ANOTHER NOTE ON EBERLEIN COMPACTS

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An Eberlein compact is a compact space that can be embedded in a Banach space with its weak topology. It is shown that: If \( X \) is compact and if \( X = M_1 \cup M_2 \) with \( M_1 \) and \( M_2 \) metrizable, then \( \bar{M}_1 \cap \bar{M}_2 \) is metrizable and \( X \) is an Eberlein compact. This answers a question of Arhangel’iskii.

1. Introduction. An Eberlein compact, or EC, is a compact space\(^1\) which can be embedded in a Banach space with its weak topology. For background and various properties of these spaces, the reader is referred to [1] or the authors’ preceding note [3].

Since every metrizable space can be embedded in a Banach space with its norm topology, every metrizable compact space is clearly an EC. The purpose of this note is to prove the following stronger result, thereby answering a question of A. V. Arhangel’iskii.

**Theorem 1.1.** If \( X \) is compact, and if \( X = M_1 \cup M_2 \) with \( M_1 \) and \( M_2 \) metrizable, then \( \bar{M}_1 \cap \bar{M}_2 \) is metrizable and \( X \) is an EC.

In contrast to Theorem 1.1, a compact space which is the union of three metrizable subsets need not be an EC, or even a Fréchet space\(^2\) (see [2, Example 6.2]\(^3\)). However, it was shown in [5] that a compact space which is the union of countably many metrizable subsets must at least be sequential (a property somewhat weaker than being a Fréchet space).

2. Proof of Theorem 1.1. We first show that \( M = \bar{M}_1 \cap \bar{M}_2 \) is metrizable. For \( i = 1, 2 \), let \( \mathcal{V}_i \) be a \( \sigma \)-discrete—hence \( \sigma \)-disjoint—base for \( M_i \). For each \( U \in \mathcal{V}_i \), choose an open set \( \phi_i(U) \) in \( X \) such that \( \phi_i(U) \cap M_i = U \). Let \( \mathcal{U} = \{ \phi_i(U) \cap M : U \in \mathcal{V}_i, i = 1, 2 \} \). Then \( \mathcal{U} \) is easily seen to be a \( \sigma \)-disjoint \( 1-m \) hence point-countable \( 1-m \) base for \( M \). Since \( M \) is compact, it must therefore be metrizable by a result of A. S. Miščenko [4].

Since \( M \) is compact and metrizable, it has a countable base \( (B_n) \). For each pair \( (m, n) \) such that \( \bar{B}_m \cap \bar{B}_n = \emptyset \), pick an open \( F_\sigma \)-set

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\(^1\) All spaces in this paper are Hausdorff.

\(^2\) \( X \) is a Fréchet space if, whenever \( x \in \bar{A} \) in \( X \), then \( x_n \to x \) for some \( x_n \in A \).

\(^3\) Every EC is a Fréchet space by a theorem of Eberlein and Šmulian (see [1, Theorem 4.1]).
$V_{m,n}$ in $X$ such that $B_m \subset V_{m,n}$ and $V_{m,n} \cap B_n = \emptyset$. Let $\mathcal{F}$ be collection of all such $V_{m,n}$.

Let $Y = X - M$. Then $Y$ is the union of the two disjoint, open metrizable subsets $X - \bar{M}_1$ and $X - \bar{M}_2$, so $Y$ is metrizable. Since $Y$ is open in $X$, it therefore has a $\sigma$-disjoint base $\mathcal{W}$ such that $\bar{W} \subset Y$ for all $W \in \mathcal{W}$. Clearly each $W \in \mathcal{W}$ is an open $F_\sigma$ in $X$.

Finally, let $\mathcal{G} = \mathcal{F} \cup \mathcal{W}$. Then $\mathcal{G}$ is a $\sigma$-disjoint, separating (in the sense of [3, Definition 1.3]) cover of $X$ by open $F_\sigma$-sets, so $X$ is an EC by a characterization of H. P. Rosenthal (see [6, Theorem 3.1] or [3, Theorem 1.4]).

3. Concluding remarks.

(3.1) The proof of Theorem 1.1 actually establishes the following somewhat sharper results.

(a) If $X$ is regular, and if $X = \bigcup_{n=1}^{\infty} X_n$ with each $X_n$ having a $\sigma$-disjoint base, then $\bigcap_{n=1}^{\infty} \bar{X}_n$ has a $\sigma$-disjoint base.

(b) If $X$ is compact, and if $X = \bigcup_{n=1}^{\infty} X_n$ with each $X_n$ metrizable, then $\bigcap_{n=1}^{\infty} \bar{X}_n$ is metrizable.

(c) If $X$ is compact, and $X = X_1 \cup X_2$ with $X_1$ and $X_2$ having $\sigma$-disjoint bases, then $X$ has a $\sigma$-disjoint, separating collection of open $F_\sigma$-subsets.

Observe that not every EC satisfies the conclusion of (c), as can be seen from the space of all points in $\{0,1\}^{\omega_1}$ which have at most two nonzero coordinates.

(3.2). Somewhat in the spirit of Theorem 1.1, one can show that if $X = \bigcup_{i=1}^{n} X_i$, and if each $X_i$ is an EC, then $X$ is an EC: In fact, $X$ is then the image under the obvious perfect map of the topological sum $\sum_{i=1}^{n} X_i$, and this sum is clearly an EC, so $X$ must be an EC by [1, Theorem 2.1] (see also [3, Theorem 1.1]).

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George E. Andrews, *Plane partitions. II. The equivalence of the Bender-Knuth and MacMahon conjectures* .............................................................. 283

Lee Wilson Badger, *An Ehrenfeucht game for the multivariable quantifiers of Malitz and some applications* .......................................................... 293

Wayne C. Bell, *A decomposition of additive set functions* .................................................. 305

Bruce Blackadar, *Infinite tensor products of C*-algebras* ............................................. 313

Arne Brøndsted, *The inner aperture of a convex set* ...................................................... 335

N. Burgoyne, *Finite groups with Chevalley-type components* ..................................... 341

Richard Dowell Byrd, Justin Thomas Lloyd and Roberto A. Mena, *On the retractability of some one-relator groups* ............................................ 351

Paul Robert Chernoff, *Schrödinger and Dirac operators with singular potentials and hyperbolic equations* ......................................................... 361

John J. F. Fournier, *Sharpness in Young’s inequality for convolution* ....................... 383

Stanley Phillip Franklin and Barbara V. Smith Thomas, *On the metrizability of \( k_\omega \)-spaces* ................................................................. 399

David Andrew Gay, Andrew McDaniel and William Yslas Vélez, *Partially normal radical extensions of the rationals* ........................................... 403

Jean-Jacques Gervais, *Sufficiency of jets* .............................................................. 419

Kenneth R. Goodearl, *Completions of regular rings. II* .............................................. 423

Sarah J. Gottlieb, *Algebraic automorphisms of algebraic groups with stable maximal tori* .............................................................. 461

Donald Gordon James, *Invariant submodules of unimodular Hermitian forms* ......... 471

J. Kyle, *\( W(T) \) is convex* .............................................................. 483

Ernest A. Michael and Mary Ellen Rudin, *A note on Eberlein compacts* .................. 487

Ernest A. Michael and Mary Ellen Rudin, *Another note on Eberlein compacts* ........ 497

Thomas Bourque Muenzenberger and Raymond Earl Smithson, *Fixed point theorems for acyclic and dendritic spaces* ..................................... 501

Budh Singh Nashier and A. R. Rajwade, *Determination of a unique solution of the quadratic partition for primes \( p \equiv 1 \pmod{7} \)* ......................... 513

Frederick J. Scott, *New partial asymptotic stability results for nonlinear ordinary differential equations* .................................................. 523

Frank Servedio, *Affine open orbits, reductive isotropy groups, and dominant gradient morphisms; a theorem of Mikio Sato* .................................... 537

D. Suryanarayana, *On the distribution of some generalized square-full integers* .......... 547

Wolf von Wahl, *Instationary Navier-Stokes equations and parabolic systems* ........... 557