COMPLEMENTED CONGRUENCES ON COMPLEMENTED LATTICES

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We prove that a congruence relation on a complemented lattice has a complement if and only if it is the minimal congruence generated by a central element. This result is then used to show that a complemented lattice has a Boolean lattice of congruence relations if and only if it is the direct product of a finite number of simple lattices. It is also used to obtain some information on the structure of complemented lattices whose lattice of congruences is a Stone lattice.

1. Introduction. What does it mean for a congruence relation \( \theta \) on a complemented lattice \( L \) to have a complement in the lattice \( \text{Con}(L) \) of congruence relations of \( L \)? The answer to this question provides the underlying theme for the paper. In case every interval \([0, a]\) is complemented, then some results of Grätzer and Schmidt ([1], Theorem 11, p. 56 and [1], Lemma 8, p. 37) can be used to show that \( \theta \) has a complement in \( \text{Con}(L) \) if and only if there is a central element \( z \) of \( L \) such that \( \theta \) is the minimal congruence generated by the ideal \([0, z]\). In §2 this result is extended to an arbitrary complemented lattice. It is then used to obtain the structure of those complemented lattices for which \( \text{Con}(L) \) is a Boolean algebra. At this point, it is shown (for a suitable class of lattices) that \( \text{Con}(L) \) being a Stone lattice is related to the existence of certain suprema in \( L \).

2. Complemented congruences. Let \( \theta, \theta' \) be congruences on the bounded lattice \( L \). Suppose \( \theta, \theta' \) are disjoint in that \( a \land (\theta \lor \theta') = b \) implies \( a = b \). The key to what is happening is provided by

**Lemma 1.** Let 0 denote the least element of \( L \). If \( 0 < a < b \) with \( 0 \land \theta \land b \), then:

1. \( (x \lor a) \land b = (x \land b) \lor a \) for every \( x \in L \).
2. \( a \) is neutral in \([0, b]\).

*If* \( L \) is complemented, we may add:

3. \( a \) is central in \([0, b]\).
4. There is an element \( c \in L \) such that \( 0 < c < b \) and \( 0 \land \theta' \land c \).

**Proof.** (1) Given \( x \in L \), we note that \( (x \lor a) \land b \land \theta \land x \land b \land (x \land b) \lor a \). Since \( (x \lor a) \land b \), \( (x \land b) \lor a \in [a, b] \) with \( a \land \theta' \land b \), it follows that \( (x \lor a) \land b = (x \land b) \lor a \).
(2) Let \(x, y \in [0, b]\), and set \(s = (a \wedge x) \vee (x \wedge y) \vee (y \wedge a)\), \(t = (a \vee x) \wedge (x \vee y) \wedge (y \vee a)\). Then \(s \theta t\) follows from \(0 \theta a\), and \(s' \theta t\) from \(a \theta' b\). Consequently, \(s = t\), and by [2], \(a\) is neutral in \([0, b]\).

(3) Let \(a'\) be a complement for \(a\) in \(L\). Then \(a \wedge (b \wedge a') = 0\), and by (1), \(a \vee (b \wedge a') = (a' \vee a) \wedge b = b\), so \(b \wedge a'\) is a complement for \(a\) in \([0, b]\). But this says that \(a\) is central in \([0, b]\).

(4) Take \(c = b \wedge a'\).

We are now ready to state our principal result.

**Theorem 2.** Let \(L\) be a complemented lattice. A congruence relation \(\theta\) has a complement in \(\text{Con}(L)\) if and only if there is a central element \(z\) of \(L\) such that \(\theta\) is the minimal congruence generated by \([0, z]\).

**Proof.** If \(z\) exists, it is clear that \(\theta\) has a complement in \(\text{Con}(L)\). Suppose conversely that \(\theta\) has a complement \(\theta'\) in \(\text{Con}(L)\). We may then find a finite chain
\[0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1\]
of minimal length such that \(x_{i-1} \theta x_i\) or \(x_{i-1} \theta' x_i\) for \(i = 1, 2, \ldots, n\). If \(n = 1\), there is nothing to prove, so we may as well assume \(n \geq 2\).

In view of Lemma 1 (4), we may also assume that \(x_i \theta x_i \theta' x_2\). If \(n \geq 3\) we must have \(x_2 \theta x_3\). We may apply Lemma 1 (4) to the interval \([0, x_2]\) to obtain an element \(c \in L\) such that \(0 < c < x_2\) and \(0 \theta' c \theta x_2\). But then the chain
\[0 = x_0 < c < x_3 < \cdots < x_{n-1} < x_n = 1\]
with \(0 \theta' c \theta x_2\) is a chain of shorter length than our original minimal length chain. From this contradiction we deduce that \(n = 2\), so there is an element \(z\) such that \(0 < z < 1\) and \(0 \theta z \theta' 1\). By Lemma 1, \(z\) is central. Evidently \(x \theta y\) is equivalent to \(x \vee z = y \vee z\), so \(\theta\) is the minimal congruence generated by the ideal \([0, z]\).

This leads immediately to

**Theorem 3.** Let \(L\) be a complemented lattice. A necessary and sufficient condition for \(\text{Con}(L)\) to be a Boolean algebra is that \(L\) be the direct product of a finite number of simple lattices.

**Proof.** Sufficiency is clear. To establish necessity, it suffices to show that if \(\text{Con}(L)\) is Boolean, then \(L\) must have a finite center. For then, if \(z_1, z_2, \ldots, z_n\) are the atoms of the center of \(L\), and if
Let \( L_i = [0, z_i] \), then \( L \) would be isomorphic to the direct product of the irreducible lattices \( L_1, L_2, \ldots, L_n \). But each \( L_i \) is a homomorphic image of \( L \), whence each \( \text{Con}(L_i) \) is Boolean. An application of Theorem 2 to the complemented lattice \( L_i \) yields \( \text{Con}(L_i) \) a 2 element chain, since the center of \( L_i \) is \( \{0, z_i\} \). In other words, each \( L_i \) is in fact simple.

We now proceed to show the center of \( L \) to be finite. Suppose this were not true. We could then find an ideal \( J \) of the center of \( L \) that is not principal. Define \( \theta \) on \( L \) by the rule \( x \theta y \) if \( x \lor z_a = y \lor z_a \) for some \( z_a \in J \), and note that \( \theta \in \text{Con}(L) \). But this forces the existence of a central element \( z \) such that \( x \theta y \) iff \( x \lor z = y \lor z \), contrary to the fact that \( J \) is not a principal ideal of the center.

3. Stone lattices. In [3] we asked what it meant for \( \text{Con}(L) \) to be a Stone lattice in the sense that for each congruence relation \( \theta, \theta^* \) and \( \theta^{**} \) are complements in \( \text{Con}(L) \). Here \( \theta^* \) denotes the pseudocomplement of \( \theta \) in \( \text{Con}(L) \). The foregoing results can be used to show that for a fairly wide class of complemented lattices, this is related to the existence of certain suprema in \( L \). The class of lattices we have in mind is the class that satisfies (A), (A*), (B) and (B*) of [4]. (Note: Axiom (X*) denotes the dual of Axiom X.) For the reader's convenience we restate (A) and (B) here:

(A) \( a/0 \rightarrow c/d \) with \( c > d \) implies \( c/d \rightarrow a_1/a_2 \) for suitable \( a_1, a_2 \) such that \( a \geq a_1 > a_2 \)

(B) \( a > b \) implies the existence of an element \( t \) such that \( t\theta_{a/b} \leq t \neq a \).

It should be noted that \( \theta_{a/b} \) denotes the smallest congruence that identifies \( a \) and \( b \). To illustrate the scope of these axioms, we mention that (A), (A*), (B) and (B*) are satisfied by each of the following types of lattices:

(i) any bounded relatively complemented lattice;
(ii) any lattice that is both atomistic and dual atomistic;
(iii) any uniquely complemented lattice;
(iv) any simple lattice;
(v) the direct product of lattices of any of the preceding types.

Here then is our result.

**Theorem 4.** (1) Let \( L \) be a complemented lattice that satisfies (A*) and (B*). If \( \text{Con}(L) \) is a Stone lattice, then the kernel of every congruence relation of \( L \) has a supremum in \( L \).

(2) Let \( L \) be a bounded lattice satisfying (A), (A*), (B) and (B*). If the kernel of each congruence relation of \( L \) has a supremum in \( L \), then \( \text{Con}(L) \) is a Stone lattice.
Proof. (1) Let $\theta \in \text{Con}(L)$ have kernel $J$. By the dual of Theorem 2, there is a central element $z$ of $L$ such that $\theta^*$ is the minimal congruence generated by the filter $[z, 1]$. By the dual of [4], Theorem 3, p. 179, $a\theta^*1$ iff $a$ is an upper bound for the kernel of $\theta$. Hence $z = \vee J$.

(2) Let $\theta \in \text{Con}(L)$ and let $z$ be the supremum of the kernel of $\theta$. By the dual of [4], Theorem 3, p. 179, $[z, 1] = \{t \in L: t\theta^*1\}$. Since $z$ is a lower bound for $\{t \in L: t\theta^*1\}$, we may apply [4], Theorem 3, p. 179 with $\theta$ replaced by $\theta^*$ to deduce that $z\theta^{**}0$. Thus, $0\theta^{**}z\theta^*1$ and so $\theta^{**}$ is a complement for $\theta^*$ in $\text{Con}(L)$.

Corollary. For $L$ a Boolean algebra, $\text{Con}(L)$ is a Stone lattice if and only if $L$ is complete.

Proof. Suppose $\text{Con}(L)$ is a Stone lattice. Then for $S$ an arbitrary nonempty subset of $L$, the ideal $J$ generated by $S$ is the kernel of a congruence. Hence $\vee J$ exists in $L$, and it is clearly effective as the supremum of $S$. The converse is clear.

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