Pacific Journal of Mathematics

GENERATING O(n) WITH REFLECTIONS

MORRIS LEROY EATON AND MICHAEL DAVID PERLMAN

Vol. 73, No. 1 March 1977

GENERATING O(n) WITH REFLECTIONS

MORRIS L. EATON AND MICHAEL PERLMAN

For $r \in C_n \equiv \{x | x \in R^n, \ ||x|| = 1\}$, let $S_r = I_n - 2rr'$ where r is a column vector. O(n) denotes the orthogonal group on R^n . If $R \subseteq C_n$, let $\mathscr{R} = \{S_r | r \in R\}$ and let G be the smallest closed subgroup of O(n) which contains \mathscr{R} . G is reducible if there is a nontrivial subspace $M \subseteq R^n$ such that $gM \subseteq M$ for all $g \in G$. Otherwise, G is irreducible.

THEOREM. If G is infinite and irreducible, then G = O(n).

In what follows, R^n denotes Euclidean n-space with the standard inner product, O(n) is the orthogonal group of R^n , and $C_n = \{x \mid x \in R^n, ||x|| = 1\}$. If U is a subset of O(n), $\langle U \rangle$ denotes the group generated algebraically by U and $\langle \overline{U} \rangle$ denotes the closure of $\langle U \rangle$. Thus, $\langle \overline{U} \rangle$ is the smallest closed subgroup of O(n) containing U. For an integer k, $1 \leq k < n$, M_k denotes a k-dimensional linear subspace of R^n . If $r \in C_n$, let $S_r = I - 2rr'$ where r is a column vector. Thus S_r is a reflection through r-henceforth called a reflection.

Suppose $R \subseteq C_n$ and let $\mathscr{R} = \{S_r | r \in R\}$. Set $G = \langle \bar{\mathscr{R}} \rangle$. The group G is reducible if there is an M_k such that $gM_k \subseteq M_k$ for all $g \in G$; otherwise, G is irreducible. The main result of this note is the following.

THEOREM 1. If G is infinite and irreducible, then G = O(n).

Proof of Theorem 1. First note that if $S_r \in \mathscr{R}$ and $g \in G$, then $gS_rg^{-1} = S_{gr} \in G$. Let $\Delta = \{gr \mid g \in G, r \in R\}$. Thus, $t \in \Delta$ implies that $S_t \in G$. Since G is infinite, Δ must be infinite (see Benson and Grove (1971), Proposition 4.1.3). Since every Γ in O(n) is a product of a finite number of reflections, to show that G = O(n), it suffices to show that G is transitive on C_n (if G is transitive on C_n , then $\Delta = C_n$ so every reflection is an element of G and hence G = O(n)).

The proof that G is transitive on C_n follows. By Lemma 1 (below), there is a subgroup $K_2 \subseteq G$ and a subspace $M_2 \subseteq R^n$ such that kx = x if $x \in M_2^\perp$ and $k \in K_2$ and K_2 is transitive on $D_2 \equiv M_2 \cap C_n$. Since G is irreducible, there is an $r_2 \in R$ such that $r_2 \notin M_2$ and $r_2 \notin M_2^\perp$. Let $M_3 = \operatorname{span}\{r_2, M_2\}$ and let $K_3 = \langle \{K_2, S_{r_2}\} \rangle > \subseteq G$. With $D_3 \equiv M_3 \cap C_n$, Lemma 3 (below) implies that kx = x for all $x \in M_3^\perp$ and $k \in K_3$, and K_3 is transitive on D_3 . Again, since G is irreducible, there is an $r_3 \in R$ such that $r_3 \notin M_3$ and $r_3 \notin M_3^\perp$. With $M_4 = \operatorname{span}\{r_3, M_3\}$, let $K_4 = \langle \{K_3, S_{r_3}\} \rangle > \subseteq G$ and let $D_4 \equiv M_4 \cap C_n$. By Lemma 3 (below)

kx = x for $x \in M_4^{\perp}$ and $k \in K_4$ and K_4 is transitive on D_4 . Applying this argument (n-2) times, we obtain $K_n \subseteq G$ and K_n is transitive on $D_n = C_n$. Thus, G is transitive on C_n and the proof is complete.

To fill in the gaps in the above argument, it remains to prove Lemmas 1, 2, and 3. Lemma 1 provides the starting point for the stepwise argument used in the proof of Theorem 1.

LEMMA 1. If G is irreducible and infinite, there is a subspace M_2 and a subgroup $K_2 \subseteq G$ such that kx = x for $x \in M_2^{\perp}$, $k \in K_2$ and K_2 acts transitively on $D_2 \equiv M_2 \cap C_n$.

Proof. As noted in the proof of Theorem 1, the set $\Delta = \{gr \mid r \in R, g \in G\}$ is infinite. Thus, there is a point $\delta_0 \in C_n$ such that every neighborhood of δ_0 contains infinitely many points in Δ . Thus we can select a sequence of pairs (r_i, t_i) , r_i , $t_i \in \Delta$, such that r_i and t_i are linearly independent and $1 - 1/i < r'_i t_i < r'_{i+1} t_{i+1} < 1$ for $i = 1, 2, \cdots$.

For $0 \le \eta < 2\pi$, set

$$\varPsi(\eta) = \begin{pmatrix} \cos \eta & \sin \eta \\ -\sin \eta & \cos \eta \end{pmatrix} \in O(2) \; .$$

Define θ_i by $\cos \theta_i = r_i' t_i$, $0 \le \theta_i < \pi$ so $\theta_i \to 0$ as $i \to \infty$. Let $\Gamma_i \in O(n)$ have first row t_i' and second row

$$(r_i - t_i'r_it_i)'/||r_i - t_i'r_it_i||$$
.

Then an easy calculation shows that

$$(\ 2\) \hspace{1cm} S_{t_i}S_{r_i}=arGamma_i^{\prime}egin{pmatrix}arPsi_i^{\prime}&0\0&I_{n-2}\end{pmatrix}arGamma_i\ , \qquad i=1,\,2,\,\cdots$$

where I_{n-2} is an $(n-2) \times (n-2)$ identity matrix. Setting $H_i = \langle \Psi(2\theta_i) \rangle \subseteq O(2)$, it is clear that

$$\left\{ arGamma_i'iggl(egin{array}{cccc} h & 0 \ 0 & I_{n-2} \end{array}iggr)arGamma_iiggr| h\in H_i
ight\} \subseteq G \;, \qquad i=1,\,2,\,\cdots \;.$$

By selecting an appropriate subsequence, we can assume without loss of generality that $\Gamma_i \to \Gamma_0 \in O(n)$, as $i \to \infty$.

If $\Psi(\eta)$ is given by (1), we now claim that

$$\Gamma_{\scriptscriptstyle 0}' \! \begin{pmatrix} \varPsi(\eta) & 0 \\ 0 & I_{\scriptscriptstyle n-2} \end{pmatrix} \! \Gamma_{\scriptscriptstyle 0} \in G \ .$$

Since G is closed and (3) holds, to establish (4), it suffices to show

there is a subsequence i_j and $h_{i_j} \in H_{i_j}$ such that $h_{i_j} \to \Psi(\eta)$ as $i_j \to \infty$. However, the existence of such a sequence is assured since $\theta_i \to 0$ as $i \to \infty$. Thus (4) holds. Hence we see that

$$K_{\scriptscriptstyle 2} \equiv \left\{ arGamma_{\scriptscriptstyle 0}^{\prime} egin{pmatrix} h & 0 \ 0 & I_{\scriptscriptstyle m-2} \end{pmatrix} arGamma_{\scriptscriptstyle 0} \middle| h \in H^*
ight\} \subseteq G$$

where H^* is the full rotation group of R^2 .

To complete the proof of Lemma 1, let M_2 be the span of the first two columns of Γ_0 . With $D_2 \equiv M_2 \cap C_n$, it is easy to check that kx = x for all $x \in M_2$, $k \in K_2$ and that K_2 acts transitively on D_2 . This completes the proof.

The following result is used in the proof of Lemma 3.

LEMMA 2. For $u_0 \in (0, 1]$, define a function $f: [0, 1] \rightarrow [0, 1]$ by

$$f(u) = egin{cases} 0 & if & 0 \leq u \leq u_0 \ 1 - [\sqrt{uu_0} + \sqrt{(1-u)(1-u_0)}]^2 & if \ u_0 \leq u \leq 1 \end{cases}.$$

Let $v_i = f(1)$ and define $v_i = f(v_{i-1})$ for $i = 2, 3, \cdots$. Then, there exists an index i_0 such that $v_i = 0$ for $i \ge i_0$.

Proof. It is not hard to verify that f is a continuous convex function. Since $0 \le v_1 < 1$, $v_2 = f(v_1) = f((1-v_1)0 + v_11) \le v_1 f(1) = v_1^2$. Proceeding by induction, $v_i \le v_1^i$ so $\lim_{i \to \infty} v_i = 0$. Since f is 0 in the interval $[0, u_0]$, there is an index i_0 such that $v_i = 0$ for $i \ge i_0$. This completes the proof.

After establishing Lemma 1, the key to Theorem 1 is Lemma 3. Although the proof of Lemma 3 is quite long, the geometric idea behind the proof is fairly simple. Consider R^3 and let $D_2 = \{x \mid x \in R^3, x_3 = 0, x_1^2 + x_2^2 = 1\}$. Also, let $H = \left\{ \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \middle| k$ is any rotation of $R^2 \right\}$. Thus H acts transitively on D_2 . Consider a fixed vector $t \in R^3$ with ||t|| = 1 such that t is not in the (x_1, x_2) plane and t is not in the x_3 -line. Let $S_t = I$ -2tt' be the reflection across the plane $\{t\}^\perp$ and let \widetilde{H} be the group generated by S_t and H. The claim is that \widetilde{H} is transitive on $D_3 = \{x \mid x \in R^3, ||x|| = 1\}$. For example, suppose the angle between t and the (x_1, x_2) plane is 45°. Geometrically, it is clear that the set $H(S_t(D_2)) \equiv \{x \mid x = hS_t u \text{ for some } h \in H, \text{ and some } u \in D_2\}$ is just D_3 —that is, $S_t(D_2)$ is a circle passing through $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and the transitivity of H implies that H moves the set $S_t(D_2)$ everywhere onto D_3 (picture this on the surface of a basketball). Thus, given

 $v_1, v_2 \in D_3, v_i = h_i S_i u_i$, for $h_i \in H$ and $u_i \in D_2$ for i = 1, 2. Since $u_1 = h_0 u_2$ for some $h_0 \in H$, it follows that $v_1 = h_1 S_t h_0 S_t h_2^{-1} v_2$ so \widetilde{H} is transitive on D_3 . For other t-vectors, D_3 does not get covered by one application of HS_t to D_2 , but D_3 is covered by a finite number of applications of HS_t to D_2 —that is, $D_3 = (H(S_t(\cdots)H)S_t)(D_2)$ for some finite string $HS_t HS_t \cdots HS_t$. Again, this implies the transitivity of \widetilde{H} on D_3 . Lemma 3 and its proof make all of the above precise.

LEMMA 3. Consider a subspace $M_m \subseteq R^n$, $2 \le m < n$, and suppose that K is a subgroup of O(n) such that

$$\begin{cases} kx = x & for \ all \quad x \in M_m^{\perp}, \ k \in K \\ K \ is \ transitive \ on \ D_m \equiv M_m \cap C_n \ . \end{cases}$$

Let $t \in C_n$ be such that $t \notin M_m$ and $t \notin M_m^{\perp}$. With $M_{m+1} = \operatorname{span} \{t, M_m\}$, let $D_{m+1} \equiv M_{m+1} \cap C_n$. Then the group $K^* \subseteq O(n)$ generated by K and $S_t = I - 2tt'$ satisfies

$$\left\{egin{array}{lll} kx=x & \textit{for all} & x\in M_{m+1}^{\perp}, \, k\in K^{st} \ K^{st} & \textit{is transitive on } D_{m+1} \end{array}
ight.$$

Proof. That kx = x for all $x \in M_{m+1}^{\perp}$ and $k \in K^*$ is not hard to verify. To establish the transitivity of K^* on D_{m+1} , define a set B_1 by

(9)
$$B_1=K(S_t(D_m))=\{x\,|\,x=kS_tu \ \text{for some}\ u\in D_m, \ \text{some}\ k\in K\}$$
 and then define B_i inductively by

$$(10) \quad B_i=K(S_i(B_{i-1}))=\{x|x=kS_iu \ \text{for some} \ u\in B_{i-1}\text{, some} \ k\in K\}$$

 $i=2,3,\cdots$. Since $K(S_t(D_{m+1}))\subseteq D_{m+1}$, it follows that $B_i\subseteq D_{m+1}$ for all i. The remainder of the proof is devoted to showing that there is an index i_0 such that $B_{i_0}=D_{m+1}$, because this implies the transitivity of K^* on D_{m+1} .

Claim 1. If $B_{i_0} = D_{m+1}$, then K^* is transitive on D_{m+1} .

Proof of Claim 1. Consider $z_1, z_2 \in D_{m+1}$. If $B_{i_0} = D_{m+1}$, then

$$\underbrace{K(S_t(K(S_t\cdots (D_m))))}_{i_0 ext{-terms}}=D_{m+1}$$
 .

Thus, there exists $k_1, \dots, k_{i_0} \in K$ and $g_1, \dots, g_{i_0} \in K$ such that

$$egin{aligned} oldsymbol{z}_1 = igg[\prod_{j=1}^{i_0} \left(k_j S_t
ight)igg] \! u_1 \equiv h_1 u_1 \end{aligned}$$

and

$$oldsymbol{z}_2 = igg\lceil \prod_{i=1}^{i_0} \left(g_i s_i
ight) igg
ceil u_2 \equiv h_2 u_2$$

for some $u_1, u_2 \in D_m$. Since K is transitive on D_m , there exists a $k_0 \in K$ such that $k_0 u_1 = u_2$. Thus, $z_2 = h_2 k_0 h_1^{-1} z_1$ which shows that K^* is transitive on D_{m+1} as $h_2 k_0 h_1^{-1} \in K^*$. This completes the proof of Claim 1.

We now continue with the proof. Let P denote the orthogonal projection onto M_m and define Z_c , $0 \le c \le 1$ by

(11)
$$Z_c = \{x \mid x \in D_{m+1}, ||Px||^2 \ge c\}.$$

Note that $Z_1 = D_m$ and $Z_0 = D_{m+1}$.

REMARK. Geometrically, Z_c is an equatorial zone (with equator D_m) which partially covers D_{m+1} . Smaller values of c correspond to more of D_{m+1} being covered.

Define φ on [0, 1] by

(12)
$$\varphi(c) = \inf_{x \in Z_c} ||PS_t x||^2, \quad 0 \leq c \leq 1,$$

and let

(13)
$$b_1 = \inf_{x \in B_1} ||Px||^2.$$

Since each $k \in K$ commutes with P, we have

$$(14) \quad b_{\scriptscriptstyle 1} = \inf_{k \in K} \inf_{x \in D_m} ||PkS_t x||^2 = \inf_{x \in D_m} ||PS_t x||^2 = \inf_{x \in Z_1} ||PS_t x||^2 = arphi(1)$$
 .

Claim 2. $B_1 = Z_{b_1}$.

Proof of Claim 2. If $x \in B_1$, $||Px||^2 \ge b_1$ which implies that $x \in Z_{b_1}$. Conversely, consider $x \in Z_{b_1}$ and let Q denote the orthogonal projection onto the one-dimensional subspace $M_m^\perp \cap M_{m+1}$ which is spanned by the vector $t^* \equiv (I-P)t/||(I-P)t||$. Since Z_c is compact and arcwise connected, the continuous function $u \to ||PS_t u||^2 (u \in Z_c)$ takes on all values between 1 and $\varphi(c)$. As $x \in Z_{b_1}$,

$$||Px||^2 \geqq b_{\scriptscriptstyle 1} = arphi(1) = \inf_{u \,\in\, D_{\scriptscriptstyle m}} ||\, PS_{\scriptscriptstyle t}u\,||^2$$
 .

Hence, there exists a $u \in D_m$ such that $||PS_tu||^2 = ||Px||^2$. Thus, $1 = ||Px||^2 + ||Qx||^2 = ||PS_tu||^2 + ||QS_tu||^2$, so $||QS_tu||^2 = ||Qx||^2$. Since Q is a projection onto a one-dimensional subspace, u can be chosen (by changing to -u if necessary) such that $Qx = QS_tu$. The transitivity of K on D_m implies there is a $k \in K$ such that $kPS_tu = Px$. Thus,

 $kS_tu = kPS_tu + kQS_tu = Px + kQS_tu = Px + QS_tu = Px + Qx = x$, so $x = kS_tu \in B_1$. This completes the proof of Claim 2.

Using Claim 2, $B_2 = K(S_t(B_1)) = K(S_t(Z_{b_1}))$. Consider

(15)
$$b_2 \equiv \inf_{x \in B_2} ||Px||^2.$$

Using (15) and the fact that each $k \in K$ commutes with P, we have

$$(16) \qquad b_2 = \inf_{x \in B_2} ||Px||^2 = \inf_{x \in Z_{b_1}} \inf_{k \in K} ||PkS_t x||^2 = \inf_{x \in Z_{b_1}} ||PS_t x||^2 = \varphi(b_1).$$

Claim 3. $B_2 = Z_{b_2}$.

Proof of Claim 3. If $x \in B_2$, then $x \in D_{m+1}$ and $||Px||^2 \ge b_2$, so $x \in Z_{b_2}$. Conversely, consider $x \in Z_{b_2}$. As u varies over Z_{b_1} , the function $u \mapsto ||PS_tu||^2$ takes on all values between 1 and b_2 . Since $||Px||^2 \ge b_2$, there is a $u \in Z_{b_1}$ such that $||PS_tu||^2 = ||Px||^2$. As in the proof of Claim 2, $1 = ||Px||^2 + ||Qx||^2 = ||PS_tu||^2 + ||QS_tu||^2$ so $||Qx||^2 = ||QS_tu||^2$, and we can choose u such that $Qx = QS_tu$. The transitivity of K implies that there is a $k \in K$ such that $kPS_tu = Px$. Thus, $x = Px + Qx = kPS_tu + QS_tu = kPS_tu + kQS_tu = kS_tu \in B_2$ since $u \in Z_{b_1} = B_1$. The proof of Claim 3 is complete.

Arguing as in the proof of Claim 3, it is an easy matter to show that $B_i=Z_{b_i}$ and $b_i=\varphi(b_{i-1})$ where

(17)
$$b_i = \inf_{x \in B_i} ||Px||^2, i = 3, 4, \cdots.$$

As noted earlier, the proof of Lemma 3 will be complete if we can show there is an index i_0 such that $B_{i_0} = Z_0 = D_{m+1}$. To establish the existence of an i_0 , we will explicitly calculate the function φ defined in (12) and then apply Lemma 2. Define $z_0 \in D_{m+1}$ by

$$z_0 = S_t t^*$$

where $t^* = (I - P)t/||(I - P)t||$. Then,

(19)
$$a \equiv ||Pz_{0}||^{2} = \frac{||PS_{t}(I-P)t||^{2}}{||(I-P)t||^{2}} = \frac{||P(I-2tt')(I-P)t||^{2}}{||(I-P)t||^{2}} = \frac{4||Pt||^{2}(t'(I-P)t)^{2}}{||(I-P)t||^{2}} = 4||Pt||^{2}(1-||Pt||^{2}).$$

Since $t \notin M_m$ and $t \notin M_m^{\perp}$, $0 < ||Pt||^2 < 1$ so $0 < a \le 1$.

Claim 4. The function φ is given by

$$(20) \varphi(c) = \begin{cases} 0 & \text{if} \quad 0 \le c \le a \\ 1 - \left[\sqrt{ac} + \sqrt{(1-a)(1-c)}\right]^2 & \text{if} \quad a \le c \le 1 \end{cases}.$$

Proof of Claim 4. Since $Q=t^*t^{*'}$ (see the proof of Claim 2), for each $x\in R^n$, $||QS_tx||^2=x'S_tQS_tx=x'S_tt^*t^{*'}S_tx=(z_0'x)^2$. Thus,

$$(21) \qquad arphi(c) = \inf_{x \in Z_c} ||PS_t x||^2 = \inf_{x \in Z_c} (1 - ||QS_t x||^2) = 1 - \sup_{x \in Z_c} (z_0' x)^2$$
 .

If a=1, then $z_0 \in D_m \subseteq Z_c$, so $\sup_{x \in Z_c} (z_0'x)^2 = 1$ and $\varphi(c) = 0$ for all $c \in [0, 1]$.

Now, consider $a \in (0, 1)$. For $x \in Z_c$, let $\gamma = ||Px||^2 \ge c$. Then, by the Cauchy-Schwarz inequality, we have

(22)
$$z_0'x = z_0'Px + z_0'Qx = (Pz_0)'Px + (Qz_0)'Qx \\ \leq ||Pz_0|| ||Px|| + ||Qz_0|| ||Qx|| = \sqrt{a}\sqrt{\gamma} + \sqrt{1-a}\sqrt{1-\gamma} .$$

Further, there is equality in the above inequality for $x = x_0$ where

$$(23) x_0 = \sqrt{\gamma/a} P z_0 + \sqrt{(1-\gamma)/(1-a)} Q z_0.$$

Clearly, $||Px_0||^2 = \gamma \ge c$ so $x_0 \in Z_c$. Thus,

(24)
$$\varphi(c) = 1 - \sup_{\gamma \in [c,1]} \left[\sqrt{a\gamma} + \sqrt{(1-a)(1-\gamma)} \right]^2.$$

If $c \le a$, then $\gamma = a \in [c, 1]$ and $\varphi(c) = 0$. If c > a, then the sup in (24) is achieved at $\gamma = c$. Thus φ is given by (20) and the proof of Claim 4 is complete.

Now, by Lemma 2, there is an index i_0 such that $b_{i_0}=0$ since $b_1=\varphi(1)$ and $b_i=\varphi(b_{i-1})$. Thus, $B_{i_0}=Z_0=D_{m+1}$ and by Claim 1, K^* is transitive on D_{m+1} . This completes the proof of Lemma 3.

The following is an immediate consequence of Theorem 1.

COROLLARY 1. Let $G_1 = \langle \mathscr{R} \rangle$ where $\mathscr{R} = \{S_r | r \in R\}$. If G_1 is infinite and irreducible, then the closure of G_1 is O(n). Also, for each $x \in C_n$, $\{gx | g \in G_1\}$ is dense in C_n .

REMARK. The assumption that G is generated by reflections cannot be removed since $O^+(n)$, $n \ge 2$ is infinite, closed and irreducible but $O^+(n) \ne O(n)$. Our interest in Theorem 1 arose in connection with results for G-monotone functions when G is generated by reflections (see Eaton and Perlman (1976)).

REFERENCES

1. C. T. Benson and L. C. Grove, *Finite Reflection Groups*, Bogden and Quigley, Tarrytown-on-Hudson, New York, 1971.

2. M. L. Eaton and M. D. Perlman, Reflection Groups, Generalized Schur Functions and the Geometry of Majorization. To appear in the Annals of Probability, 1976.

Received February 11, 1977 and in revised form July 28, 1977. The research for the first author was supported in part by a grant from the National Science Foundation—NSF-GP-34482.

The research for the second author was supported in part by a grant from the National Science Foundation—NSF-MCS-72-04364-A03.

University of Copenhagen 5, Universitetsparken DK 2100 Copenhagen, Ø Denmark

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California Los Angeles, California 90024

C. W. CURTIS

University of Oregon Eugene, OR 97403

C.C. MOORE

University of California Berkeley, CA 94720 J. Dugundji

Department of Mathematics University of Southern California Los Angeles, California 90007

R. FINN AND J. MILGRAM Stanford University

Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. Wolf

K. Yoshida

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA STANFORD UNIVERSITY UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON OSAKA UNIVERSITY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 73, No. 1 March, 1977

Thomas Robert Berger, Hall-Higman type theorems. V	1
Frank Peter Anthony Cass and Billy E. Rhoades, Mercerian theorems via	
spectral theory	63
Morris Leroy Eaton and Michael David Perlman, Generating $O(n)$ with	70
reflections	73
Frank John Forelli, Jr., A necessary condition on the extreme points of a class of holomorphic functions	81
Melvin F. Janowitz, Complemented congruences on complemented	
lattices	87
Maria M. Klawe, Semidirect product of semigroups in relation to	0.4
amenability, cancellation properties, and strong Fø lner conditions	91
Theodore Willis Laetsch, Normal cones, barrier cones, and the "spherical	
image" of convex surfaces in locally convex spaces	107
Chao-Chu Liang, Involutions fixing codimension two knots	125
Joyce Longman, On generalizations of alternative algebras	131
Giancarlo Mauceri, Square integrable representations and the Fourier	
algebra of a unimodular group	143
J. Marshall Osborn, Lie algebras with descending chain condition	155
John Robert Quine, Jr., Tangent winding numbers and branched	
mappings	161
Louis Jackson Ratliff, Jr. and David Eugene Rush, Notes on ideal covers	
and associated primes	169
H. B. Reiter and N. Stavrakas, On the compactness of the hyperspace of	
faces	193
Walter Roth, A general Rudin-Carlson theorem in Banach-spaces	197
Mark Andrew Smith, Products of Banach spaces that are uniformly rotund	
in every direction	215
Roger R. Smith, The R-Borel structure on a Choquet simplex	221
Gerald Stoller, The convergence-preserving rearrangements of real infinite	
series	227
Graham H. Toomer, Generalized homotopy excision theorems modulo a	
Serre class of nilpotent groups	233
Norris Freeman Weaver, Dehn's construction and the Poincaré	
conjecture	247
Steven Howard Weintraub, Topological realization of equivariant	
intersection forms	257