

Pacific Journal of Mathematics

ANOTHER NOTE ON EBERLEIN COMPACTS

ERNEST A. MICHAEL AND MARY ELLEN RUDIN

ANOTHER NOTE ON EBERLEIN COMPACTS

E. MICHAEL AND M. E. RUDIN

An Eberlein compact is a compact space that can be embedded in a Banach space with its weak topology. It is shown that: If X is compact and if $X = M_1 \cup M_2$ with M_1 and M_2 metrizable, then $\bar{M}_1 \cap \bar{M}_2$ is metrizable and X is an Eberlein compact. This answers a question of Arhangel'skii.

1. Introduction. An *Eberlein compact*, or *EC*, is a compact space¹⁾ which can be embedded in a Banach space with its weak topology. For background and various properties of these spaces, the reader is referred to [1] or the authors' preceding note [3].

Since every metrizable space can be embedded in a Banach space with its norm topology, every metrizable compact space is clearly an *EC*. The purpose of this note is to prove the following stronger result, thereby answering a question of A. V. Arhangel'skii.

THEOREM 1.1. *If X is compact, and if $X = M_1 \cup M_2$ with M_1 and M_2 metrizable, then $\bar{M}_1 \cap \bar{M}_2$ is metrizable and X is an *EC*.*

In contrast to Theorem 1.1, a compact space which is the union of *three* metrizable subsets need *not* be an *EC*, or even a Fréchet space²⁾ (see [2, Example 6.2]³⁾). However, it was shown in [5] that a compact space which is the union of countably many metrizable subsets must at least be sequential (a property somewhat weaker than being a Fréchet space).

2. Proof of Theorem 1.1. We first show that $M = \bar{M}_1 \cap \bar{M}_2$ is metrizable. For $i = 1, 2$, let \mathcal{U}_i be a σ -discrete—hence σ -disjoint—base for M_i . For each $U \in \mathcal{U}_i$, choose an open set $\phi_i(U)$ in X such that $\phi_i(U) \cap M_i = U$. Let $\mathcal{U} = \{\phi_i(U) \cap M : U \in \mathcal{U}_i, i = 1, 2\}$. Then \mathcal{U} is easily seen to be a σ -disjoint 1— m hence point-countable 1— m base for M . Since M is compact, it must therefore be metrizable by a result of A. S. Miščenko [4].

Since M is compact and metrizable, it has a countable base (B_n) . For each pair (m, n) such that $\bar{B}_m \cap \bar{B}_n = \emptyset$, pick an open F_σ -set

¹ All spaces in this paper are Hausdorff.

² X is a *Fréchet* space if, whenever $x \in \bar{A}$ in X , then $x_n \rightarrow x$ for some $x_n \in A$. Every *EC* is a Fréchet space by a theorem of Eberlein and Šmulian (see [1, Theorem 4.1]).

³ In this example, the three metrizable subsets are actually discrete, and one of them is an open set whose complement is (necessarily, by Theorem 1.1) an *EC*.

$V_{m,n}$ in X such that $B_m \subset V_{m,n}$ and $V_{m,n} \cap B_n = \emptyset$. Let \mathcal{V} be collection of all such $V_{m,n}$.

Let $Y = X - M$. Then Y is the union of the two disjoint, open metrizable subsets $X - \bar{M}_1$ and $X - \bar{M}_2$, so Y is metrizable. Since Y is open in X , it therefore has a σ -disjoint base \mathcal{W} such that $\bar{W} \subset Y$ for all $W \in \mathcal{W}$. Clearly each $W \in \mathcal{W}$ is an open F_σ in X .

Finally, let $\mathcal{S} = \mathcal{V} \cup \mathcal{W}$. Then \mathcal{S} is a σ -disjoint, separating (in the sense of [3, Definition 1.3]) cover of X by open F_σ -sets, so X is an EC by a characterization of H. P. Rosenthal (see [6, Theorem 3.1] or [3, Theorem 1.4]).

3. Concluding remarks.

(3.1) The proof of Theorem 1.1 actually establishes the following somewhat sharper results.

(a) If X is regular, and if $X = \bigcup_{n=1}^{\infty} X_n$ with each X_n having a σ -disjoint base, then $\bigcap_{n=1}^{\infty} \bar{X}_n$ has a σ -disjoint base.

(b) If X is compact, and if $X = \bigcup_{n=1}^{\infty} X_n$ with each X_n metrizable, then $\bigcap_{n=1}^{\infty} \bar{X}_n$ is metrizable.

(c) If X is compact, and $X = X_1 \cup X_2$ with X_1 and X_2 having σ -disjoint bases, then X has a σ -disjoint, separating collection of open F_σ -subsets.

Observe that not every EC satisfies the conclusion of (c), as can be seen from the space of all points in $\{0, 1\}^{\omega_1}$ which have at most two nonzero coordinates.

(3.2). Somewhat in the spirit of Theorem 1.1, one can show that if $X = \bigcup_{i=1}^n X_i$, and if each X_i is an EC , then X is an EC : In fact, X is then the image under the obvious perfect map of the topological sum $\sum_{i=1}^n X_i$, and this sum is clearly an EC , so X must be an EC by [1, Theorem 2.1] (see also [3, Theorem 1.1]).

REFERENCES

1. Y. Benyamini, M. E. Rudin and M. Wage, *Continuous images of weakly compact subsets of Banach spaces*, to appear in Pacific J. Math.
2. S. P. Franklin, *Spaces in which sequences suffice. II*, Fund. Math., **61** (1967), 51-56.
3. E. Michael and M. E. Rudin, *A note on Eberlein compacts*, Pacific J. Math., **73** (1977), 487-495.
4. A. S. Miščenko, *Spaces with pointwise denumerable basis*, Dokl. Akad. Nauk SSSR, **145** (1962), 985-988. (Soviet Math. Dokl. **3** (1962), 855-858.)
5. A. Ostaszewski, *Compact σ -metric spaces are sequential*, Proc. Amer. Math. Soc. (to appear.)
6. H. P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math., **81** (1974), 83-111.

Received May 20, 1977. The second author was partly supported by N. S. F. grant MPS-73-08825.

UNIVERSITY OF WASHINGTON
SEATTLE, WA 98195
AND
UNIVERSITY OF WISCONSIN
MADISON, WI 53706

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, CA 90024

CHARLES W. CURTIS

University of Oregon
Eugene, OR 97403

C. C. MOORE

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

George E. Andrews, <i>Plane partitions. II. The equivalence of the Bender-Knuth and MacMahon conjectures</i>	283
Lee Wilson Badger, <i>An Ehrenfeucht game for the multivariable quantifiers of Malitz and some applications</i>	293
Wayne C. Bell, <i>A decomposition of additive set functions</i>	305
Bruce Blackadar, <i>Infinite tensor products of C^*-algebras</i>	313
Arne Brøndsted, <i>The inner aperture of a convex set</i>	335
N. Burgoyne, <i>Finite groups with Chevalley-type components</i>	341
Richard Dowell Byrd, Justin Thomas Lloyd and Roberto A. Mena, <i>On the retractability of some one-relator groups</i>	351
Paul Robert Chernoff, <i>Schrödinger and Dirac operators with singular potentials and hyperbolic equations</i>	361
John J. F. Fournier, <i>Sharpness in Young's inequality for convolution</i>	383
Stanley Phillip Franklin and Barbara V. Smith Thomas, <i>On the metrizability of k_ω-spaces</i>	399
David Andrew Gay, Andrew McDaniel and William Yslas Vélez, <i>Partially normal radical extensions of the rationals</i>	403
Jean-Jacques Gervais, <i>Sufficiency of jets</i>	419
Kenneth R. Goodearl, <i>Completions of regular rings. II</i>	423
Sarah J. Gottlieb, <i>Algebraic automorphisms of algebraic groups with stable maximal tori</i>	461
Donald Gordon James, <i>Invariant submodules of unimodular Hermitian forms</i>	471
J. Kyle, <i>$W_\delta(T)$ is convex</i>	483
Ernest A. Michael and Mary Ellen Rudin, <i>A note on Eberlein compacts</i>	487
Ernest A. Michael and Mary Ellen Rudin, <i>Another note on Eberlein compacts</i>	497
Thomas Bourque Muenzenberger and Raymond Earl Smithson, <i>Fixed point theorems for acyclic and dendritic spaces</i>	501
Budh Singh Nashier and A. R. Rajwade, <i>Determination of a unique solution of the quadratic partition for primes $p \equiv 1 \pmod{7}$</i>	513
Frederick J. Scott, <i>New partial asymptotic stability results for nonlinear ordinary differential equations</i>	523
Frank Servedio, <i>Affine open orbits, reductive isotropy groups, and dominant gradient morphisms; a theorem of Mikio Sato</i>	537
D. Suryanarayana, <i>On the distribution of some generalized square-full integers</i>	547
Wolf von Wahl, <i>Instationary Navier-Stokes equations and parabolic systems</i>	557