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Let T_1 and T_2 be maximal tori of a connected linear algebraic group $G \subseteq GL(n, \kappa)$, and suppose some (algebraic group) automorphism σ of G stabilizes both T_1 and T_2. Suppose further that σ also stabilizes two Borel subgroups, B_1 and B_2, of G. This paper is about the following natural questions:

(1) Are T_1 and T_2 conjugate by a σ-fixed point of G?

(2) Are B_1 and B_2 conjugate by a σ-fixed point of G?

(3) If $T_i \subseteq B_i$, $(i = 1, 2)$, are the T_i and B_i respectively conjugate by a single σ-fixed point of G?

(4) Are at least T_1 and T_2 described in (3) above conjugate by a σ-fixed point of G?

In this paper is treated the case in which σ is an algebraic automorphism. If either $p = \text{char } \kappa = 0$ or σ is semisimple, then the answer to (4) above is yes; but there are counterexamples for (1), (2), and (3). (See below, Counterexamples A-1 and B.) Moreover, if both $p > 0$ and σ is not semisimple, then there is also a counterexample for question (4). (See below, Counterexample C.)

Incidental in the proofs is the simple result that when σ is algebraic, a σ-stable maximal torus is pointwise fixed by some finite power of σ, and by σ itself for $p = 0$, σ unipotent (Theorem 1).

Robert Steinberg has studied the questions above in [3], for the case that σ has finite fixed-point set in G, finding that the answers to questions (2), (3), and (4) are all yes. There is a counterexample for question (1) in the finite fixed-point set case, when the σ-stable maximal tori are not respectively contained in σ-stable Borel subgroups. (See below, Counterexample A-2.)

When σ is an algebraic automorphism of a general algebraic group G, its fixed-point set may be infinite. In fact, Steinberg shows (by [3], 10.10) that if σ is algebraic with finite fixed-point set, then G is necessarily solvable.

Throughout the paper the (now standard) terminology and basic results of Borel ([1] and [2]) are used, including the name *Borel subgroup* for a maximal solvable connected subgroup. In addition the mnemonic *clag* is used for a connected linear algebraic group, and the expression "the pair $T \subseteq B$" for a maximal torus T and a Borel subgroup B containing T.

In all of the following theorems, G is a clag and σ an algebraic automorphism of G.

THEOREM 1. *If G has a σ-stable maximal torus T, then T is pointwise-fixed by some power σ^n of σ. If $p = 0$ and σ is unipotent, then T is pointwise fixed by σ.*

Proof. Since σ is an algebraic automorphism of G, there is a closed linear algebraic group $\mathscr{G}$ with $G \Delta \mathscr{G}$ and an element $s \in \mathscr{G}$ such that $\sigma(g) = sgs^{-1}$ for each $g \in G$. (In fact this may be taken as the definition for an algebraic automorphism of G.)

Form the algebraic group generated by T and s, $\mathscr{A}(T, s) = \mathscr{A}$ in $\mathscr{G}$ (see [1], §3). T is normalized by s, so T is normal in $\mathscr{A}$. Moreover, T is a torus in $\mathscr{A}_0$, and so is contained in a maximal torus of $\mathscr{A}_0$. Thus T is contained in every maximal torus of $\mathscr{A}_0$, hence is contained centrally in every Borel subgroup of $\mathscr{A}_0$ by ([1], §18, 18.1). T is therefore central in $\mathscr{A}_0$ by ([1], §18, 18.5).

Now $s \in \mathscr{A} \Rightarrow$ some power s^n of s is in $\mathscr{A}_0$, whence s^n centralizes T. Equivalently, σ^n fixes T pointwise.

Suppose now that $p = 0$ and σ is unipotent. Since s^n centralizes T; so does $\mathscr{A}(s^n)$. Now σ unipotent $\Rightarrow s$ unipotent; and for $p = 0$, $\mathscr{A}(s^n) = \mathscr{A}(s)$ (see [1], 8.2). Thus s also centralizes T, i.e., σ fixes T pointwise.

THEOREM 2. *Let G be solvable, and let either $p = 0$ or σ be semisimple. Then two σ-stable maximal tori T_i of G $(i = 1, 2)$ are conjugate by a σ-fixed point of G.*

Proof. (1) Since σ has finite order, say n (n is prime to p when $p > 0$), on T_1 and T_2, it may be assumed without change in hypothesis that σ has such finite order on all of G, by replacing G with $(G_{\sigma^n})_0$, the connected component of the set of σ^n-fixed points in G.

(2) Let U be the unipotent part of G, and let V be the unipotent part of $C(T_1)$, the Cartan subgroup of T_1. There exists $u \in U$ such that $uT_1u^{-1} = T_2$, and for any such u, $u^{-1}\cdot\sigma(u) \in V$. Therefore it suffices to show that whenever there is a u with $u^{-1}\cdot\sigma(u) \in V$, then there must exist $v \in V$ with $u^{-1}\cdot\sigma(u) = v^{-1}\cdot\sigma(v)$. For in that case, uv^{-1} is σ-fixed with $uv^{-1}T_1vu^{-1} = T_2$.

In view of (1) and (2) it suffices to prove the following lemma:

LEMMA 3. *Let G be a unipotent clag with automorphism σ of finite order n* (prime to p when $p > 0$). *If G has a σ-stable subclag H, and an element $g \in G$ such that $g^{-1}\cdot\sigma(g) \in H$, then $\exists h \in H$ such that $g^{-1}\cdot\sigma(g) = h^{-1}\cdot\sigma(h)$.*

Proof. For any subset X of G, denote by X_σ the σ-fixed point set of X; and for any element $x \in G$, set $\alpha(x) = x^{-1} \cdot \sigma(x)$. There is no non-identity element of the form $\alpha(x)$ in G_σ, because if $\alpha(x) \in G_\sigma$ for some $x \in G$, then

$$\begin{aligned}(\alpha(x))^n &= \alpha(x) \cdot \sigma(\alpha(x)) \cdot \sigma^2(\alpha(x)) \cdots \sigma^{n-1}(\alpha(x)) \\ &= x^{-1} \cdot \sigma(x) \cdot \sigma(x^{-1}) \cdot \sigma^2(x) \sigma^2(x^{-1}) \cdots \sigma^{n-1}(x^{-1}) \sigma^n(x) \\ &= x^{-1} \sigma^n(x) = e \ ;\end{aligned}$$

but only the identity element can be both unipotent and of order n.

Case I. H normal in G. H is unipotent, hence nilpotent, so one may use induction on the length l of the lower central series for H.

If $l = 1$, then H is commutative, so $\alpha|_H$ is an endomorphism of H with kernel H_σ and image $\alpha(H)$. Therefore $\dim H = \dim H_0 + \dim \alpha(H)$, and $H_\sigma \cap \alpha(H) = \{e\}$. So $H = H_\sigma \cdot \alpha(H)$ as a direct product.

Thus $\exists h_1 \in H_\sigma$, $h_2 \in H$ such that $\alpha(g) = h_1 \cdot \alpha(h_2)$. That is, $g^{-1} \cdot \sigma(g) = h_1 \cdot h_2^{-1} \cdot \sigma(h_2) = h_2^{-1} \cdot h_1 \cdot \sigma(h_2)$; and this implies that

$$h_2 \cdot g^{-1} \cdot \sigma(g) \sigma(h_2^{-1}) = (gh_2^{-1})^{-1} \cdot \sigma(gh_2^{-1}) = \alpha(gh_2^{-1}) = h_1 \in H_\sigma \subseteqq G_\sigma \, .$$

So $\alpha(gh_2^{-1}) = e = h_1$ and $\alpha(g) = \alpha(h_2)$.

Now suppose $l > 1$. If $\alpha(g) \in H^1$, then by induction $\exists h \in H^1$ with $\alpha(g) = \alpha(h)$. So suppose $\alpha(g) \notin H^1$. Then $\overline{\alpha(g)} \neq \bar{e}$ in $\bar{H} = \pi_{H^1}(H)$, where π_{H^1} is the projection of G with kernel H^1. $\bar{H}$ is commutative, and $\overline{\alpha(g)} = \overline{g^{-1} \cdot \sigma(g)} = \bar{g}^{-1} \cdot \bar{\sigma}(\bar{g}) = \bar{\alpha}(\bar{g})$, so as in the case for $l = 1$, $\exists \bar{h} \in \bar{H}$ such that $\overline{\alpha(g)} = \overline{\alpha(h)}$. That is,

$$\bar{g}^{-1} \cdot \bar{\sigma}(\bar{g}) = \bar{h}^{-1} \cdot \bar{\sigma}(\bar{h}) \ , \quad \text{and} \quad \overline{(gh^{-1})^{-1} \cdot \sigma(gh^{-1})} = \bar{e} \, .$$

In other words, $\alpha(gh^{-1}) \in H^1$, whence by induction $\exists h' \in H^1$ such that $\alpha(gh^{-1}) = \alpha(h')$. We now have $(gh^{-1})^{-1} \sigma(gh^{-1}) = hg^{-1} \sigma(g) \sigma(h)^{-1} = h'^{-1} \cdot \sigma(h')$, implying $g^{-1} \cdot \sigma(g) = h^{-1} h'^{-1} \cdot \sigma(h') \sigma(h) = (h'h)^{-1} \sigma(h'h)$. Hence $\alpha(g) = \alpha(h'h) \in \alpha(H)$.

Case II. If H is not normal in G, set $H = G_1$, and let G_i be the connected normalizer in G of G_{i-1}, for $i \geqq 2$. Since a proper subclag of a nilpotent clag is properly contained in its connected normalizer by ([1], 20.3), there is a chain of σ-stable subclags of G:

$$H = G_1 \underset{+}{\Delta} G_2 \underset{+}{\Delta} \cdots \underset{+}{\Delta} G_r = G \ ,$$

each of which is a normal and proper subclag of the following one.

Now the element $g \in G$ with which we are concerned is contained in G_i for some (minimal) i, with $i \geqq 2$. Since $\alpha(g) \in H \subseteqq G_{i-1}$, and G_{i-1}

is normal in G_i, there is by Case I an element $g_{i-1} \in G_{i-1}$ for which $\alpha(g) = \alpha(g_{i-1})$.

If $(i-1) \geqq 2$, apply Case I again to obtain an element $g_{i-2} \in G_{i-2}$ for which $\alpha(g_{i-1}) = \alpha(g_{i-2})$, since $\alpha(g_{i-1}) \in H \subseteqq G_{i-2}$, and G_{i-2} is normal in G_{i-1}.

Similarly, by a total of $(i-1)$ application of Case I, one obtains an element $h \in H = G_1 = G_{i-(i-1)}$, for which $\alpha(h) = \alpha(g_2) = \alpha(g_3) = \cdots = \alpha(g_{i-1}) = \alpha(g)$.

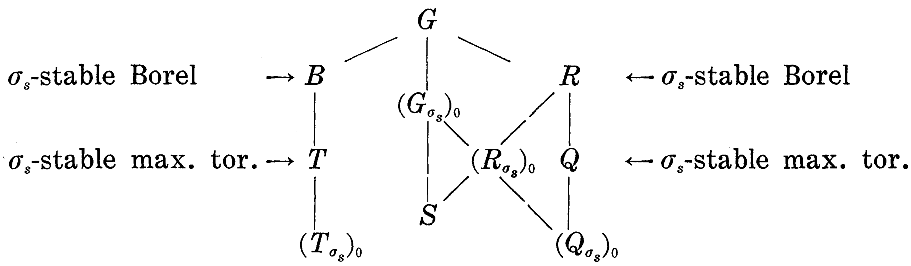
This completes the proof of Theorem 2.

THEOREM 4. *Let G have two σ-stable pairs, $T_i \subseteqq B_i$ $(i = 1, 2)$. If $p = 0$, or if σ is semisimple, then the T_i $(i = 1, 2)$ are conjugate by a σ-fixed point of G.*

Proof. Let $T \subseteqq B$ be any σ-stable pair of G.

First consider σ_s, the semisimple component of σ. (Any σ-stable clag is also σ_s-stable.)

Let S be a maximal torus of $(G_{\sigma_s})_0$. By ([3], 7.4), $S \subseteqq a$ σ_s-stable Borel subgroup R of G. S is also a maximal torus of $(R_{\sigma_s})_0$.



By ([3], 7.6), R has a σ_s-stable maximal torus Q. Now $R = Q \cdot V$ (semi-direct product), where V is the unipotent part of R. So any σ_s-fixed point $f \in R$ has Jordan decomposition $f = q \cdot v$ for some $q \in Q$, $v \in V$. Thus $f = \sigma_s(f) = \sigma_s(q)\sigma_s(v)$, with $\sigma_s(q) \in Q$, $\sigma_s(v) \in V$, whence $\sigma_s(q) = q$ and $\sigma_s(v) = v$. Hence $(R_{\sigma_s})_0 = (Q_{\sigma_s})_0 \cdot (V_{\sigma_s})_0$, and $(Q_{\sigma_s})_0$ is a maximal torus of $(R_{\sigma_s})_0$. Thus $\dim (Q_{\sigma_s})_0 = \dim S$, so $(Q_{\sigma_s})_0$ is also a maximal torus of $(G_{\sigma_s})_0$.

Now $\exists g \in G$ such that $gRg^{-1} = B$, $gQg^{-1} = T$, and (since $Q \subseteqq R$, $T \subseteqq B$ are all σ_s-stable), $g^{-1} \cdot \sigma_s(g) \in N_G(R) \cap N_G(Q) = R \cap N_G(Q) = C(Q)$, the Cartan subgroup of Q in G. This implies that $g(Q_{\sigma_s})_0 g^{-1} = (T_{\sigma_s})_0$, so that $\dim (T_{\sigma_s})_0 = \dim (Q_{\sigma_s})_0$, and $(T_{\sigma_s})_0$ is itself a maximal torus of $(G_{\sigma_s})_0$.

Moreover, $(T_{\sigma_s})_0$ is a torus of $(G_{\sigma_u})_0$, because $T \subseteqq (G_{\sigma_u})_0$. Therefore $(T_{\sigma_s})_0$ is a maximal torus of $(G_{\sigma_s})_0 \cap (G_{\sigma_u})_0 = (G_\sigma)_0$. Thus the $[(T_i)_{\sigma_s}]_0$ are both maximal tori of $(G_\sigma)_0$; so they are conjugate by a fixed point $y \in (G_\sigma)_0$, that is, $y(T_{1_{\sigma_s}})_0 y^{-1} = (T_{2_{\sigma_s}})_0$. Set $T_3 = yT_1y^{-1}$.

Both T_2 and T_3 belong to the connected centralizer Z of $(T_{2_{\sigma_S}})_0$ in G. By ([4], Cor. 4), Z is solvable. Also, Z is σ-stable with maximal tori T_2 and T_3, so by (Thm. 2), T_2 and T_3 are conjugate under a σ-fixed point $z \in Z$; that is, $zT_2z^{-1} = T_3$. Then for $g = y^{-1}z$, g is a σ-fixed point of G for which $gT_2g^{-1} = T_1$.

[Note on the field of definition κ: If κ is algebraically closed, the point of conjugacy in Theorems 2 and 4 may be taken to be κ-rational; and theorems analogous to Theorems 2 and 4 hold for κ-groups. The proofs are mechanical glosses on those here and are found in the author's Ph. D. thesis.]

Counterexample A-1. σ is semisimple; G has two σ-stable maximal tori which are not both contained in σ-stable Borel subgroups, and are not conjugate by a σ-fixed point:

Take $G = SL(2, \Omega)$, $p \neq 2$. Let T_1 consist of matrices of the form

$$\begin{bmatrix} \alpha & 0 \\ 0 & \dfrac{1}{\alpha} \end{bmatrix}, \quad \alpha \neq 0;$$

and let T_2 be given by matrices of the form

$$\begin{bmatrix} \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right), & \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right) \\ \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right), & \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right) \end{bmatrix}, \quad \alpha \neq 0 .$$

$\Bigg($ T_1 is the maximal torus of G which has diagonal form; T_2 is the conjugate of T_1 by the element

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \in G . \Bigg)$$

Take $\sigma = \operatorname{Inn}_G g$, where $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The effect of σ is to interchange diagonally the corner entries in each matrix of G. The σ-fixed point set G_σ of G is therefore

$$G_\sigma = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a^2 - b^2 = 1 \right\} .$$

G_σ is infinite; and since $\sigma^2 = 1$ and $p \neq 2$, σ is semisimple.

Now T_2 is pointwise σ-fixed, and T_1 is not, although it is σ-stable. So T_1 and T_2 cannot be conjugate by a σ-fixed point of G.

(*Note.* The only Borel subgroups of G containing T_1 are the upper and lower triangular matrix groups in G, and σ leaves neither of these stable, but maps one onto the other.)

Counterexample A-2. σ (nonalgebraic) is the Frobenius map for $p = 2$, having finite fixed-point set; G has two σ-stable maximal tori which are not both contained in a σ-stable Borel subgroup, and are not conjugate by a σ-fixed point.

Take $G = SL(2, \Omega)$, $p = 2$. Let

$$T_1 = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \dfrac{1}{\alpha} \end{bmatrix} : 0 \neq \alpha \in \Omega \right\} ;$$

and let

$$T_2 = \left\{ \begin{bmatrix} \alpha + a\left(\alpha + \dfrac{1}{\alpha}\right), & \left(\alpha + \dfrac{1}{\alpha}\right) \\ \left(\alpha + \dfrac{1}{\alpha}\right), & \dfrac{1}{\alpha} + a\left(\alpha + \dfrac{1}{\alpha}\right) \end{bmatrix} : 0 \neq \alpha \in \Omega ; \quad a \text{ fixed such that } a^2 + a + 1 = 0. \right\}$$

For σ take the Frobenius map $\sigma\colon (x_{ij}) \longrightarrow (x_{ij}^2)$. T_1 is clearly σ-stable. $T_2 = xT_1x^{-1}$, where

$$x = \begin{bmatrix} a & (a+1) \\ (a+1) & a \end{bmatrix}, \quad \text{and} \quad x^{-1}\cdot\sigma(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in N(T_1) ,$$

so T_2 is σ-stable too.

It can easily be seen that T_1 and T_2 are not conjugate by a σ-fixed point of G, since there are only 6 fixed points.

Counterexample B. σ is semisimple; G has two σ-stable pairs; but the σ-stable Borel subgroups are not conjugate by a σ-fixed point.

Take G and σ as in Counterexample A-1 ($p \neq 2$). Let

$$T = T_2 = \left\{ \begin{bmatrix} \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right) & \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right) \\ \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right) & \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right) \end{bmatrix} : 0 \neq \alpha \in \Omega \right\} .$$

Set $\Delta = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{bmatrix} : \alpha, \beta \in \Omega,\ \alpha \neq 0 \right\} \subseteqq G$, a Borel subgroup of G. Set

$$x = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ 1 & \dfrac{1}{2} \end{bmatrix} \in G \,, \quad \text{and} \quad y = \begin{bmatrix} 1 & \dfrac{1}{2} \\ -1 & \dfrac{1}{2} \end{bmatrix} \in G \,.$$

Take

$$B_1 = x\Delta x^{-1} = \left\{ \begin{bmatrix} \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right) - \beta, & \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right) + \beta \\ \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right) - \beta, & \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right) + \beta \end{bmatrix} : \alpha, \beta \in \Omega, \ \alpha \neq 0 \right\}$$

and

$$B_2 = y\Delta y^{-1} = \left\{ \begin{bmatrix} \dfrac{1}{2}\left(\dfrac{1}{\alpha} + \alpha\right) + \beta, & \dfrac{1}{2}\left(\dfrac{1}{\alpha} - \alpha\right) + \beta \\ \dfrac{1}{2}\left(\dfrac{1}{\alpha} - \alpha\right) - \beta, & \dfrac{1}{2}\left(\dfrac{1}{\alpha} + \alpha\right) - \beta \end{bmatrix} : \alpha, \beta \in \Omega, \ \alpha \neq 0 \right\} .$$

Recalling that σ diagonally interchanges the entries of a matrix, one sees that B_1 and B_2 are σ-stable, and T is pointwise σ-fixed. Moreover, T is clearly a maximal torus of both B_1 and B_2 (i.e., when $\beta = 0$). So $T \subsetneqq B_1$ and $T \subsetneqq B_2$ are σ-stable pairs.

Suppose now that B_1, B_2 are conjugate by a σ-fixed point $f \in G_\sigma$, i.e., that $B_1 = fB_2f^{-1}$. Then $B_1 = x\Delta x^{-1} = fB_2f^{-1} = fy\Delta y^{-1}f^{-1} \Rightarrow \Delta = x^{-1}fy\Delta y^{-1}f^{-1}x \Rightarrow x^{-1}fy \in N_G(\Delta) = \Delta$.

Say that $x^{-1}fy = b = \begin{bmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{bmatrix} \in \Delta$, and $f = \begin{bmatrix} \gamma & \delta \\ \delta & \gamma \end{bmatrix} \in G_\sigma$, for some α, β, γ, $\delta \in \Omega$ with $\gamma^2 - \delta^2 = 1$, and $\alpha \neq 0$. Then

$$\begin{aligned} x^{-1}fy = b &\Longrightarrow fy = xb \\ &\Longrightarrow \begin{bmatrix} \gamma & \delta \\ \delta & \gamma \end{bmatrix} \begin{bmatrix} 1 & \dfrac{1}{2} \\ -1 & \dfrac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ 1 & \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \dfrac{1}{\alpha} \end{bmatrix} \\ &\Longrightarrow \begin{bmatrix} \gamma - \delta & \dfrac{1}{2}(\gamma + \delta) \\ \delta - \gamma & \dfrac{1}{2}(\gamma + \delta) \end{bmatrix} = \begin{bmatrix} \alpha & \beta - \dfrac{1}{2\alpha} \\ \alpha & \beta + \dfrac{1}{2\alpha} \end{bmatrix} \\ &\Longrightarrow \gamma - \delta = \delta - \gamma \Rightarrow \gamma = \delta \Rightarrow \gamma^2 - \delta^2 = 0 \,, \end{aligned}$$

a contradiction of the fact that $\gamma^2 - \delta^2 = 1$.

Thus B_1 and B_2 cannot be conjugate by a σ-fixed point of G.

Counterexample C. G solvable, σ unipotent, and $p > 0$. G has two σ-stable maximal tori which are not conjugate by a σ-fixed point.

Take $p = 2$.

Let T be the torus $\subseteqq GL(6, \Omega)$ consisting of diagonal matrices t of the form

$$t = \left[\begin{array}{ccc|ccc} \tau_1 & & & & & \\ & \tau_2 & & & & \\ & & \tau_1 & & & \\ \hline & & & \tau_2 & & \\ & & & & \tau_1 & \\ & & & & & \tau_2 \end{array}\right], \quad \tau_1, \tau_2 \in \Omega, \quad \tau_1\tau_2 \neq 0 .$$

Let U be the unipotent clag consisting of upper triangular matrices u of the form

$$u = \left[\begin{array}{ccc|ccc} 1 & \alpha & x & & & \\ & 1 & \beta & & & \\ & & 1 & & & \\ \hline & & & 1 & \beta & y \\ & & & & 1 & \alpha \\ & & & & & 1 \end{array}\right], \quad \alpha, \beta, x, y \in \Omega ,$$

satisfying: $x + y - \alpha\beta = 0$.

The reader may verify that U is closed under multiplication, and since $u^4 = e$, $\forall u \in U$, U is also closed under inverses. Hence U is well-defined.

Moreover, U is normalized by T, as the reader again may verify.

One may therefore form the solvable clag $G = T \cdot U$ (semi-direct product).

Let the automorphism σ on $t \cdot u \in G$ be given by the following action on the entries of t and u

$$\begin{aligned} \sigma\colon \tau_1 &\longleftrightarrow \tau_2 \\ \alpha &\longleftrightarrow \beta \\ x &\longleftrightarrow y \end{aligned}$$

σ is thus conjugation by the permutation matrix:

$$s = \left[\begin{array}{ccc|ccc} & & & 1 & & 0 \\ & 0 & & & 1 & \\ & & & 0 & & 1 \\ \hline 1 & & 0 & & & \\ & 1 & & & 0 & \\ 0 & & 1 & & & \end{array}\right].$$

So s and σ are unipotent of order 2.

T is a σ-stable maximal torus of G, whose Cartan subgroup is $C(T) = T \times C(T)_u$, where

$$C(T)_u = \left\{ \left[\begin{array}{ccc|ccc} 1 & 0 & x & & & \\ & 1 & 0 & & 0 & \\ 0 & & 1 & & & \\ \hline & & & 1 & 0 & x \\ & 0 & & & 1 & 0 \\ & & & 0 & & 1 \end{array}\right] : x \in \Omega \right\}.$$

Now if $u \in U$, then uTu^{-1} is σ-stable if and only if $u^{-1}\cdot\sigma(u) \in C(T)_u$. Moreover, $\exists$ a σ-fixed element $f \in U_\sigma$ such that $uTu^{-1} = fTf^{-1}$ if and only if $f^{-1}u \in C(T)_u$; i.e., if and only if $\exists c \in C(T)_u$ such that $uc^{-1} = f$ is σ-fixed.

However, all $c \in C(T)_u$ are σ-fixed; So a σ-stable maximal torus uTu^{-1} of G is conjugate to T by a σ-fixed point if and only if u itself is σ-fixed.

However, for the unipotent matrix

$$u = \left[\begin{array}{ccc|ccc} 1 & \alpha & x & & & \\ & 1 & \alpha & & 0 & \\ 0 & & 1 & & & \\ \hline & & & 1 & \alpha & y \\ & 0 & & & 1 & \alpha \\ & & & 0 & & 1 \end{array}\right],$$

satisfying $x + y - \alpha^2 = 0$, $\alpha \neq 0$, one gets

$$u^{-1}\cdot\sigma(u) = \left[\begin{array}{ccc|ccc} 1 & 0 & -x+y & & & \\ & 1 & 0 & & 0 & \\ 0 & & 1 & & & \\ \hline & & & 1 & 0 & -x+y \\ & 0 & & & 1 & 0 \\ & & & 0 & & 1 \end{array}\right].$$

That is, $u^{-1}\cdot\sigma(u) \in C(T)_u$, so uTu^{-1} is σ-stable. But u is not σ-fixed, so T and uTu^{-1} are not conjugate by a σ-fixed element of G.

(*Note.* This counterexample in $p = 2$ is due to D. Winter. The present author has generalized it in a separate paper for all $p > 0$. The resulting group may be of some interest in itself.)

REFERENCES

1. Armand Borel, *Groupes lineares algebriques*, Annals of Math., **64** (1956), 20-82.
2. ———, *Linear Algebraic Groups*, W. A. Benjamin, New York, 1969.
3. Robert Steinberg, *Endomorphisms of linear algebraic groups*, A. M. S. Memoirs, **80** (1968).
4. David Winter, *On automorphisms of algebraic groups*, Bull. A. M. S., **72** (1966), 706-708.
5. ———, *Algebraic group automorphisms having finite-fixed point sets*, preprint, 1966.

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