdot \mathcal{E}^p_n$.

(ii) $\Rightarrow$ (i)

(a) Let $w_1, \ldots, w_k$ be homogeneous jets of degree $r+1, \ldots, r+k$ respectively and put $z' = \sum_{i=1}^k w_i$. Let $t_0 \in [0,1]$. By hypothesis,

$$m \cdot \text{Im } M_{z+t_0w_1} \supset m^{r+1} \cdot \mathcal{E}^p_n.$$ 

Hence we have

$$m^{r+1} \cdot \mathcal{E}^p_n \subset m \cdot \text{Im } M_{z+t_0w_1} \subset m \cdot \text{Im } M_{z+t_0z'} + m^{r+2} \cdot \mathcal{E}^p_n.$$ 

Nakayama's lemma implies

$$m^{r+1} \cdot \mathcal{E}^p_n \subset m \cdot \text{Im } M_{z+t_0z'}.$$ 

Then the range of the mapping $\mathcal{E}^{r+k} \ni \gamma \mapsto \gamma \cdot (z + t_0z')$ contains all $r+k$-jets $z + z''$, where $z''$ is an $r+k$-jet in a neighborhood of $t_0z'$ such that $j^r(z'') = 0$. In particular, there exist $t_1 < t_0 < t_2$ such that for all $t'$ and $t'' \in [t_1, t_2]$, there exists $(g, \tau) \in \mathcal{G}(n)$ such that $j^{r+k}((g, \tau) \cdot (z + t'z')) = z + t''z'$. Since $[0,1]$ is compact, it follows that there exists $(g, \tau) \in \mathcal{G}(n)$ such that $j^{r+k}((g, \tau) \cdot (z + z')) = z + 0 \cdot z' = z$.

(b) Let $f \in \bigoplus_p m$ such that $j^r(f) = z$, we must prove that there exists $(g, \tau) \in \mathcal{G}(n)$ such that $(g, \tau) \cdot f = z$. We have

$$m^{r+1} \cdot \mathcal{E}^p_n \subset m \cdot \text{Im } M_{j^{r+1}(f)}.$$ 

Hence

$$m^{r+1} \cdot \mathcal{E}^p_n \subset m \cdot \text{Im } M_{j^{r+1}(f)} \subset m \cdot \text{Im } M_f + m^{r+2} \cdot \mathcal{E}^p_n.$$ 

Nakayama's lemma implies

$$m^{r+1} \cdot \mathcal{E}^p_n \subset m \cdot \text{Im } M_f.$$ 

It follows from a result of J. C. Tougeron [2, Théorème VIII 3.6] that there exists $N \in N$ such that $j^N(f)$ is $G$-sufficient. If $N \leq r$ the proof is finished. Suppose $N > r$. By (a), there exist $(g_i, \tau_i) \in \mathcal{G}(n)$ and $\phi \in m^{N+1} \cdot \mathcal{E}^p_n$ such that

$$z = (g_i, \tau_i) \cdot j^N(f) + \phi; \text{ hence }$$

$$z = (g_i, \tau_i) \cdot [j^N(f) + (g_i, \tau_i)^{-1} \cdot \phi].$$

Since $\phi \in m^{N+1} \cdot \mathcal{E}^p_n$, $(g_i, \tau_i)^{-1} \cdot \phi \in m^{N+1} \cdot \mathcal{E}^p_n$. But $j^N(f)$ is $G$-sufficient, consequently there exists $(g_2, \tau_2) \in \mathcal{G}(n)$ such that

$$j^N(f) + (g_i, \tau_i)^{-1} \cdot \phi = (g_2, \tau_2) \cdot f.$$ 

Hence
Let \( f \in m \). We say that \( f \) is \( r \)-determined if \( j^r(f) \) is \( C^\infty \)-sufficient (i.e., \( G \)-sufficient with \( G = \{e\} \)).

From Theorem 1 we deduce the following two results of J. N. Mather [1], stated as follows in [3, Theorem 2.6 and Corollary 2.10]:

**Theorem 2.** Let \( f \in m \) and \( I_f \) be the ideal generated in \( \mathcal{E}_n \) by the partial derivatives of \( f \). If
\[
m^r \subset m \cdot I_f + m^{r+1},
\]
then \( f \) is \( r \)-determined.

**Theorem 3.** Let \( f \in m \) be \( r \)-determined. Then
\[
m^{r+1} \subset m \cdot I_f.
\]

**REFERENCES**


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