

# Pacific Journal of Mathematics

**SUFFICIENCY OF JETS**

JEAN-JACQUES GERVAIS

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**We give a necessary and sufficient condition for the  $C^\infty$ -sufficiency of a jet: this generalizes and improves some results of J. N. Mather and J. C. Tougeron. Our result, given in terms of  $G$ -sufficiency which is a generalization of the ordinary sufficiency, can be applied to many cases.**

NOTATIONS. Let  $G$  be a  $q$ -dimensional Lie subgroup of  $Gl_p(\mathbf{R})$ . Let  $G(n) = C_{0,e}^\infty(\mathbf{R}^n, G)$  be the group of germs at 0 of smooth mappings  $g$  from  $\mathbf{R}^n$  to  $G$  such that  $g(0) = e$  (where  $e$  is the identity of  $G$ ) and  $\text{Diff}(n)$  the group of germs at 0 of smooth diffeomorphisms  $\tau$  from a neighborhood of 0 in  $\mathbf{R}^n$  on a neighborhood of 0 in  $\mathbf{R}^n$  such that  $\tau(0) = 0$ . Let  $\mathcal{S}_n$  be the ring of germs at 0 of smooth functions from  $\mathbf{R}^n$  to  $\mathbf{R}$  and  $m$  its maximal ideal. For  $f \in \bigoplus_p m$ ,  $j^r(f)$  will denote the  $r$ -jet of  $f$  at 0. The set  $\mathcal{S}(n) = G(n) \times \text{Diff}(n)$  is a group with the following multiplication:  $(g_1, \tau_1) \cdot (g_2, \tau_2) = (g_1 \cdot (g_2 \circ \tau_1^{-1}), \tau_1 \circ \tau_2)$ . Then we may define an action of  $\mathcal{S}(n)$  on  $\bigoplus_p m$  by the formula: for  $(g, \tau) \in \mathcal{S}(n)$  and  $f \in \bigoplus_p m$ ,  $(g, \tau) \cdot f$  is the germ at 0 of the mapping  $x \mapsto \tilde{g}(x) \cdot (\tilde{f} \circ \tilde{\tau}^{-1}(X))$  where  $\tilde{g}, \tilde{f}$ , and  $\tilde{\tau}$  are representatives of  $g, f$ , and  $\tau$  respectively.

DEFINITION 1. An  $r$ -jet  $z$  of an element of  $\bigoplus_p m$  is  $G$ -sufficient if for any  $f \in \bigoplus_p m$  such that  $j^r(f) = z$  there exists  $(g, \tau) \in \mathcal{S}(n)$  such that  $(g, \tau) \cdot f = z$ .

REMARK. When  $G = \{e\}$  and  $p = 1$  the  $G$ -sufficiency is the ordinary  $C^\infty$ -sufficiency of jets.

We will use the well known:

NAKAYAMA'S LEMMA. *Let  $A$  be a commutative ring with identity and let  $I$  be an ideal in  $A$  such that  $1 + a$  is invertible for any  $a \in I$ . Let  $M$  and  $N$  be submodules of an  $A$ -module  $P$  such that  $M$  is finitely generated and  $M \subset N + I \cdot M$ . Then  $M \subset N$ .*

*Jets  $G$ -sufficient.* Let  $\{A_1, \dots, A_q\}$  be a base over  $\mathbf{R}$  of the Lie algebra  $T_e G$  of  $G$ . For every  $g \in G(n)$  there exists  $u = (u_1, \dots, u_q) \in \bigoplus_q m$  such that

$$g(x) = e^{\sum_{i=1}^q u_i(x) \cdot A_i}.$$

Hence we may identify  $G(n)$  with  $\bigoplus_q m$ .

Let  $\mathcal{G}^r$  be the analytic Lie group of the  $r$ -jets of the elements of  $\mathcal{G}(n)$  and let  $X^r$  be the space of  $r$ -jets of the elements of  $\bigoplus_p m$ . The group action of  $\mathcal{G}(n)$  on  $\bigoplus_p m$  induces, for each  $r$ , a well defined group action of  $\mathcal{G}^r$  on  $X^r$ . One easily sees that this group action is analytic for each  $r$ .

For  $f \in \bigoplus_p m$ , let  $M_f$  be the  $\mathcal{E}_n$ -linear mapping:

$$M_f: \mathcal{E}_n^{p+q} \longrightarrow \mathcal{E}_n^p,$$

where  $M_f$  is given by the  $p \times (q + n)$ -matrix with  $A_1 \cdot f, \dots, A_q \cdot f, \partial f / \partial x_1, \dots, \partial f / \partial x_n$  as columns. It is easily seen that for  $f \in \bigoplus_p m$  the mapping

$$\bar{M}_f^r: \bigoplus_{q+n} \left( \frac{m}{m^{r+1}} \right) \longrightarrow \bigoplus_p \left( \frac{m}{m^{r+1}} \right),$$

derived from  $M_f$ , is the tangent mapping at the identity of the mapping

$$\mathcal{G}^r \ni \gamma \longrightarrow \gamma \cdot j^r(f) \in X^r.$$

**THEOREM 1.** *Let  $z \in X^r$ . The following statements are equivalent:*

- (i)  $z$  is  $G$ -sufficient.
- (ii) For any homogeneous jet  $w$  of degree  $r + 1$  we have  $m \cdot \text{Im } M_{z+w} \supset m^{r+1} \cdot \mathcal{E}_n^p$  (where  $\text{Im } M_{z+w}$  is the range of  $M_{z+w}$ ).

*Proof.*

(i)  $\implies$  (ii) Let  $w$  and  $w'$  be two homogeneous jets of degree  $r + 1$ . Since  $z$  is  $G$ -sufficient, there exist  $(g, \tau)$  and  $(g', \tau') \in \mathcal{G}(n)$  such that  $(g, \tau) \cdot z = z + w$  and  $(g', \tau') \cdot z = z + w'$ ; hence  $(g', \tau') \cdot (g, \tau)^{-1} \cdot (z + w) = z + w'$ .

Consequently, if we put  $\gamma = j^{r+1}((g', \tau') \cdot (g, \tau)^{-1})$ , we have  $\gamma \cdot (z + w) = z + w'$ . We have thus shown that for any homogeneous jet  $w$  of degree  $r + 1$  the  $\mathcal{G}^{r+1}$ -orbit of  $z + w$  in  $X^{r+1}$  contains  $\{z + w' \mid w' \text{ is a homogeneous jet of degree } r + 1\}$ . Since the tangent mapping at the identity of the mapping  $\mathcal{G}^{r+1} \ni \gamma \mapsto \gamma \cdot (z + w) \in X^{r+1}$  is

$$\bar{M}_{z+w}^{r+1}: \bigoplus_{q+n} \left( \frac{m}{m^{r+2}} \right) \longrightarrow \bigoplus_p \left( \frac{m}{m^{r+2}} \right),$$

derived from  $M_{z+w}$ , we have  $\text{Im } \bar{M}_{z+w}^{r+1} \supset \bigoplus_p (m^{r+1} / m^{r+2})$ , i.e.,  $m \cdot \text{Im } M_{z+w} + m^{r+2} \cdot \mathcal{E}_n^p \supset m^{r+1} \cdot \mathcal{E}_n^p$ . From the Nakayama's lemma, we

conclude that  $m \cdot \text{Im } M_{z+w} \supset m^{r+1} \cdot \mathcal{E}_n^p$ .

(ii)  $\Rightarrow$  (i)

(a) Let  $w_1, \dots, w_k$  be homogeneous jets of degree  $r+1, \dots, r+k$  respectively and put  $z' = \sum_{i=1}^k w_i$ . Let  $t_0 \in [0, 1]$ . By hypothesis,

$$m \cdot \text{Im } M_{z+t_0 w_1} \supset m^{r+1} \cdot \mathcal{E}_n^p.$$

Hence we have

$$m^{r+1} \cdot \mathcal{E}_n^p \subset m \cdot \text{Im } M_{z+t_0 w_1} \subset m \cdot \text{Im } M_{z+t_0 z'} + m^{r+2} \cdot \mathcal{E}_n^p.$$

Nakayama's lemma implies

$$m^{r+1} \cdot \mathcal{E}_n^p \subset m \cdot \text{Im } M_{z+t_0 z'}.$$

Then the range of the mapping  $\mathcal{E}^{r+k} \ni \gamma \mapsto \gamma \cdot (z + t_0 z')$  contains all  $r+k$ -jets  $z + z''$ , where  $z''$  is an  $r+k$ -jet in a neighborhood of  $t_0 z'$  such that  $j^r(z'') = 0$ . In particular, there exist  $t_1 < t_0 < t_2$  such that for all  $t'$  and  $t'' \in [t_1, t_2]$ , there exists  $(g, \tau) \in \mathcal{E}(n)$  such that  $j^{s+k}((g, \tau) \cdot (z + t' z')) = z + t'' z'$ . Since  $[0, 1]$  is compact, it follows that there exists  $(g, \tau) \in \mathcal{E}(n)$  such that  $j^{s+k}((g, \tau) \cdot (z + z')) = z + 0 \cdot z' = z$ .

(b) Let  $f \in \bigoplus_p m$  such that  $j^r(f) = z$ , we must prove that there exists  $(g, \tau) \in \mathcal{E}(n)$  such that  $(g, \tau) \cdot f = z$ . We have

$$m^{r+1} \cdot \mathcal{E}_n^p \subset m \cdot \text{Im } M_{j^{r+1}(f)}.$$

Hence

$$m^{r+1} \cdot \mathcal{E}_n^p \subset m \cdot \text{Im } M_{j^{r+1}(f)} \subset m \cdot \text{Im } M_f + m^{r+2} \cdot \mathcal{E}_n^p.$$

Nakayama's lemma implies

$$m^{r+1} \cdot \mathcal{E}_n^p \subset m \cdot \text{Im } M_f.$$

It follows from a result of J. C. Tougeron [2, Théorème VIII 3.6] that there exists  $N \in \mathbb{N}$  such that  $j^N(f)$  is  $G$ -sufficient. If  $N \leq r$  the proof is finished. Suppose  $N > r$ . By (a), there exist  $(g_1, \tau_1) \in \mathcal{E}(n)$  and  $\phi \in m^{N+1} \cdot \mathcal{E}_n^p$  such that

$$\begin{aligned} z &= (g_1, \tau_1) \cdot j^N(f) + \phi; \text{ hence} \\ z &= (g_1, \tau_1) \cdot [j^N(f) + (g_1, \tau_1)^{-1} \cdot \phi]. \end{aligned}$$

Since  $\phi \in m^{N+1} \cdot \mathcal{E}_n^p$ ,  $(g_1, \tau_1)^{-1} \cdot \phi \in m^{N+1} \cdot \mathcal{E}_n^p$ . But  $j^N(f)$  is  $G$ -sufficient, consequently there exists  $(g_2, \tau_2) \in \mathcal{E}(n)$  such that

$$j^N(f) + (g_1, \tau_1)^{-1} \cdot \phi = (g_2, \tau_2) \cdot f.$$

Hence

$$z = (g_1, \tau_1) \cdot (g_2, \tau_2) \cdot f .$$

DEFINITION 2. Let  $f \in m$ . We say that  $f$  is  $r$ -determined if  $j^r(f)$  is  $C^\infty$ -sufficient (i.e.,  $G$ -sufficient with  $G = \{e\}$ ).

From Theorem 1 we deduce the following two results of J. N. Mather [1], stated as follows in [3, Theorem 2.6 and Corollary 2.10]:

THEOREM 2. Let  $f \in m$  and  $I_f$  be the ideal generated in  $\mathcal{E}_n$  by the partial derivatives of  $f$ . If

$$m^r \subset m \cdot I_f + m^{r+1} ,$$

then  $f$  is  $r$ -determined.

THEOREM 3. Let  $f \in m$  be  $r$ -determined. Then

$$m^{r+1} \subset m \cdot I_f .$$

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