ALGEBRAIC AUTOMORPHISMS OF ALGEBRAIC GROUPS WITH STABLE MAXIMAL TORI

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Let $T_1$ and $T_2$ be maximal tori of a connected linear algebraic group $G \subseteq GL(n, \kappa)$, and suppose some (algebraic group) automorphism $\sigma$ of $G$ stabilizes both $T_1$ and $T_2$. Suppose further that $\sigma$ also stabilizes two Borel subgroups, $B_1$ and $B_2$, of $G$. This paper is about the following natural questions:

1. Are $T_1$ and $T_2$ conjugate by a $\sigma$-fixed point of $G$?
2. Are $B_1$ and $B_2$ conjugate by a $\sigma$-fixed point of $G$?
3. If $T_i \subseteq B_i$, $(i = 1, 2)$, are the $T_i$ and $B_i$ respectively conjugate by a single $\sigma$-fixed point of $G$?
4. Are at least $T_1$ and $T_2$ described in (3) above conjugate by a $\sigma$-fixed point of $G$?

In this paper is treated the case in which $\sigma$ is an algebraic automorphism. If either $p = \text{char } \kappa = 0$ or $\sigma$ is semisimple, then the answer to (4) above is yes; but there are counterexamples for (1), (2), and (3). (See below, Counterexamples A-1 and B.) Moreover, if both $p > 0$ and $\sigma$ is not semisimple, then there is also a counterexample for question (4). (See below, Counterexample C.)

Incidental in the proofs is the simple result that when $\sigma$ is algebraic, a $\sigma$-stable maximal torus is pointwise fixed by some finite power of $\sigma$, and by $\sigma$ itself for $p = 0$, $\sigma$ unipotent (Theorem 1).

Robert Steinberg has studied the questions above in [3], for the case that $\sigma$ has finite fixed-point set in $G$, finding that the answers to questions (2), (3), and (4) are all yes. There is a counterexample for question (1) in the finite fixed-point set case, when the $\sigma$-stable maximal tori are not respectively contained in $\sigma$-stable Borel subgroups. (See below, Counterexample A-2.)

When $\sigma$ is an algebraic automorphism of a general algebraic group $G$, its fixed-point set may be infinite. In fact, Steinberg shows (by [3], 10.10) that if $\sigma$ is algebraic with finite fixed-point set, then $G$ is necessarily solvable.

Throughout the paper the (now standard) terminology and basic results of Borel ([1] and [2]) are used, including the name Borel subgroup for a maximal solvable connected subgroup. In addition the mnemonic clag is used for a connected linear algebraic group, and the expression "the pair $T \subseteq B$" for a maximal torus $T$ and a Borel subgroup $B$ containing $T$.

In all of the following theorems, $G$ is a clag and $\sigma$ an algebraic automorphism of $G$. 
THEOREM 1. If $G$ has a $\sigma$-stable maximal torus $T$, then $T$ is pointwise-fixed by some power $\sigma^n$ of $\sigma$. If $p = 0$ and $\sigma$ is unipotent, then $T$ is pointwise fixed by $\sigma$.

Proof. Since $\sigma$ is an algebraic automorphism of $G$, there is a closed linear algebraic group $G$ with $G \Delta G$ and an element $s \in G$ such that $\sigma(g) = sgs^{-1}$ for each $g \in G$. (In fact this may be taken as the definition for an algebraic automorphism of $G$.)

Form the algebraic group generated by $T$ and $s$, $\mathcal{A}(T, s) = \mathcal{A}$ in $G$ (see [1], §3). $T$ is normalized by $s$, so $T$ is normal in $\mathcal{A}$. Moreover, $T$ is a torus in $\mathcal{A}$, and so is contained in a maximal torus of $\mathcal{A}$. Thus $T$ is contained in every maximal torus of $\mathcal{A}$, hence is contained centrally in every Borel subgroup of $\mathcal{A}$ by ([1], §18, 18.1). $T$ is therefore central in $\mathcal{A}$ by ([1], §18, 18.5).

Now $s \in \mathcal{A}$ implies some power $s^n$ of $s$ is in $\mathcal{A}$, whence $s^n$ centralizes $T$. Equivalently, $\sigma^n$ fixes $T$ pointwise.

Suppose now that $p = 0$ and $\sigma$ is unipotent. Since $s^n$ centralizes $T$; so does $\mathcal{A}(s^n)$. Now $\sigma$ unipotent implies $s$ unipotent; and for $p = 0$, $\mathcal{A}(s^n) = \mathcal{A}(s)$ (see [1], 8.2). Thus $s$ also centralizes $T$, i.e., $\sigma$ fixes $T$ pointwise.

THEOREM 2. Let $G$ be solvable, and let either $p = 0$ or $\sigma$ be semisimple. Then two $\sigma$-stable maximal tori $T_i$ of $G$ ($i = 1, 2$) are conjugate by a $\sigma$-fixed point of $G$.

Proof. (1) Since $\sigma$ has finite order, say $n$ ($n$ is prime to $p$ when $p > 0$), on $T_1$ and $T_2$, it may be assumed without change in hypothesis that $\sigma$ has such finite order on all of $G$, by replacing $G$ with $(G_{\sigma^n})$, the connected component of the set of $\sigma^n$-fixed points in $G$.

(2) Let $U$ be the unipotent part of $G$, and let $V$ be the unipotent part of $C(T_1)$, the Cartan subgroup of $T_1$. There exists $u \in U$ such that $uT_1u^{-1} = T_2$, and for any such $u$, $u^{-1}\cdot \sigma(u) \in V$. Therefore it suffices to show that whenever there is a $u$ with $u^{-1}\cdot \sigma(u) \in V$, then there must exist $v \in V$ with $u^{-1}\cdot \sigma(u) = v^{-1}\cdot \sigma(v)$. For in that case, $uv^{-1}$ is $\sigma$-fixed with $uv^{-1}T_1uv^{-1} = T_2$.

In view of (1) and (2) it suffices to prove the following lemma:

LEMMA 3. Let $G$ be a unipotent cay with automorphism $\sigma$ of finite order $n$ (prime to $p$ when $p > 0$). If $G$ has a $\sigma$-stable subcay $H$, and an element $g \in G$ such that $g^{-1}\cdot \sigma(g) \in H$, then $\exists h \in H$ such that $g^{-1}\cdot \sigma(h) = h^{-1}\cdot \sigma(h)$.

Proof. For any subset $X$ of $G$, denote by $X_\sigma$ the $\sigma$-fixed point
set of $X$; and for any element $x \in G$, set $\alpha(x) = x^{-1} \cdot \sigma(x)$. There is no non-identity element of the form $\alpha(x)$ in $G_\sigma$, because if $\alpha(x) \in G_\sigma$ for some $x \in G$, then

$$(\alpha(x))^n = \alpha(x) \cdot \sigma(\alpha(x)) \cdot \sigma^2(\alpha(x)) \cdots \sigma^{n-1}(\alpha(x))$$
$$= x^{-1} \cdot \sigma(x) \cdot \sigma(x^{-1}) \cdot \sigma^2(x) \sigma^2(x^{-1}) \cdots \sigma^{n-1}(x^{-1}) \sigma^n(x)$$
$$= x^{-1} \sigma^n(x) = e;$$

but only the identity element can be both unipotent and of order $n$.

**Case I.** $H$ normal in $G$. $H$ is unipotent, hence nilpotent, so one may use induction on the length $l$ of the lower central series for $H$.

If $l = 1$, then $H$ is commutative, so $\alpha|_H$ is an endomorphism of $H$ with kernel $H_0$ and image $\alpha(H)$. Therefore $\dim H = \dim H_0 + \dim \alpha(H)$, and $H_0 \cap \alpha(H) = \{e\}$. So $H = H_0 \cdot \alpha(H)$ as a direct product.

Thus $\exists h_1 \in H_0$, $h_2 \in H$ such that $\alpha(g) = h_1 \cdot \alpha(h_2)$. That is, $g^{-1} \cdot \sigma(g) = h_1 \cdot h_2^{-1} \cdot \sigma(h_2) = h_2^{-1} \cdot h_1 \cdot \sigma(h_2)$; and this implies that

$$h_2 \cdot g^{-1} \cdot \sigma(g) \sigma(h_2^{-1}) = (g h_2^{-1})^{-1} \cdot \sigma(g h_2^{-1}) = \alpha(g h_2^{-1}) = h_1 \in H_0 \subseteq G_\sigma.$$

So $\alpha(g h_2^{-1}) = e = h_1$ and $\alpha(g) = \alpha(h_2)$.

Now suppose $l > 1$. If $\alpha(g) \in H^1$, then by induction $\exists h \in H^1$ with $\alpha(g) = \alpha(h)$. So suppose $\alpha(g) \not\in H^1$. Then $\overline{\alpha(g)} \neq \overline{e}$ in $\overline{H} = \pi_{H^1}(H)$, where $\pi_{H^1}$ is the projection of $G$ with kernel $H^1$. $\overline{H}$ is commutative, and $\overline{\alpha(g)} = \overline{g^{-1} \cdot \sigma(g)} = \overline{g^{-1}} \cdot \overline{\sigma(g)} = \overline{\alpha(g)}$, so as in the case for $l = 1$, $\exists \overline{h} \in \overline{H}$ such that $\overline{\alpha(g)} = \overline{\alpha(h)}$. That is,

$$\overline{g^{-1} \cdot \sigma(g)} = \overline{h^{-1} \cdot \sigma(h)}$$

and $\overline{(g h^{-1})^{-1} \cdot \sigma(g h^{-1})} = \overline{e}$.

In other words, $\alpha(g h^{-1}) \in H^1$, whence by induction $\exists h' \in H^1$ such that $\alpha(g h^{-1}) = \alpha(h')$. We now have $\overline{(g h^{-1})^{-1} \cdot \sigma(g h^{-1})} = \overline{h^{-1} \cdot \sigma(g) \sigma(h^{-1})} = h'^{-1} \cdot \sigma(h')$, implying $g^{-1} \cdot \sigma(g) = h'^{-1} \cdot h'^{-1} \cdot \sigma(h') \sigma(h) = (h'h)^{-1} \sigma(h'h)$. Hence $\alpha(g) = \alpha(h'h) \in \alpha(H)$.

**Case II.** If $H$ is not normal in $G$, set $H = G_i$, and let $G_i$ be the connected normalizer in $G$ of $G_{i-1}$, for $i \geq 2$. Since a proper subclag of a nilpotent clag is properly contained in its connected normalizer by ([1], 20.3), there is a chain of $\sigma$-stable subclags of $G$:

$$H = G_1 \Delta G_2 \Delta \cdots \Delta G_r = G,$$

each of which is a normal and proper subclag of the following one.

Now the element $g \in G$ with which we are concerned is contained in $G_i$ for some (minimal) $i$, with $i \geq 2$. Since $\alpha(g) \in H \subseteq G_{i-1}$, and $G_{i-1}$
is normal in $G_i$, there is by Case I an element $g_{i-1} \in G_{i-1}$ for which $\alpha(g) = \alpha(g_{i-1})$.

If $(i - 1) \geq 2$, apply Case I again to obtain an element $g_{i-2} \in G_{i-2}$ for which $\alpha(g_{i-1}) = \alpha(g_{i-2})$, since $\alpha(g_{i-1}) \in H \subseteq G_{i-2}$, and $G_{i-2}$ is normal in $G_{i-1}$.

Similarly, by a total of $(i - 1)$ application of Case I, one obtains an element $h \in H = G_i = G_{i-(i-1)}$, for which $\alpha(h) = \alpha(g) = \alpha(g_0) = \cdots = \alpha(g_{i-1}) = \alpha(g)$.

This completes the proof of Theorem 2.

**Theorem 4.** Let $G$ have two $\sigma$-stable pairs, $T_i \subseteq B_i$ $(i = 1, 2)$. If $p = 0$, or if $\sigma$ is semisimple, then the $T_i$ $(i = 1, 2)$ are conjugate by a $\sigma$-fixed point of $G$.

**Proof.** Let $T \subseteq B$ be any $\sigma$-stable pair of $G$.

First consider $\sigma_s$, the semisimple component of $\sigma$. (Any $\sigma$-stable $\theta$-flag is also $\sigma_s$-stable.)

Let $S$ be a maximal torus of $(G_{\sigma_s})_0$. By ([3], 7.4), $S \subseteq \sigma_s$-stable Borel subgroup $R$ of $G$. $S$ is also a maximal torus of $(R_{\sigma_s})_0$.

By ([3], 7.6), $R$ has a $\sigma_s$-stable maximal torus $Q$. Now $R = Q \cdot V$ (semi-direct product), where $V$ is the unipotent part of $R$. So any $\sigma_s$-fixed point $f \in R$ has Jordan decomposition $f = q \cdot v$ for some $q \in Q$, $v \in V$. Thus $f = \sigma_s(f) = \sigma_s(q)\sigma_s(v)$, with $\sigma_s(q) \in Q$, $\sigma_s(v) \in V$, whence $\sigma_s(q) = q$ and $\sigma_s(v) = v$. Hence $(R_{\sigma_s})_0 = (Q_{\sigma_s})_0 \cdot (V_{\sigma_s})_0$, and $(Q_{\sigma_s})_0$ is a maximal torus of $(R_{\sigma_s})_0$. Thus $\dim (Q_{\sigma_s})_0 = \dim S$, so $(Q_{\sigma_s})_0$ is also a maximal torus of $(G_{\sigma_s})_0$.

Now $\exists g \in G$ such that $gRg^{-1} = B$, $gQg^{-1} = T$, and (since $Q \subseteq R$, $T \subseteq B$ are all $\sigma_s$-stable), $g^{-1} \cdot \sigma_s(g) \in N_\theta(R) \cap N_\theta(Q) = R \cap N_\theta(Q) = C(Q)$, the Cartan subgroup of $Q$ in $G$. This implies that $g(Q_{\sigma_s})_0g^{-1} = (T_{\sigma_s})_0$, so that $\dim (T_{\sigma_s})_0 = \dim (Q_{\sigma_s})_0$, and $(T_{\sigma_s})_0$ is itself a maximal torus of $(G_{\sigma_s})_0$.

Moreover, $(T_{\sigma_s})_0$ is a torus of $(G_{\sigma_s})_0$, because $T \subseteq (G_{\sigma_s})_0$. Therefore $(T_{\sigma_s})_0$ is a maximal torus of $(G_{\sigma_s})_0 \cap (G_{\sigma_s})_0 = (G_{\sigma_s})_0$. Thus the $[(T_{\sigma_s})_0$ are both maximal tori of $(G_{\sigma_s})_0$; so they are conjugate by a fixed point $y \in (G_{\sigma_s})_0$, that is, $y(T_{\sigma_s})_0y^{-1} = (T_{\sigma_s})_0$. Set $T_z = yT_zy^{-1}$.


Both $T_2$ and $T_3$ belong to the connected centralizer $Z$ of $(T_{\sigma g})_0$ in $G$. By ([4], Cor. 4), $Z$ is solvable. Also, $Z$ is $\sigma$-stable with maximal tori $T_2$ and $T_3$, so by (Thm. 2), $T_2$ and $T_3$ are conjugate under a $\sigma$-fixed point $z \in Z$; that is, $zT_2z^{-1} = T_3$. Then for $g = y^{-1}z$, $g$ is a $\sigma$-fixed point of $G$ for which $gT_3g^{-1} = T_1$.

[Note on the field of definition $\kappa$: If $\kappa$ is algebraically closed, the point of conjugacy in Theorems 2 and 4 may be taken to be $\kappa$-rational; and theorems analogous to Theorems 2 and 4 hold for $\kappa$-groups. The proofs are mechanical glosses on those here and are found in the author’s Ph. D. thesis.]

Counterexample A-1. $\sigma$ is semisimple; $G$ has two $\sigma$-stable maximal tori which are not both contained in $\sigma$-stable Borel subgroups, and are not conjugate by a $\sigma$-fixed point:

Take $G = SL(2, \Omega)$, $p \neq 2$. Let $T_1$ consist of matrices of the form

$$\begin{bmatrix}
\alpha & 0 \\
0 & \frac{1}{\alpha}
\end{bmatrix}, \quad \alpha \neq 0;$$

and let $T_2$ be given by matrices of the form

$$\begin{bmatrix}
\frac{1}{2}(\alpha + \frac{1}{\alpha}), & \frac{1}{2}(\alpha - \frac{1}{\alpha}) \\
\frac{1}{2}(\alpha - \frac{1}{\alpha}), & \frac{1}{2}(\alpha + \frac{1}{\alpha})
\end{bmatrix}, \quad \alpha \neq 0.$$

($T_1$ is the maximal torus of $G$ which has diagonal form; $T_2$ is the conjugate of $T_1$ by the element

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \in G.$$  

Take $\sigma = \text{Inn}_G g$, where $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The effect of $\sigma$ is to interchange diagonally the corner entries in each matrix of $G$. The $\sigma$-fixed point set $G_\sigma$ of $G$ is therefore

$$G_\sigma = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a^2 - b^2 = 1 \right\}.$$  

$G_\sigma$ is infinite; and since $\sigma^2 = 1$ and $p \neq 2$, $\sigma$ is semisimple.

Now $T_3$ is pointwise $\sigma$-fixed, and $T_1$ is not, although it is $\sigma$-stable. So $T_1$ and $T_3$ cannot be conjugate by a $\sigma$-fixed point of $G$.

(Note. The only Borel subgroups of $G$ containing $T_1$ are the
upper and lower triangular matrix groups in $G$, and $\sigma$ leaves neither of these stable, but maps one onto the other.)

**Counterexample A-2.** $\sigma$ (nonalgebraic) is the Frobenius map for $p = 2$, having finite fixed-point set; $G$ has two $\sigma$-stable maximal tori which are not both contained in a $\sigma$-stable Borel subgroup, and are not conjugate by a $\sigma$-fixed point.

Take $G = SL(2, \mathbb{Q})$, $p = 2$. Let

$$T_1 = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} : 0 \neq \alpha \in \Omega \right\};$$

and let

$$T_2 = \left\{ \begin{bmatrix} \alpha + a(\alpha + \frac{1}{\alpha}), \left(\alpha + \frac{1}{\alpha}\right) \\ (\alpha + \frac{1}{\alpha}), \frac{1}{\alpha} + a\left(\alpha + \frac{1}{\alpha}\right) \end{bmatrix} : 0 \neq \alpha \in \Omega \right\};$$

$a$ fixed such that $a^2 + a + 1 = 0$.

For $\sigma$ take the Frobenius map $\sigma: (x_{ij}) \rightarrow (x_{ij}^p)$. $T_1$ is clearly $\sigma$-stable. $T_2 = xT_1x^{-1}$, where

$$x = \begin{bmatrix} a & (a + 1) \\ (a + 1) & a \end{bmatrix}, \text{ and } x^{-1} \cdot \sigma(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in N(T_1),$$

so $T_2$ is $\sigma$-stable too.

It can easily be seen that $T_1$ and $T_2$ are not conjugate by a $\sigma$-fixed point of $G$, since there are only 6 fixed points.

**Counterexample B.** $\sigma$ is semisimple; $G$ has two $\sigma$-stable pairs; but the $\sigma$-stable Borel subgroups are not conjugate by a $\sigma$-fixed point.

Take $G$ and $\sigma$ as in Counterexample A-1 ($p \neq 2$). Let

$$T = T_2 = \left\{ \begin{bmatrix} \frac{1}{2}(\alpha + \frac{1}{\alpha}) & \frac{1}{2}(\alpha - \frac{1}{\alpha}) \\ \frac{1}{2}(\alpha - \frac{1}{\alpha}) & \frac{1}{2}(\alpha + \frac{1}{\alpha}) \end{bmatrix} : 0 \neq \alpha \in \Omega \right\}.$$

Set $\Delta = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \frac{1}{\alpha} \end{bmatrix} : \alpha, \beta \in \Omega, \alpha \neq 0 \right\} \subseteq G$, a Borel subgroup of $G$. Set
$x = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \in G$, and $y = \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \in G$.

Take

$$B_i = x \Delta x^{-1} = \begin{cases} \frac{1}{2}(\alpha + \frac{1}{\alpha}) - \beta, \frac{1}{2}(\alpha - \frac{1}{\alpha}) + \beta \\ \frac{1}{2}(\alpha - \frac{1}{\alpha}) - \beta, \frac{1}{2}(\alpha + \frac{1}{\alpha}) + \beta \end{cases} : \alpha, \beta \in \Omega, \alpha \neq 0$$

and

$$B_2 = y \Delta y^{-1} = \begin{cases} \frac{1}{2}(\frac{1}{\alpha} + \alpha) + \beta, \frac{1}{2}(\frac{1}{\alpha} - \alpha) + \beta \\ \frac{1}{2}(\frac{1}{\alpha} - \alpha) - \beta, \frac{1}{2}(\frac{1}{\alpha} + \alpha) - \beta \end{cases} : \alpha, \beta \in \Omega, \alpha \neq 0$$.

Recalling that $\sigma$ diagonally interchanges the entries of a matrix, one sees that $B_1$ and $B_2$ are $\sigma$-stable, and $T$ is pointwise $\sigma$-fixed. Moreover, $T$ is clearly a maximal torus of both $B_1$ and $B_2$ (i.e., when $\beta = 0$). So $T \nparallel B_1$ and $T \nparallel B_2$ are $\sigma$-stable pairs.

Suppose now that $B_1$, $B_2$ are conjugate by a $\sigma$-fixed point $f \in G_\sigma$, i.e., that $B_1 = fB_2f^{-1}$. Then $B_i = x \Delta x^{-1} = fB_2f^{-1} = fy \Delta y^{-1}f^{-1} \Rightarrow \Delta = x^{-1}fy \Delta y^{-1}f^{-1}x \Rightarrow x^{-1}fy \in N_\sigma(\Delta) = \Delta$.

Say that $x^{-1}fy = b = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \begin{bmatrix} \beta \\ 1/\alpha \end{bmatrix} \in \Delta$, and $f = \begin{bmatrix} \gamma & \delta \\ \bar{\delta} & \bar{\gamma} \end{bmatrix} \in G_\sigma$, for some $\alpha, \beta, \gamma, \delta \in \Omega$ with $\gamma^2 - \delta^2 = 1$, and $\alpha \neq 0$. Then

$$x^{-1}fy = b \implies fy = xb$$

$$\begin{bmatrix} \gamma & \delta \\ \bar{\delta} & \bar{\gamma} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \begin{bmatrix} \beta \\ 1/\alpha \end{bmatrix}$$

$$\begin{bmatrix} \gamma - \delta & \frac{1}{2}(\gamma + \delta) \\ \delta - \gamma & \frac{1}{2}(\gamma + \delta) \end{bmatrix} = \begin{bmatrix} \alpha & \beta - \frac{1}{2\alpha} \\ \alpha & \beta + \frac{1}{2\alpha} \end{bmatrix}$$

$$\implies \gamma - \delta = \delta - \gamma \implies \gamma = \delta \implies \gamma^2 - \delta^2 = 0,$$

a contradiction of the fact that $\gamma^2 - \delta^2 = 1$.

Thus $B_i$ and $B_2$ cannot be conjugate by a $\sigma$-fixed point of $G$.

**Counterexample C.** $G$ solvable, $\sigma$ unipotent, and $p > 0$. $G$ has
two $\sigma$-stable maximal tori which are not conjugate by a $\sigma$-fixed point.

Take $p = 2$.

Let $T$ be the torus $\subseteq GL(6, \Omega)$ consisting of diagonal matrices $t$ of the form

$$
t = \begin{bmatrix}
\tau_1 \\
\tau_2 \\
\tau_1 \\
\tau_2 \\
\tau_1 \\
\tau_2 \\
\end{bmatrix}, \quad \tau_1, \tau_2 \in \Omega, \quad \tau_1 \tau_2 \neq 0.
$$

Let $U$ be the unipotent clag consisting of upper triangular matrices $u$ of the form

$$
u = \begin{bmatrix}
1 & \alpha & x \\
1 & \beta & y \\
1 & \alpha & 1 \\
\end{bmatrix}, \quad \alpha, \beta, x, y \in \Omega,
$$
satisfying: $x + y - \alpha \beta = 0$.

The reader may verify that $U$ is closed under multiplication, and since $u^t = e$, $\forall u \in U$, $U$ is also closed under inverses. Hence $U$ is well-defined.

Moreover, $U$ is normalized by $T$, as the reader again may verify.

One may therefore form the solvable clag $G = T \cdot U$ (semi-direct product).

Let the automorphism $\sigma$ on $t \cdot u \in G$ be given by the following action on the entries of $t$ and $u$

$$
\sigma: \tau_1 \longleftrightarrow \tau_2 \\
\alpha \longleftrightarrow \beta \\
x \longleftrightarrow y
$$

$\sigma$ is thus conjugation by the permutation matrix:

$$
s = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 1 \\
1 & 0 \\
\end{bmatrix}.
$$
So $s$ and $\sigma$ are unipotent of order 2.

$T$ is a $\sigma$-stable maximal torus of $G$, whose Cartan subgroup is $C(T) = T \times C(T)_u$, where

$$C(T)_u = \begin{pmatrix} 1 & 0 & x & 0 \\ 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} : x \in \Omega$$

Now if $u \in U$, then $uTu^{-1}$ is $\sigma$-stable if and only if $u^{-1}\cdot\sigma(u) \in C(T)_u$. Moreover, there exists a $\sigma$-fixed element $f \in U_u$ such that $uTu^{-1} = fTf^{-1}$ if and only if $f^{-1}u \in C(T)_u$; i.e., if and only if $\exists c \in C(T)_u$ such that $uc^{-1} = f$ is $\sigma$-fixed.

However, all $c \in C(T)_u$ are $\sigma$-fixed; So a $\sigma$-stable maximal torus $uTu^{-1}$ of $G$ is conjugate to $T$ by a $\sigma$-fixed point if and only if $u$ itself is $\sigma$-fixed.

However, for the unipotent matrix

$$u = \begin{pmatrix} 1 & \alpha & x & 0 \\ 1 & \alpha & 0 & x \\ 0 & 1 & 1 & \alpha \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

satisfying $x + y - \alpha^2 = 0, \alpha \neq 0$, one gets

$$u^{-1} \cdot \sigma(u) = \begin{pmatrix} 1 & 0 & -x + y & 0 \\ 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$ 

That is, $u^{-1} \cdot \sigma(u) \in C(T)_u$, so $uTu^{-1}$ is $\sigma$-stable. But $u$ is not $\sigma$-fixed, so $T$ and $uTu^{-1}$ are not conjugate by a $\sigma$-fixed element of $G$.

(Note. This counterexample in $p = 2$ is due to D. Winter. The present author has generalized it in a separate paper for all $p > 0$. The resulting group may be of some interest in itself.)
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