

Pacific Journal of Mathematics

$W_\delta(T)$ IS CONVEX

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Stampfli introduced a generalization of the numerical range for any bounded linear operator T on a Hilbert space \mathcal{H} . This is denoted by $W_\delta(T)$ and is defined by

$$W_\delta(T) = \text{closure } \{ \langle Tx, x \rangle : \|x\| = 1 \text{ and } \|Tx\| \geq \delta \} .$$

Stampfli asked whether $W_\delta(T)$ is convex. In this short note we provide an affirmative answer to this question.

$\mathcal{L}(\mathcal{H})$ will denote the set of bounded linear operators on the Hilbert space \mathcal{H} .

LEMMA 1. *Suppose S and A belong to $\mathcal{L}(\mathcal{H})$, and that $S = S^*$. Then*

$$S(A, \delta) = \{ x \in \mathcal{H} : \|x\| = 1 \text{ and } \|Ax\| \geq \delta \text{ and } \langle Sx, x \rangle = 0 \}$$

is path connected.

Proof. Suppose x and y belong to $S(A, \delta)$. We may assume that x and y are linearly independent. (If not, they both lie on an arc of

$$\{ e^{i\theta}x : 0 \leq \theta \leq 2\pi \}$$

which lies in $S(A, \delta)$ if x does.)

Choose θ in \mathbf{R} such that $e^{i\theta}\langle Sx, y \rangle$ is purely imaginary and let $a = e^{i\theta}x$.

Choose n such that $(-1)^n \mathbf{Re} \langle (A^*A - \delta^2 I)a, y \rangle$ is positive and let $b = (-1)^n y$. Then a and b may be joined by a path in $S(A, \delta)$ to x and y respectively. Thus we need only find a path connecting a to b . Let $y(t) = ta + (1-t)b$ and let $x(t) = \|y(t)\|^{-1}y(t)$. Then $\langle Sx(t), x(t) \rangle = 0 \Leftrightarrow \langle Sy(t), y(t) \rangle = 0$ and

$$\begin{aligned} \langle Sy(t), y(t) \rangle &= t^2 \langle Sa, a \rangle + (1-t)^2 \langle Sb, b \rangle \\ &\quad + 2\mathbf{Ret}(1-t) \langle Sa, b \rangle \\ &= 2(-1)^n t(1-t) \mathbf{Re} e^{i\theta} \langle Sx, y \rangle \\ &= 0 . \end{aligned}$$

Also

$$\begin{aligned} \|Ay(t)\|^2 &= \langle A^*Ay(t), y(t) \rangle \\ &= t^2 \|Aa\|^2 + (1-t)^2 \|Ab\|^2 \\ &\quad + 2t(1-t) \mathbf{Re} \langle A^*Aa, b \rangle \end{aligned}$$

$$\begin{aligned} &\geq \delta^2(t^2 + (1 - t)^2 + 2\mathbf{Re}t(1 - t)\langle a, b \rangle) \\ &\quad + 2t(1 - t)\mathbf{Re}\langle (A^*A - \delta^2I)a, b \rangle \\ &= \delta^2\|y(t)\|^2 \\ &\quad + 2t(1 - t)(-1)^*\mathbf{Re}\langle (A^*A - \delta^2I)a, y \rangle \\ &\geq \delta^2\|y(t)\|^2. \end{aligned}$$

Hence $\|Ax(t)\| \geq \delta$ and so $t \rightarrow x(t)$ is a path connecting a to b in $S(A, \delta)$ as required.

LEMMA 2. *Suppose H and K are self-adjoint elements in $\mathcal{L}(\mathcal{H})$. Let*

$$V(A, \delta) = \{(\langle Hx, x \rangle, \langle Kx, x \rangle) : \|x\| = 1 \text{ and } \|Ax\| \geq \delta\}.$$

Then $V(A, \delta)$ is a convex subset of \mathbf{R}^2 .

Proof. We need only show that $V(A, \delta) \cap L$ is connected whenever L is a straight line in \mathbf{R}^2 . Suppose L is given by

$$\alpha\xi + \beta\eta + \gamma = 0.$$

Let

$$S = \alpha H + \beta K + \gamma I.$$

Then the mapping π , given by

$$\begin{aligned} \pi(x) &= (\langle Hx, x \rangle, \langle Kx, x \rangle) \text{ is continuous, and} \\ S(A, \delta) &= \{x : \|x\| = 1; \|Ax\| \geq \delta \text{ and } \pi(x) \in L\}. \end{aligned}$$

Thus $V(A, \delta) \cap L = \pi(S(A, \delta))$ is connected.

THEOREM 3. *Suppose T and A are in $\mathcal{L}(\mathcal{H})$. Then*

$$V(T; A, \delta) = \{(\langle Tx, x \rangle) : \|x\| = 1 \text{ and } \|Ax\| \geq \delta\}$$

is convex.

Proof. Suppose $T = H + iK$ with H and K both self-adjoint. Then

$$V(T; A, \delta) = \{\xi + i\eta : (\xi, \eta) \in V(A, \delta)\}.$$

Hence $V(T; A, \delta)$ is convex.

COROLLARY 4. $W_i(T)$ is convex.

Proof. Take $A = T$. Indeed we have shown that

$$\{\langle Tx, x \rangle: \|x\| = 1 \text{ and } \|Tx\| \geq \delta\}$$

is convex.

REMARK. It will be noticed that the ideas here are improvements on basic ideas in 1.

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