Pacific Journal of Mathematics

INVOLUTIONS FIXING CODIMENSION TWO KNOTS

CHAO-CHU LIANG

Vol. 73, No. 1 March 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol. 73, No. 1, 1977

INVOLUTIONS FIXING CODIMENSION TWO KNOTS

CHAO-CHU LIANG

1. Involution. An m-knot (Σ^{m+2}, M^m) consists of an $(m+2)$-homotopy sphere Σ^{m+2} and an m-homotopy sphere M^m differentiably (or piecewise linearly) embedded in it. A $(2n-1)$-knot is callsd simple if $\pi_j(\Sigma - M) = \pi_j(S^1)$ for $j < n$. It is well known that each knot cobordism class contains a simple knot, [5] or [7].

Associated to each $(2n-1)$-knot, we have Seifert matrices B, with $B + \varepsilon B'$ unimodular, where $\varepsilon = (-1)^n$ and B' denotes the transpose of B. For $n \geqq 2$, the isotopy class of simple knot is completely determined by its Seifert matrices [8].

In [1, §11], Cappell and Shaneson used their algebraic K-theoretic obstruction groups to determine which knot cobordism classes admit semifree Z_p actions fixing the knots. In §3 below, we will prove the following theorem from the viewpoint of [5] and [8].

THEOREM 1. *A simple knot $(\Sigma^{2n+1}, M^{2n-1})$, $n \geqq 3$, admits a p. l. involution T fixing M^{2n-1} if and only if it has an associated Seifert matrix B of the form $B = A(A - \varepsilon A')^{-1}A$ for some matrix A with both $A + \varepsilon A'$ and $A - \varepsilon A'$ being unimodular.*

We will also discuss the differentiable case in the last section.

2. A technical lemma. Recall that $\varepsilon = (-1)^n$.

LEMMA 2. *Let A be an $(r \times r)$-matrix with both $A + \varepsilon A'$ and $A - \varepsilon A'$ being unimodular. Then the following system of equations has a unique solution for the pair of $(r \times r)$-matrices C_1 and C_2.*

$$C_1 A + \varepsilon C_2 A' = A + \varepsilon A' \tag{1}$$

$$\varepsilon C_1 A' + C_2 A = 0 . \tag{2}$$

Proof.

$$(1)+(2) C_1(A + \varepsilon A') + C_2(A + \varepsilon A') = A + \varepsilon A' . \tag{3}$$

Since $A + \varepsilon A'$ is unimodular, (3) becomes

$$C_1 + C_2 = I, \text{ the identity} \tag{4}$$

$$(1)-(2) C_1(A - \varepsilon A') - C_2(A - \varepsilon A') = A + \varepsilon A' . \tag{5}$$

Since $A - \varepsilon A'$ is unimodular, (5) becomes

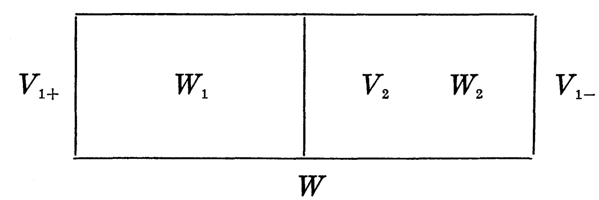
$$(6) \qquad C_1 - C_2 = (A + \varepsilon A')(A - \varepsilon A')^{-1} .$$

From (4) and (6), we have

$$C_1 = A(A - \varepsilon A')^{-1} \quad \text{and} \quad C_2 = -\varepsilon A'(A - \varepsilon A')^{-1} .$$

3. Proof of Theorem 1. If a simple knot $(\Sigma^{2n+1}, M^{2n-1})$, $n \geqq 3$, admits a $p.$l. (or differentiable) involution T fixing M^{2n-1}, then it is easy to see that $\Sigma_1 = \Sigma/T$ is a $(2n+1)$-homotopy sphere, and (Σ_1, M) is again a simple knot. Let Y be the closure of $(\Sigma_1 - M \times D^2)$, and $V^{2n} \subseteqq Y$ be an $(n-1)$-connected Seifert manifold for (Σ_1, M) with $\partial V = M \times e^{i0}$, (we consider $S^1 = \{e^{i\theta}\}$), [5], [7]. Lifting V to Σ, we have two equivariant Seifert manifolds V_1 and V_2 with $TV_1 = V_2$, $\partial V_1 = M \times e^{i0}$, and $\partial V_2 = M \times e^{i\pi}$, [9]. We then cut $X =$ closure of $(\Sigma - M \times D^2)$ along V_1 to get a manifold W.

(*)



We see immediately that W_1 is the manifold obtained from Y (in Σ_1) by cutting it along V and $W_2 = TW_1$. Let $\{e_1, \cdots, e_r\}$ be a basis for $H_n V_{1+}$, and $\{f_1, \cdots, f_r\}$ a basis for $H_n W$ determined by the Alexander duality (in Σ). Similarly, viewing $\{e_i\}$ as a basis for $H_n V$, we have a basis $\{d_i\}$ for $H_n W_1$ by using the Alexander duality in Σ_1. Let A and B be the Seifert matrices associated to (Σ_1, M) and (Σ, M) respectively (with respect to the basis $\{e_i\}$) [5], [7].

From [5], we know that A represents the map $H_n V_{1+} \to H_n W_1$ with respect to the bases $\{e_i\}$ and $\{d_i\}$, also the map $H_n V_2 \to H_n W_2$ with respect to the bases $\{T_* e_i\}$ and $\{T_* d_i\}$. The matrix $-\varepsilon A'$ represents the map $H_n V_2 \to H_n W_1$ with respect to the bases $\{T_* e_i\}$ and $\{d_i\}$, also the map $H_n V_{1-} \to H_n W_2$ with respect to the bases $\{e_i\}$ and $\{T_* d_i\}$. The matrix B represents $H_n V_{1+} \to H_n W$ with respect to the bases $\{e_i\}$ and $\{f_i\}$, and $-\varepsilon B'$ represents $H_n V_{1-} \to H_n W$ with respect to the bases $\{e_i\}$ and $\{f_i\}$. All the maps here are induced by inclusions. Finally, let C_1 and C_2 denote the matrices represent the maps $H_n W_1 \to H_n W$ and $H_n W_2 \to H_n W$ with respect to the appropriate bases respectively. From (*), we have the following equation:

$$B = C_1 A, \ -\varepsilon B' = C_2(-\varepsilon A'), \ C_1(-\varepsilon A') = C_2 A .$$

These, together with the fact that $A + \varepsilon A' = B + \varepsilon B' =$ intersection form on $H_n V$, [5], give us the two equations in Lemma 2. Also, we have proved in [9] that both $A+\varepsilon A'$ and $A-\varepsilon A'$ are unimodular. Thus it follows from Lemma 2 that $B = C_1 A = A(A - \varepsilon A')^{-1}A$.

Conversely, given a knot $(\Sigma^{2n+1}, M^{2n-1})$ with its Seifert matrix B satisfying the condition in Theorem 1, we can construct a simple knot (Σ_1, M) with an $(n-1)$-connected Seifert manifold V and associ ated Seifert matrix A, [5]. Then we construct the 2-fold branched covering of (Σ_1, M) to obtain a simple knot (Σ_2, M) as in [4], [9], [12]. If we are in the $p.$l. category, then both Σ and Σ_2 are the standard sphere S^{2n+1}. Both (Σ, M) and (Σ_2, M) have the same Seifert matrix, hence they are actually equivalent, [8]. The involution T is given by the covering translation for the branched covering.

4. **Free involutions.** Since the study of knots invariant under free involutions on spheres is very similar to that of knots fixed under involutions, [9], [10], the following theorem can be proved in a similar fashion.

THEOREM 1′. *A simple knot* $(\Sigma^{2n+1}, M^{2n-1})$, $n \geqq 3$, *admits a free* $p.$l. *involution* T *leaving* M *invariant if and only if it has an associated Seifert matrix* B *of the form* $B = A(A - \varepsilon A')^{-1}A$ *for some matrix* A *with both* $A + \varepsilon A'$ *and* $A - \varepsilon A'$ *being unimodular.*

5. **The differentiable case.** Let T denote a differentiable involution on Σ^{2n+1} fixing M^{2n-1}, $n \geqq 3$. We want to study the relation between the differentiable structure of Σ and $\Sigma_1 = \Sigma/T$. If $\Sigma_1 \neq S^{2n+1}$, then we may view (Σ_1, M) as the connected sum of (S^{2n+1}, M) and Σ_1 along a disk disjoint from the Seifert manifold V and M. We then construct the 2-fold branched covering (Σ_3, M) of (S^{2n+1}, M) with branched point set M. By the uniqueness of differentiable structure of the cyclic branched covering ([2] or [4]), it is easy to see that $\Sigma = 2\Sigma_1 + \Sigma_3$, where the sum denotes the connected sum in the group of homotopy spheres Γ_{2n+1}, [6].

In the case n is odd, we let Σ_0 denote the generator of $bP_{4k} = \{y \in \Gamma_{4k-1} \mid y$ bounds parallelizable manifolds$\}$. Then we have the following proposition.

PROPOSITION 3. $\Sigma = 1/8(\text{index}\,(A + A'))\Sigma_0 + 2\Sigma_1$.

Proof. We first note that $A + A'$ is a unimodular, symmetric, even matrix, hence its index is divided by 8, [6]. According to the remark in the preceeding paragraphs, we only have to determine the differentiable structure of Σ_3, the 2-fold branched covering of (S^{4k-1}, M).

We push the Seifert manifold V into D^{4k}, a disk having S^{4k-1} as its boundary; then use V as the branched point set to construct a 2-fold branched covering N of D^{4k} with $\partial N = \Sigma_3$, [4, §4]. Proposition 5.6 in [4] tells us that the intersection form on $H_{2k}(N)$ is given by $A + A'$. All we have to do now is to show that N is parallelizable. The Seifert manifold V^{4k-2} has the homotopy type of a wedge of r copies of S^{2k-1}, hence we may represent each of the basis element of $H_{2k-1}(V) = r$ copies of Z by an embedded $(2k-1)$-sphere S_i. Each S_i bounds a $2k$-disk D_i in D^{4k}. Let x denote the covering translation in the 2-fold branched covering N over D^{4k}. Then $Q_i = xD_i \cup (-D_i)$ represent a basis for $H_{2k}(N)$, [4, p. 155]. N has the homotopy type of a wedge of r copies of S^{2k}, represented by the Q_i's. Then the argument used in Lemma 4 (i) of [12] shows that the normal bundle of each Q_i in N is stably trivial. Thus N is parallelizable, and it follows that $\Sigma_3 = 1/8(\text{index}\,(A + A'))\Sigma_0$.

In particular, we see that Σ_0 does not admit an involution T fixing a codimension 2 simple knot M with $1/8(\text{index}\,(A + A'))$ = even integer. In contrast, if G is a free differentiable involution acting on Σ^{4k-1} leaving M invariant, and A a Seifert matrix for the equivariant knot complement $(\Sigma - M \times D^2)/G$; then we proved in [10] that $1/8(\text{index}\,(A + A')) = \sigma(G, \Sigma)$ = the Browder-Livesay index desuspension invariant, [11]. But we know that Σ_0^7, the generator of bP_8, admits a free involution G with $\sigma(G, \Sigma_0^7) = 0$, [3], [11, p. 63]. Thus (G, Σ_0^7) admits an unknotted invariant S^5, [11], which implies $1/8(\text{index}\,(A + A')) = 0$.

In the case n is odd, we know that $bP_{4k+2} = Z_2$ or 0, [6]. Recall that $\Sigma_1 = \Sigma/T$, where the involution T fixes a simple knot M in Σ^{4k+1}. Then we have the following proposition.

PROPOSITION 4. $\Sigma = 2\Sigma_1$.

Proof. As in Proposition 3, we only have to determine the differentiable structure of Σ_3, the 2-fold branched cover of (S^{4k+1}, M). The proof of Proposition 3 shows that Σ_3 bounds a $2k$-connected parallelizable manifold N^{4k+2} with intersection form $A - A'$. Then the argument in [5, p. 256–257] enables us to embed N in S^{4k+3} in such a way that (S^{4k+3}, Σ_3) is a simple knot with Seifert manifold N and Seifert matrix A (see [4, p. 153] and [5, p. 256]). We know from [7, p. 544] that the Kervaire invariant of N is the Arf invariant of A. Since $A + A'$ is a symmetric, even, unimodular matrix, Lemma 2 in [11, p. 36] shows that the Arf invariant of A is zero. Hence Σ_3 is the standard sphere.

The author wishes to thank the referee for pointing out Proposition 4 to him.

REFERENCES

1. S. Cappell and J. Shaneson, *The codimension* 2 *placement problem and homology equivalent manifolds*, Ann. of Math., **99** (1974), 277–348.
2. A. Durfee and L. Kauffman, *Periodicity of branched cyclic covers*, Math. Ann., **218** (1975), 157–174.
3. M. Hirsch and J. Milnor, *Some curious involutions of spheres*, Bull. Amer. Math. Soc., **70** (1964), 372–377.
4. L. Kauffman, *Branched coverings, open books and knot periodicity*, Topology, **13** (1974), 143–160.
5. M. A. Kervaire, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. France, **93** (1965), 225–271.
6. M. A. Kervaire and J. Milnor, *Groups of homotopy spheres*, Ann. of Math., **77** (1963), 504–537.
7. J. Levine, *Polynomial invariants of knots of codimension two*, Ann. of Math., **84** (1966), 537–544.
8. ———, *An algebraic classification of some knots of codimension two*, Comment. Math. Helv., **45** (1970), 185–198.
9. C. C. Liang, *Semifree involutions on sphere knots*, Michigan Math. J., **22** (1975), 161–163.
10. ———, *Browder-Livesay index invariant and equivariant knots*, Michigan Math. J., **23** (1976), 325–326.
11. S. López de Medrano, *Involutions on Manifolds*, Erge. der Math. Band, **59**, Springer-Verlag, New York, 1971.
12. I. Tamura, *Fixed point sets of differentiable periodic transformations on spheres*, J. of the Faculty of Sci., Univ. of Tokyo, **XVI** (1969), 101–114.

Received January 1977. Partially supported by the University of Kansas General Research Fund.

UNIVERSITY OF KANSAS
LAWRENCE, KS 66045

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 73, No. 1 March, 1977