Pacific Journal of Mathematics

GENERATING O(n) WITH REFLECTIONS

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Vol. 73, No. 1

March 1977

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For $r \in C_n \equiv \{x | x \in \mathbb{R}^n, ||x|| = 1\}$, let $S_r = I_n - 2rr'$ where r is a column vector. O(n) denotes the orthogonal group on \mathbb{R}^n . If $R \subseteq C_n$, let $\mathscr{R} = \{S_r | r \in R\}$ and let G be the smallest closed subgroup of O(n) which contains \mathscr{R} . G is reducible if there is a nontrivial subspace $M \subseteq \mathbb{R}^n$ such that $gM \subseteq M$ for all $g \in G$. Otherwise, G is *irreducible*.

THEOREM. If G is infinite and irreducible, then G = O(n).

In what follows, \mathbb{R}^n denotes Euclidean *n*-space with the standard inner product, O(n) is the orthogonal group of \mathbb{R}^n , and $C_n = \{x \mid x \in \mathbb{R}^n, ||x|| = 1\}$. If U is a subset of O(n), $\langle U \rangle$ denotes the group generated algebraically by U and $\langle \overline{U} \rangle$ denotes the closure of $\langle U \rangle$. Thus, $\langle \overline{U} \rangle$ is the smallest closed subgroup of O(n) containing U. For an integer $k, 1 \leq k < n, M_k$ denotes a k-dimensional linear subspace of \mathbb{R}^n . If $r \in C_n$, let $S_r = I - 2rr'$ where r is a column vector. Thus S_r is a reflection through r-henceforth called a reflection.

Suppose $R \subseteq C_n$ and let $\mathscr{R} = \{S_r | r \in R\}$. Set $G = \langle \overline{\mathscr{R}} \rangle$. The group G is *reducible* if there is an M_k such that $gM_k \subseteq M_k$ for all $g \in G$; otherwise, G is *irreducible*. The main result of this note is the following.

THEOREM 1. If G is infinite and irreducible, then G = O(n).

Proof of Theorem 1. First note that if $S_r \in \mathscr{R}$ and $g \in G$, then $gS_rg^{-1} = S_{gr} \in G$. Let $\Delta = \{gr | g \in G, r \in R\}$. Thus, $t \in \Delta$ implies that $S_t \in G$. Since G is infinite, Δ must be infinite (see Benson and Grove (1971), Proposition 4.1.3). Since every Γ in O(n) is a product of a finite number of reflections, to show that G = O(n), it suffices to show that G is transitive on C_n (if G is transitive on C_n , then $\Delta = C_n$ so every reflection is an element of G and hence G = O(n)).

The proof that G is transitive on C_n follows. By Lemma 1 (below), there is a subgroup $K_2 \subseteq G$ and a subspace $M_2 \subseteq R^n$ such that kx = x if $x \in M_2^{\perp}$ and $k \in K_2$ and K_2 is transitive on $D_2 \equiv M_2 \cap C_n$. Since G is irreducible, there is an $r_2 \in R$ such that $r_2 \notin M_2$ and $r_2 \notin M_2^{\perp}$. Let $M_3 = \text{span} \{r_2, M_2\}$ and let $K_3 = \langle \{K_2, S_{r_2}\} \rangle \subseteq G$. With $D_3 \equiv M_3 \cap C_n$, Lemma 3 (below) implies that kx = x for all $x \in M_3^{\perp}$ and $k \in K_3$, and K_3 is transitive on D_3 . Again, since G is irreducible, there is an $r_3 \in R$ such that $r_3 \notin M_3$ and $r_3 \notin M_3^{\perp}$. With $M_4 = \text{span} \{r_3, M_3\}$, let $K_4 = \langle \{K_3, S_{r_3}\} \rangle \subseteq G$ and let $D_4 \equiv M_4 \cap C_n$. By Lemma 3 (below)

kx = x for $x \in M_4^{\perp}$ and $k \in K_4$ and K_4 is transitive on D_4 . Applying this argument (n-2) times, we obtain $K_n \subseteq G$ and K_n is transitive on $D_n = C_n$. Thus, G is transitive on C_n and the proof is complete.

To fill in the gaps in the above argument, it remains to prove Lemmas 1, 2, and 3. Lemma 1 provides the starting point for the stepwise argument used in the proof of Theorem 1.

LEMMA 1. If G is irreducible and infinite, there is a subspace M_2 and a subgroup $K_2 \subseteq G$ such that kx = x for $x \in M_2^{\perp}$, $k \in K_2$ and K_2 acts transitively on $D_2 \equiv M_2 \cap C_n$.

Proof. As noted in the proof of Theorem 1, the set $\Delta = \{gr | r \in R, g \in G\}$ is infinite. Thus, there is a point $\delta_0 \in C_n$ such that every neighborhood of δ_0 contains infinitely many points in Δ . Thus we can select a sequence of pairs $(r_i, t_i), r_i, t_i \in \Delta$, such that r_i and t_i are linearly independent and $1 - 1/i < r'_i t_i < r'_{i+1} t_{i+1} < 1$ for $i = 1, 2, \cdots$.

For $0 \leq \eta < 2\pi$, set

(1)
$$\Psi(\eta) = \begin{pmatrix} \cos \eta & \sin \eta \\ -\sin \eta & \cos \eta \end{pmatrix} \in O(2)$$
.

Define θ_i by $\cos \theta_i = r'_i t_i$, $0 \leq \theta_i < \pi$ so $\theta_i \to 0$ as $i \to \infty$. Let $\Gamma_i \in O(n)$ have first row t'_i and second row

$$(r_i - t'_i r_i t_i)' / || r_i - t'_i r_i t_i ||$$
.

Then an easy calculation shows that

(2)
$$S_{i_i}S_{r_i} = \Gamma'_i \begin{pmatrix} \Psi(2\theta_i) & 0 \\ 0 & I_{n-2} \end{pmatrix} \Gamma_i, \quad i = 1, 2, \cdots$$

where I_{n-2} is an $(n-2) \times (n-2)$ identity matrix. Setting $H_i = \langle \Psi(2\theta_i) \rangle \subseteq O(2)$, it is clear that

$$(3) \qquad \left\{ \Gamma'_i \begin{pmatrix} h & 0 \\ 0 & I_{n-2} \end{pmatrix} \Gamma_i \middle| h \in H_i \right\} \subseteq G, \qquad i = 1, 2, \cdots$$

By selecting an appropriate subsequence, we can assume without loss of generality that $\Gamma_i \to \Gamma_0 \in O(n)$, as $i \to \infty$.

If $\Psi(\eta)$ is given by (1), we now claim that

(4)
$$\Gamma_0' \begin{pmatrix} \Psi(\gamma) & 0 \\ 0 & I_{n-2} \end{pmatrix} \Gamma_0 \in G .$$

Since G is closed and (3) holds, to establish (4), it suffices to show

there is a subsequence i_j and $h_{ij} \in H_{ij}$ such that $h_{ij} \to \Psi(\eta)$ as $i_j \to \infty$. However, the existence of such a sequence is assured since $\theta_i \to 0$ as $i \to \infty$. Thus (4) holds. Hence we see that

(5)
$$K_{2} \equiv \left\{ \Gamma_{0}' \begin{pmatrix} h & 0 \\ 0 & I_{n-2} \end{pmatrix} \Gamma_{0} \middle| h \in H^{*} \right\} \subseteq G$$

where H^* is the full rotation group of R^2 .

To complete the proof of Lemma 1, let M_2 be the span of the first two columns of Γ'_0 . With $D_2 \equiv M_2 \cap C_n$, it is easy to check that kx = x for all $x \in M_2^{\perp}$, $k \in K_2$ and that K_2 acts transitively on D_2 . This completes the proof.

The following result is used in the proof of Lemma 3.

LEMMA 2. For
$$u_0 \in (0, 1]$$
, define a function $f: [0, 1] \rightarrow [0, 1]$ by

$$(6) f(u) = \begin{cases} 0 & if \quad 0 \le u \le u_0 \\ 1 - [\sqrt{uu_0} + \sqrt{(1-u)(1-u_0)}]^2 & if \ u_0 \le u \le 1 \end{cases}$$

Let $v_1 = f(1)$ and define $v_i = f(v_{i-1})$ for $i = 2, 3, \cdots$. Then, there exists an index i_0 such that $v_i = 0$ for $i \ge i_0$.

Proof. It is not hard to verify that f is a continuous convex function. Since $0 \leq v_1 < 1$, $v_2 = f(v_1) = f((1 - v_1)0 + v_11) \leq v_1f(1) = v_1^3$. Proceeding by induction, $v_i \leq v_1^i$ so $\lim_{i \to \infty} v_i = 0$. Since f is 0 in the interval $[0, u_0]$, there is an index i_0 such that $v_i = 0$ for $i \geq i_0$. This completes the proof.

After establishing Lemma 1, the key to Theorem 1 is Lemma 3. Although the proof of Lemma 3 is quite long, the geometric idea behind the proof is fairly simple. Consider R^3 and let $D_2 = \{x | x \in R^3, x_3 = 0, x_1^2 + x_2^2 = 1\}$. Also, let $H = \left\{ \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \middle| k$ is any rotation of $R^2 \right\}$. Thus H acts transitively on D_2 . Consider a fixed vector $t \in R^3$ with ||t|| = 1 such that t is not in the (x_1, x_2) plane and t is not in the x_3 -line. Let $S_t = I$ -2tt' be the reflection across the plane $\{t\}^{\perp}$ and let \widetilde{H} be the group generated by S_t and H. The claim is that \widetilde{H} is transitive on $D_3 = \{x | x \in R^3, ||x|| = 1\}$. For example, suppose the angle between t and the (x_1, x_2) plane is 45° . Geometrically, it is clear that the set $H(S_t(D_2)) \equiv \{x | x = hS_t u$ for some $h \in H$, and some $u \in D_2$ is just D_3 —that is, $S_t(D_2)$ is a circle passing through $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and the transitivity of H implies that H moves the set $S_t(D_2)$ everywhere onto D_3 (picture this on the surface of a basketball). Thus, given $v_1, v_2 \in D_3, v_i = h_i S_i u_i$, for $h_i \in H$ and $u_i \in D_2$ for i = 1, 2. Since $u_1 = h_0 u_2$ for some $h_0 \in H$, it follows that $v_1 = h_1 S_t h_0 S_t h_2^{-1} v_2$ so \tilde{H} is transitive on D_3 . For other t-vectors, D_3 does not get covered by one application of HS_t to D_2 , but D_3 is covered by a finite number of applications of HS_t to D_2 —that is, $D_3 = (H(S_t(\cdots)H)S_t)(D_2)$ for some finite string $HS_tHS_t\cdots HS_t$. Again, this implies the transitivity of \tilde{H} on D_3 . Lemma 3 and its proof make all of the above precise.

LEMMA 3. Consider a subspace $M_m \subseteq R^n$, $2 \leq m < n$, and suppose that K is a subgroup of O(n) such that

(7)
$$\begin{cases} kx = x \quad for \quad all \quad x \in M_m^{\perp}, k \in K \\ K \quad is \quad transitive \quad on \quad D_m \equiv M_m \cap C_n \end{cases}$$

Let $t \in C_n$ be such that $t \notin M_m$ and $t \notin M_m^{\perp}$. With $M_{m+1} = \text{span} \{t, M_m\}$, let $D_{m+1} \equiv M_{m+1} \cap C_n$. Then the group $K^* \subseteq O(n)$ generated by K and $S_t = I - 2tt'$ satisfies

$$(8) \qquad \qquad \begin{cases} kx = x \quad for \ all \quad x \in M_{m+1}^{\perp}, \ k \in K^* \\ K^* \quad is \ transitive \ on \ D_{m+1} \ . \end{cases}$$

Proof. That kx = x for all $x \in M_{m+1}^{\perp}$ and $k \in K^*$ is not hard to verify. To establish the transitivity of K^* on D_{m+1} , define a set B_1 by (9) $B_1 = K(S_i(D_m)) = \{x | x = kS_i u \text{ for some } u \in D_m, \text{ some } k \in K\}$ and then define B_i inductively by (10) $B_i = K(S_i(B_{i-1})) = \{x | x = kS_i u \text{ for some } u \in B_{i-1}, \text{ some } k \in K\}$ $i = 2, 3, \cdots$. Since $K(S_i(D_{m+1})) \subseteq D_{m+1}$, it follows that $B_i \subseteq D_{m+1}$ for all *i*. The remainder of the proof is devoted to showing that there is an index i_0 such that $B_{i_0} = D_{m+1}$, because this implies the transitivity of K^* on D_{m+1} .

Claim 1. If
$$B_{i_0} = D_{m+1}$$
, then K^* is transitive on D_{m+1} .
Proof of Claim 1. Consider $z_1, z_2 \in D_{m+1}$. If $B_{i_0} = D_{m+1}$, then
$$\underbrace{K(S_t(K(S_t \cdots (D_m))))}_{i_0 \text{-terms}} = D_{m+1}.$$

Thus, there exists $k_1, \dots, k_{i_0} \in K$ and $g_1, \dots, g_{i_0} \in K$ such that

$$oldsymbol{z}_{\scriptscriptstyle 1} = igg[\prod_{j=1}^{i_0} \left(k_j oldsymbol{S}_{oldsymbol{t}}
ight) igg] oldsymbol{u}_{\scriptscriptstyle 1} \equiv h_{\scriptscriptstyle 1} oldsymbol{u}_{\scriptscriptstyle 1}$$

and

$$oldsymbol{z}_{\scriptscriptstyle 2} = iggl[\prod_{j=1}^{i_0} \left(g_j s_t
ight) iggr] oldsymbol{u}_{\scriptscriptstyle 2} \equiv h_{\scriptscriptstyle 2} oldsymbol{u}_{\scriptscriptstyle 2}$$

for some $u_1, u_2 \in D_m$. Since K is transitive on D_m , there exists a $k_0 \in K$ such that $k_0 u_1 = u_2$. Thus, $z_2 = h_2 k_0 h_1^{-1} z_1$ which shows that K^* is transitive on D_{m+1} as $h_2 k_0 h_1^{-1} \in K^*$. This completes the proof of Claim 1.

We now continue with the proof. Let P denote the orthogonal projection onto M_m and define Z_c , $0 \leq c \leq 1$ by

(11)
$$Z_{c} = \{x \mid x \in D_{m+1}, ||Px||^{2} \geq c\}.$$

Note that $Z_1 = D_m$ and $Z_0 = D_{m+1}$.

REMARK. Geometrically, Z_c is an equatorial zone (with equator D_m) which partially covers D_{m+1} . Smaller values of c correspond to more of D_{m+1} being covered.

Define φ on [0, 1] by

(12)
$$\varphi(c) = \inf_{x \in Z_c} ||PS_i x||^2$$
 , $0 \leq c \leq 1$,

and let

(13)
$$b_1 = \inf_{x \in B_1} ||Px||^2$$
.

Since each $k \in K$ commutes with P, we have

 $(14) \quad b_{1} = \inf_{k \in K} \inf_{x \in D_{m}} ||PkS_{t}x||^{2} = \inf_{x \in D_{m}} ||PS_{t}x||^{2} = \inf_{x \in Z_{1}} ||PS_{t}x||^{2} = \varphi(1).$

Claim 2. $B_1 = Z_{b_1}$.

Proof of Claim 2. If $x \in B_1$, $||Px||^2 \ge b_1$ which implies that $x \in Z_{b_1}$. Conversely, consider $x \in Z_{b_1}$ and let Q denote the orthogonal projection onto the one-dimensional subspace $M_m^{\perp} \cap M_{m+1}$ which is spanned by the vector $t^* \equiv (I - P)t/||(I - P)t||$. Since Z_c is compact and arcwise connected, the continuous function $u \to ||PS_tu||^2 (u \in Z_c)$ takes on all values between 1 and $\varphi(c)$. As $x \in Z_{b_1}$,

$$||Px||^{2} \ge b_{1} = \varphi(1) = \inf_{u \in D_{m}} ||PS_{t}u||^{2}$$
.

Hence, there exists a $u \in D_m$ such that $||PS_tu||^2 = ||Px||^2$. Thus, $1 = ||Px||^2 + ||Qx||^2 = ||PS_tu||^2 + ||QS_tu||^2$, so $||QS_tu||^2 = ||Qx||^2$. Since Q is a projection onto a one-dimensional subspace, u can be chosen (by changing to -u if necessary) such that $Qx = QS_tu$. The transitivity of K on D_m implies there is a $k \in K$ such that $kPS_tu = Px$. Thus,

 $kS_iu = kPS_iu + kQS_iu = Px + kQS_iu = Px + QS_iu = Px + Qx = x$, so $x = kS_iu \in B_1$. This completes the proof of Claim 2.

Using Claim 2, $B_2 = K(S_t(B_1)) = K(S_t(Z_{b_1}))$. Consider

(15)
$$b_2 \equiv \inf_{x \in B_2} ||Px||^2$$
 .

Using (15) and the fact that each $k \in K$ commutes with P, we have

(16)
$$b_2 = \inf_{x \in B_2} ||Px||^2 = \inf_{x \in Z_{b_1}} \inf_{k \in K} ||PkS_tx||^2 = \inf_{x \in Z_{b_1}} ||PS_tx||^2 = \varphi(b_1)$$

Claim 3. $B_2 = Z_{b_2}$.

Proof of Claim 3. If $x \in B_2$, then $x \in D_{m+1}$ and $||Px||^2 \ge b_2$, so $x \in Z_{b_2}$. Conversely, consider $x \in Z_{b_2}$. As u varies over Z_{b_1} , the function $u \to ||PS_tu||^2$ takes on all values between 1 and b_2 . Since $||Px||^2 \ge b_2$, there is a $u \in Z_{b_1}$ such that $||PS_tu||^2 = ||Px||^2$. As in the proof of Claim 2, $1 = ||Px||^2 + ||Qx||^2 = ||PS_tu||^2 + ||QS_tu||^2 = ||QS_tu||^2$ so $||Qx||^2 = ||QS_tu||^2$, and we can choose u such that $Qx = QS_tu$. The transitivity of K implies that there is a $k \in K$ such that $kPS_tu = Px$. Thus, $x = Px + Qx = kPS_tu + QS_tu = kPS_tu + kQS_tu = kS_tu \in B_2$ since $u \in Z_{b_1} = B_1$. The proof of Claim 3 is complete.

Arguing as in the proof of Claim 3, it is an easy matter to show that $B_i = Z_{b_i}$ and $b_i = \varphi(b_{i-1})$ where

(17)
$$b_i = \inf_{x \in B_i} ||Px||^2, i = 3, 4, \cdots$$

As noted earlier, the proof of Lemma 3 will be complete if we can show there is an index i_0 such that $B_{i_0} = Z_0 = D_{m+1}$. To establish the existence of an i_0 , we will explicitly calculate the function φ defined in (12) and then apply Lemma 2. Define $z_0 \in D_{m+1}$ by

$$(18) z_0 = S_t t^*$$

where $t^* = (I - P)t/||(I - P)t||$. Then,

(19)
$$a \equiv ||Pz_0||^2 = \frac{||PS_t(I-P)t||^2}{||(I-P)t||^2} = \frac{||P(I-2tt')(I-P)t||^2}{||(I-P)t||^2} = \frac{4||Pt||^2(t'(I-P)t)|^2}{||(I-P)t||^2} = 4||Pt||^2(1-||Pt||^2).$$

Since $t \notin M_m$ and $t \notin M_m^{\scriptscriptstyle \perp}$, $0 < ||Pt||^2 < 1$ so $0 < a \leq 1$.

Claim 4. The function φ is given by

(20)
$$\varphi(c) = \begin{cases} 0 & \text{if } 0 \leq c \leq a \\ 1 - \left[\sqrt{ac} + \sqrt{(1-a)(1-c)}\right]^2 & \text{if } a \leq c \leq 1 \end{cases}$$

Proof of Claim 4. Since $Q = t^*t^{*'}$ (see the proof of Claim 2), for each $x \in R^*$, $||QS_tx||^2 = x'S_tQS_tx = x'S_tt^*t^*S_tx = (z_0'x)^2$. Thus,

(21)
$$\varphi(c) = \inf_{x \in Z_c} ||PS_t x||^2 = \inf_{x \in Z_c} (1 - ||QS_t x||^2) = 1 - \sup_{x \in Z_c} (z_0' x)^2.$$

If a = 1, then $z_0 \in D_m \subseteq Z_c$, so $\sup_{x \in Z_c} (z'_0 x)^2 = 1$ and $\varphi(c) = 0$ for all $c \in [0, 1]$.

Now, consider $a \in (0, 1)$. For $x \in Z_c$, let $\gamma = ||Px||^2 \ge c$. Then, by the Cauchy-Schwarz inequality, we have

(22)
$$\begin{array}{c} z_0'x = z_0'Px + z_0'Qx = (Pz_0)'Px + (Qz_0)'Qx \\ \leq ||Pz_0|| ||Px|| + ||Qz_0|| ||Qx|| = \sqrt{a}\sqrt{\gamma} + \sqrt{1-a}\sqrt{1-\gamma} \ . \end{array}$$

Further, there is equality in the above inequality for $x = x_0$ where

(23)
$$x_0 = \sqrt{\gamma/a} P z_0 + \sqrt{(1-\gamma)/(1-a)} Q z_0.$$

Clearly, $||Px_0||^2 = \gamma \ge c$ so $x_0 \in Z_c$. Thus,

(24)
$$\varphi(c) = 1 - \sup_{\gamma \in [c,1]} \left[\sqrt{a\gamma} + \sqrt{(1-a)(1-\gamma)} \right]^2.$$

If $c \leq a$, then $\gamma = a \in [c, 1]$ and $\varphi(c) = 0$. If c > a, then the sup in (24) is achieved at $\gamma = c$. Thus φ is given by (20) and the proof of Claim 4 is complete.

Now, by Lemma 2, there is an index i_0 such that $b_{i_0} = 0$ since $b_1 = \varphi(1)$ and $b_i = \varphi(b_{i-1})$. Thus, $B_{i_0} = Z_0 = D_{m+1}$ and by Claim 1, K^* is transitive on D_{m+1} . This completes the proof of Lemma 3.

The following is an immediate consequence of Theorem 1.

COROLLARY 1. Let $G_1 = \langle \mathscr{R} \rangle$ where $\mathscr{R} = \{S_r | r \in R\}$. If G_1 is infinite and irreducible, then the closure of G_1 is O(n). Also, for each $x \in C_n$, $\{gx | g \in G_1\}$ is dense in C_n .

REMARK. The assumption that G is generated by reflections cannot be removed since $O^+(n)$, $n \ge 2$ is infinite, closed and irreducible but $O^+(n) \ne O(n)$. Our interest in Theorem 1 arose in connection with results for G-monotone functions when G is generated by reflections (see Eaton and Perlman (1976)).

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Received February 11, 1977 and in revised form July 28, 1977. The research for the first author was supported in part by a grant from the National Science Foundation—NSF-GP-34482.

The research for the second author was supported in part by a grant from the National Science Foundation—NSF-MCS-72-04364-A03.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

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