INVOLUTIONS FIXING CODIMENSION TWO KNOTS

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1. Involution. An $m$-knot $(\Sigma^{m+2}, M^m)$ consists of an $(m + 2)$-homotopy sphere $\Sigma^{m+2}$ and an $m$-homotopy sphere $M^m$ differentiably (or piecewise linearly) embedded in it. A $(2n - 1)$-knot is called simple if $\pi_j(\Sigma - M) = \pi_j(S^1)$ for $j < n$. It is well known that each knot cobordism class contains a simple knot, [5] or [7].

Associated to each $(2n - 1)$-knot, we have Seifert matrices $B$, with $B + \varepsilon B'$ unimodular, where $\varepsilon = (-1)^n$ and $B'$ denotes the transpose of $B$. For $n \geq 2$, the isotopy class of simple knot is completely determined by its Seifert matrices [8].

In [1, §11], Cappell and Shaneson used their algebraic K-theoretic obstruction groups to determine which knot cobordism classes admit semifree $\mathbb{Z}_p$ actions fixing the knots. In §3 below, we will prove the following theorem from the viewpoint of [5] and [8].

**THEOREM 1.** A simple knot $(\Sigma^{2n+1}, M^{2n-1})$, $n \geq 3$, admits a p. 1. involution $T$ fixing $M^{2n-1}$ if and only if it has an associated Seifert matrix $B$ of the form $B = A(A - \varepsilon A')^{-1}A$ for some matrix $A$ with both $A + \varepsilon A'$ and $A - \varepsilon A'$ being unimodular.

We will also discuss the differentiable case in the last section.

2. A technical lemma. Recall that $\varepsilon = (-1)^n$.

**LEMMA 2.** Let $A$ be an $(r \times r)$-matrix with both $A + \varepsilon A'$ and $A - \varepsilon A'$ being unimodular. Then the following system of equations has a unique solution for the pair of $(r \times r)$-matrices $C_1$ and $C_2$.

(1) \[ C_1A + \varepsilon C_2A' = A + \varepsilon A' \]

(2) \[ \varepsilon C_1A' + C_2A = 0. \]

**Proof.**

(3) \[ (1)+(2)C_1(A + \varepsilon A') + C_2(A + \varepsilon A') = A + \varepsilon A'. \]

Since $A + \varepsilon A'$ is unimodular, (3) becomes

(4) \[ C_1 + C_2 = I, \text{ the identity} \]

(5) \[ (1)-(2)C_1(A - \varepsilon A') - C_2(A - \varepsilon A') = A + \varepsilon A'. \]

Since $A - \varepsilon A'$ is unimodular, (5) becomes
\( C_1 - C_2 = (A + \varepsilon A')(A - \varepsilon A')^{-1} \).

From (4) and (6), we have
\[ C_1 = A(A - \varepsilon A')^{-1} \quad \text{and} \quad C_2 = -\varepsilon A'(A - \varepsilon A')^{-1}. \]

3. Proof of Theorem 1. If a simple knot \((\Sigma^{2n+1}, M^{2n-1})\), \(n \geq 3\), admits a p.l. (or differentiable) involution \(T\) fixing \(M^{2n-1}\), then it is easy to see that \(\Sigma_1 = \Sigma/T\) is a \((2n + 1)\)-homotopy sphere, and \((\Sigma_1, M)\) is again a simple knot. Let \(Y\) be the closure of \((\Sigma_1 - M \times D^2)\), and \(V^{2n} \subseteq Y\) be an \((n-1)\)-connected Seifert manifold for \((\Sigma_1, M)\) with \(\partial V = M \times e^{i\theta}\), (we consider \(S^1 = \{e^{i\theta}\}\), [5], [7]. Lifting \(V\) to \(\Sigma\), we have two equivariant Seifert manifolds \(V_1\) and \(V_2\) with \(TV_1 = V_2, \partial V_1 = M \times e^{i\theta}\), and \(\partial V_2 = M \times e^{i\varepsilon}\), [9]. We then cut \(X = \text{closure of } (\Sigma - M \times D^2)\) along \(V_1\) to get a manifold \(W\).

\[
(*) \quad \begin{array}{ccc}
V_{1+} & W_1 & V_2 \\
W_2 & V_{1-} & W
\end{array}
\]

We see immediately that \(W_1\) is the manifold obtained from \(Y\) (in \(\Sigma_1\)) by cutting it along \(V\) and \(W_2 = TW_1\). Let \(\{e_1, \ldots, e_r\}\) be a basis for \(H_nV_{1+}\), and \(\{f_1, \ldots, f_r\}\) a basis for \(H_nW\) determined by the Alexander duality (in \(\Sigma\)). Similarly, viewing \(\{e_i\}\) as a basis for \(H_nV\), we have a basis \(\{d_i\}\) for \(H_nW_1\) by using the Alexander duality in \(\Sigma_1\). Let \(A\) and \(B\) be the Seifert matrices associated to \((\Sigma_1, M)\) and \((\Sigma, M)\) respectively (with respect to the basis \(\{e_i\}\)) [5], [7].

From [5], we know that \(A\) represents the map \(H_nV_{1+} \to H_nW_1\) with respect to the bases \(\{e_i\}\) and \(\{d_i\}\), also the map \(H_nV_2 \to H_nW_2\) with respect to the bases \(\{T_*e_i\}\) and \(\{T_*d_i\}\). The matrix \(-\varepsilon A'\) represents the map \(H_nV_2 \to H_nW_1\) with respect to the bases \(\{T_*e_i\}\) and \(\{d_i\}\), also the map \(H_nV_{1-} \to H_nW_2\) with respect to the bases \(\{e_i\}\) and \(\{T_*d_i\}\). The matrix \(B\) represents \(H_nV_{1+} \to H_nW\) with respect to the bases \(\{e_i\}\) and \(\{f_i\}\), and \(-\varepsilon B'\) represents \(H_nV_{1-} \to H_nW\) with respect to the bases \(\{e_i\}\) and \(\{f_i\}\). All the maps here are induced by inclusions. Finally, let \(C_1\) and \(C_2\) denote the matrices represent the maps \(H_nW_{1-} \to H_nW\) and \(H_nW_2 \to H_nW\) with respect to the appropriate bases respectively. From (*), we have the following equation:

\[ B = C_1A, \quad -\varepsilon B' = C_2(-\varepsilon A'), \quad C_1(-\varepsilon A') = C_2A. \]

These, together with the fact that \(A + \varepsilon A' = B + \varepsilon B' = \text{intersection form on } H_nV, \) [5], give us the two equations in Lemma 2. Also,
we have proved in [9] that both $A + \varepsilon A'$ and $A - \varepsilon A'$ are unimodular. Thus it follows from Lemma 2 that $B = C, A = A(A - \varepsilon A')^{-1}A$.

Conversely, given a knot $(\Sigma^{2n+1}, M^{2n-1})$ with its Seifert matrix $B$ satisfying the condition in Theorem 1, we can construct a simple knot $(\Sigma, M)$ with an $(n - 1)$-connected Seifert manifold $V$ and associated Seifert matrix $A$, [5]. Then we construct the 2-fold branched covering of $(\Sigma, M)$ to obtain a simple knot $(\Sigma_2, M)$ as in [4], [9], [12]. If we are in the $p.l.$ category, then both $\Sigma$ and $\Sigma_2$ are the standard sphere $S^{2n+1}$. Both $(\Sigma, M)$ and $(\Sigma_2, M)$ have the same Seifert matrix, hence they are actually equivalent, [8]. The involution $T$ is given by the covering translation for the branched covering.

4. Free involutions. Since the study of knots invariant under free involutions on spheres is very similar to that of knots fixed under involutions, [9], [10], the following theorem can be proved in a similar fashion.

THEOREM 1'. A simple knot $(\Sigma^{2n+1}, M^{2n-1})$, $n \geq 3$, admits a free $p.l.$ involution $T$ leaving $M$ invariant if and only if it has an associated Seifert matrix $B$ of the form $B = A(A - \varepsilon A')^{-1}A$ for some matrix $A$ with both $A + \varepsilon A'$ and $A - \varepsilon A'$ being unimodular.

5. The differentiable case. Let $T$ denote a differentiable involution on $\Sigma^{2n+1}$ fixing $M^{2n-1}$, $n \geq 3$. We want to study the relation between the differentiable structure of $\Sigma$ and $\Sigma_1 = \Sigma/T$. If $\Sigma_1 \neq S^{2n+1}$, then we may view $(\Sigma_1, M)$ as the connected sum of $(S^{2n+1}, M)$ and $\Sigma_1$ along a disk disjoint from the Seifert manifold $V$ and $M$. We then construct the 2-fold branched covering $(\Sigma_3, M)$ of $(S^{2n+1}, M)$ with branched point set $M$. By the uniqueness of differentiable structure of the cyclic branched covering ([2] or [4]), it is easy to see that $\Sigma = 2\Sigma_1 + \Sigma_3$, where the sum denotes the connected sum in the group of homotopy spheres $\Gamma_2^{2n+1}$, [6].

In the case $n$ is odd, we let $\Sigma_0$ denote the generator of $bP_{4k} = \{y \in \Gamma_{4k-1} \mid y$ bounds parallelizable manifolds$\}$. Then we have the following proposition.

PROPOSITION 3. $\Sigma = 1/8(\text{index } (A + A'))\Sigma_0 + 2\Sigma_1$.

Proof. We first note that $A + A'$ is a unimodular, symmetric, even matrix, hence its index is divided by 8, [6]. According to the remark in the preceeding paragraphs, we only have to determine the differentiable structure of $\Sigma_3$, the 2-fold branched covering of $(S^{4k-1}, M)$. 
We push the Seifert manifold $V$ into $D^{4k}$, a disk having $S^{4k-1}$ as its boundary; then use $V$ as the branched point set to construct a 2-fold branched covering $N$ of $D^{4k}$ with $\partial N = \Sigma_3$. Proposition 5.6 in [4, §4] tells us that the intersection form on $H_{2k}(N)$ is given by $A + A'$. All we have to do now is to show that $N$ is parallelizable. The Seifert manifold $V^{4k-2}$ has the homotopy type of a wedge of $r$ copies of $S^{2k-1}$, hence we may represent each of the basis element of $H_{2k}^*(V)$ by an embedded $(2k - 1)$-sphere $S_i$. Each $S_i$ bounds a 2$k$-disk $D_i$ in $D^{4k}$. Let $x$ denote the covering translation in the 2-fold branched covering $N$ over $D^{4k}$. Then $Q_i = xD_i \cup (-D_i)$ represent a basis for $H_{2k}(N)$, [4, p. 155]. $N$ has the homotopy type of a wedge of $r$ copies of $S^{2k}$, represented by the $Q_i$'s. Then the argument used in Lemma 4 (i) of [12] shows that the normal bundle of each $Q_i$ in $N$ is stably trivial. Thus $N$ is parallelizable, and it follows that $\Sigma_3 = 1/8(\text{index } (A + A'))\Sigma_6$.

In particular, we see that $\Sigma_6$ does not admit an involution $T$ fixing a codimension 2 simple knot $M$ with $1/8(\text{index } (A + A')) = \text{even integer}$. In contrast, if $G$ is a free differentiable involution acting on $\Sigma^{4k-1}$ leaving $M$ invariant, and $A$ a Seifert matrix for the equivariant knot complement $(\Sigma - M \times D^3)/G$; then we proved in [10] that $1/8(\text{index } (A + A')) = \sigma(G, \Sigma) = \text{the Browder-Livesay index de-suspension invariant}$, [11]. But we know that $\Sigma_6$, the generator of $bP_8$, admits a free involution $G$ with $\sigma(G, \Sigma_6) = 0$, [3], [11, p. 63]. Thus $(G, \Sigma_6)$ admits an unknotted invariant $S^6$, [11], which implies $1/8(\text{index } (A + A')) = 0$.

In the case $n$ is odd, we know that $bP_{4k+2} = \mathbb{Z}_2$ or $0$, [6]. Recall that $\Sigma_1 = \Sigma/T$, where the involution $T$ fixes a simple knot $M$ in $\Sigma^{4k+1}$. Then we have the following proposition.

**Proposition 4.** $\Sigma = 2\Sigma_1$.

**Proof.** As in Proposition 3, we only have to determine the differentiable structure of $\Sigma_1$, the 2-fold branched cover of $(S^{4k+1}, M)$. The proof of Proposition 3 shows that $\Sigma_3$ bounds a 2$k$-connected parallelizable manifold $N^{4k+2}$ with intersection form $A - A'$. Then the argument in [5, p. 256-257] enables us to embed $N$ in $S^{4k+3}$ in such a way that $(S^{4k+3}, \Sigma_3)$ is a simple knot with Seifert manifold $N$ and Seifert matrix $A$ (see [4, p. 153] and [5, p. 256]). We know from [7, p. 544] that the Kervaire invariant of $N$ is the Arf invariant of $A$. Since $A + A'$ is a symmetric, even, unimodular matrix, Lemma 2 in [11, p. 36] shows that the Arf invariant of $A$ is zero. Hence $\Sigma_3$ is the standard sphere.

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REFERENCES


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