SQUARE INTEGRABLE REPRESENTATIONS AND THE FOURIER ALGEBRA OF A UNIMODULAR GROUP

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Let $G$ be a unimodular group, and let $\lambda_d$ be the sub-
representation of the left regular representation $\lambda$, which
is the sum of the square integrable representations. The
purpose of this paper is to study the representation $\lambda_d$ with
special emphasis on the closed subspace $A_d(G)$ of the Fourier
algebra $A(G)$ of the group which is generated by the coeffici-
ents of $\lambda_d$. In the last part of the paper we study in detail
a particular noncompact group for which $\lambda = \lambda_d$.

We denote, as in [4], by $A(G)$ the algebra of the coefficients of
$\lambda$, that is the algebra of continuous functions on $G$ of the type
$(\lambda(x)f, g)$, with $f, g \in L^1(G)$. The first section contains results of a
general nature: we show that $A_d(G)$ is the dual space of a $C^*$-algebra
contained in $VN(G)$, the von Neumann algebra generated by the
operators $\lambda(x)$, $x \in G$, and that its unit ball is the weak closure of
the extreme points of the unit ball of $A(G)$. We also show that
$A_d(G) \subset L^p(G)$ if and only if the formal degrees of the square integrable
representations of $G$ are bounded away from zero.

In the last section we make a closer study of an example due
to J. Fell of a noncompact group $G$ for which $A_d(G) = A(G)$. We
show that the traces of the square integrable representations of this
group are bounded measures and we construct a kind of Dirichlet
kernels, which also turn out to be bounded measures.

We prove that summation with respect to these kernels converges
in $L^p(G)$ for $1 < p < \infty$, but not for $L^1(G)$.

We conclude the paper with some remarks on the Wiener-Pitt
phenomenon for bounded measures on this group.

1. We refer the reader to [4] for the definitions and the pro-
erties of the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra
$B(G)$ of a locally compact group $G$, and to [3] for the basic facts
about $C^*$-algebras, von Neumann algebras and square integrable
representations of unimodular groups. Throughout the paper “group”
will always mean “locally compact unimodular group” and “repre-
sentation” will mean “unitary continuous representation.”

Following Arsac [1], given a representation $\pi$ of $G$ on a Hilbert
space $H_\pi$, we denote by $A_\pi$ the closed subspace of $B(G)$ spanned by
the coefficients of $\pi$, i.e., the functions $(\pi(x)\mu|\nu)$, $x \in G$, $\mu, \nu \in H_\pi$. Let
$\lambda_d$ be the subrepresentation of the left regular representation of $G$
which is the sum of all the irreducible square integrable representations of \(G\). Then \(A_x(G) = A_x^G\) is a closed subspace of \(A(G)\). From the results of [1] it follows easily that there exists a closed subspace \(A_x(G) \subset A(G)\) such that \(A(G) = A_x(G) \oplus A_x(G)\). Moreover \(A_x(G)\) itself is the direct sum \(\bigoplus_{\pi \in \hat{\mathcal{G}}_x} A_x\), where \(\hat{\mathcal{G}}_x\) denotes the family of all equivalence classes of irreducible square integrable representations of \(G\).

Now, given a representation \(\pi\) of \(G\) on the Hilbert \(H_\pi\), we denote by \(\bar{\pi}\) its conjugate representation on \(\overline{H_\pi}\) the Hilbert space conjugate to \(H_\pi\). We remember that \(A(G)\) is endowed with a structure of left \(VN(G)\)-module [3, Prop. 3.17]. For \(\pi \in \hat{\mathcal{G}}_x\) let \(P_\pi\) and \(K_\pi\) denote respectively the minimal central projection and the minimal biinvariant subspace corresponding to \(\pi\). Then the following facts are more or less immediate consequence of [3, Ch. 14]. For every \(\pi \in \hat{\mathcal{G}}_x\) \(A_x = P_\pi A(G)\) is contained in \(K_\pi\). Moreover the mapping \(u \mapsto d_\pi \pi(u)\), where \(d_\pi\) is the formal degree of \(\pi\) and \(\pi(u) = \int_\mathcal{G} \pi(x)u(x)dx\), is an isometric isomorphism of \(A_x\) onto the Banach space \(TC(H_\pi)\) of all trace class operators on \(H_\pi\). Any function \(u \in A_x(G)\) is the sum of its Fourier series:

\[
u(x) = \sum_{\pi \in \hat{\mathcal{G}}_x} d_\pi \text{tr} (\pi(x^{-1})\pi(u))\]

where the series converges absolutely as well as in \(A(G)\).

If \(G\) is a compact abelian group it is well known that its dual group \(\hat{G}\) is a discrete measure space. Therefore \(A(G)\), being isometric to \(\ell^1(\hat{G})\), can be identified with the dual of the Banach space \(c_0(\hat{G})\) of all bounded complex functions on \(\hat{G}\), vanishing at infinity. The following lemma shows that for \(G\) nonabelian a similar result holds for \(A_x(G)\).

**Lemma 1.1.** Let \(c_0(\hat{G}_x)\) be the direct sum, in the \(C^*\)-algebra theoretical sense, of the algebras \(C^* = \pi(C^*(G))\) for all \(\pi \in \hat{G}_x\). Then \(A_x(G)\) can be isometrically identified with the dual space of \(c_0(\hat{G}_x)\) via the following pairing:

\[\langle T, u \rangle = \sum_{\pi \in \hat{G}_x} d_\pi \text{tr} (\pi(T)\pi(u))\]

for \(T \in c_0(\hat{G}_x), u \in A_x(G)\).

**Proof.** For every \(\pi \in \hat{G}_x\), \(C^*\) is isometric to the \(C^*\)-algebra \(LC(H_\pi)\) of all compact operators on \(H_\pi\) [3, 4.1.11, and 18.4.1]. Since \(A_x\) is isometric to \(TC(H_\pi)\), which is the dual of \(LC(H_\pi)\), the lemma easily follows.

Now, to find the extreme points of the unit ball of \(A(G)\), we need the following lemma.
**Lemma 1.2.** Let $T$ be an operator in the unit ball of the space $TC(H)$ of trace class operators on a Hilbert space $H$. Then these are equivalent:

(i) $T$ is an extreme point.

(ii) $|T|$ is a projection of rank one.

(iii) There exist $\mu, \nu \in H$, $|\mu| = |\nu| = 1$ such that $T\phi = (\phi | \nu)\mu$ for every $\phi \in H$.

**Proof.** Let us denote by $TC_1(H)$ the unit ball of the space $TC(H)$. It is obvious that (ii) and (iii) are equivalent. We prove only the equivalence between (i) and (ii). Let $T$ be an extreme point in $TC_1(H)$ and suppose that there exist $R, S$ in $TC_1(H)$ such that $|T| = 1/2(R + S)$; then $1 = 1/2(tr(R) + tr(S))$.

It follows that $tr(R) = tr(S) = 1$, so $R$ and $S$ are positive operators. Since $|T| = 1/2(R + S)$ and since $T$ is an extreme point we get: $T = UR = US$. Then $|T| = U^*UR = R$ and $T = U^*US = S$ because $U^*U$ is greater than or equal to the supports of $R$ and $S$. So $|T|$ is an extreme point of $TC_1(H)$. Since $TC(H)$ is the dual space of the $C^*$-algebra $LC(H)$ \cite{3, 4.1.2}, the extreme points in its positive unit ball are just zero and the pure states, i.e., the positive operators $P$ in $TC_1(H)$ such that $0 \leq P' \leq P$ implies $P' = \lambda P$, $0 \leq \lambda \leq 1$ \cite{3, 2.5.5}. This proves that $|T|$ must be a projection of rank one.

So (i) implies (ii).

To show that (ii) implies (i) we shall prove that if $P$ is a projection of rank one and $U$ is a partial isometry such that $U^*U = P$, then $UP$ is an extreme point in $TC_1(H)$. Since the final projection of $U$ is one-dimensional there is an isometry $W$ which coincides with $U$ on its support. Therefore $UP = WP$. Now let $R, S$ be in $CT_1(H)$ such that $WP = 1/2(R + S)$; then $P = 1/2(W^*R + W*S)$ and $W^*R, W^*S \in TC_1(H)$. Since $P$ is one-dimensional, $P$ defines a pure state on $TC(H)$. Therefore $P$ is an extreme point in $TC_1(H)$ and $P = W^*R = W*S$.

So $UP = WP = R = S$ and $UP$ is extreme in $TC_1(H)$.

**Theorem 1.1.** Let $u$ be in the unit ball of $A(G)$. Then $u$ is an extreme point if and only if there exist $\pi \in \hat{G}_d$ and vectors $\mu, \nu \in H_z$, $|\mu| = |\nu| = 1$, such that $u(x) = (\pi(x)\mu | \nu)$. Moreover $A_d(G)$ is the closed subspace of $A(G)$ spanned by the extreme points of the unit ball of $A(G)$.

**Proof.** Let $u$ be an extreme point in the unit ball of $A(G)$. Let $P \in VN(G)$ be the central support of $u$, considered as an ultraweakly continuous form on $VN(G)$. We claim that $P$ is a minimal central
projection in $VN(G)$. Suppose to the contrary that there exists a central projection $Q$ in $VN(G)$ such that $0 < Q < P$. Let $\alpha = \|Qu\|_A$ and $\beta = \|(P - Q)u\|_A$. Then $\alpha + \beta = 1$ and $\alpha > 0$, $\beta > 0$ because $P$ is the minimal central projection such that $Pu = u$. Hence $u = \alpha u_1 + \beta u_2$, where $u_1 = Qu/\|Qu\|_A$ and $u_2 = (P - Q)u/\|(P - Q)u\|_A$ are in the unit ball of $A(G)$. But this contradicts the extremality of $u$. Hence there exists $\pi \in \hat{G}_d$ such that $u$ is an extreme point in the unit ball of $A_\pi$. Therefore $d_\pi \pi(u)$ is an extreme point in the unit ball of $TC(H_\pi)$. Then, by Lemma 1.2, there exist $\mu, \nu \in \tilde{H}_\pi$, $|\mu| = |\nu| = 1$, such that $d_\pi \pi(u)\phi = (\phi | \mu)\nu$ for every $\phi \in \tilde{H}_\pi$. We may assume that $\pi$ is a subrepresentation of the left regular representation $\lambda$ of $G$. Then, denoting by $[\cdot | \cdot]$ the inner product in $H_\pi$ and by $(\cdot | \cdot)$ the inner product in $H_\pi$, and remembering that $(\phi | \psi) = [\psi | \phi]$, we have by [3, 14.3.3]:

$$
\int_G u(x)(\pi(x)\phi | \psi)dx = \pi(u)\phi | \psi) = d_\pi^{-1}(\phi | \mu)(\psi | \nu)
$$

$$
\times \int_G (\pi(x)\phi | \psi)(\pi(x)\mu | \nu)dx = \int_G (\pi(x)\phi | \psi)(\pi(x)\mu | \nu)dx
$$

for every $\phi, \psi \in \tilde{H}_\pi$. Since, by [3, 14.3.1], the functions $(\pi(x)\phi | \psi)$, $\phi, \psi \in \tilde{H}_\pi$ are dense in $K_\pi$, the identity $u(x) = [\pi(x)\mu | \nu]$ follows at once. The last assertion of the theorem follows by Lemma 1.1 and the Krein Milman theorem.

As we have seen in the introductory remarks, $A_\pi \subset L^2(G)$ for every $\pi \in \hat{G}_d$. It is natural to ask whether $A_d(G)$ is contained in $L^2(G)$ or not. Since we have that $\|u\|_A = \sum_{\pi} d_\pi \text{tr} (|\pi(u)|)$ for every $u \in A_\pi(G)$ and $\|f\|_2^2 = \sum_{\pi} d_\pi \text{tr} (\pi(f)\pi(f))$ for every $f \in \bigoplus_{\pi} K_\pi$, $\pi \in \hat{G}_d$, a comparison of the two formulas, together with the closed graph theorem, yields at once that $A_d(G)$ is contained in $L^2(G)$ if and only if the formal degrees of the square integrable representations of $G$ are bounded away from zero.

It is well known that if $G$ is a locally compact abelian group then $A(G) = A_d(G)$ if and only if $G$ is compact. If $G$ is not abelian the situation is more complicated, because there exist noncompact groups such that $A(G) = A_d(G)$. In the next section we shall study in detail an example of such groups. A more detailed discussion of the structure and properties of unimodular groups, whose regular representation is the direct sum of irreducible subrepresentations, will appear in a forthcoming paper of M. Picardello and the author [7]. Here we bound ourselves to the following few remarks.

**Remark 1.** Let $G$ be a noncompact unimodular group such that
A(G) = A_d(G). Then inf \{d_\pi; \pi \in \hat{G}\} = 0. This is an easy consequence of the remark following Theorem 1.1 and the fact \(A(G)\) cannot be contained in \(L^2(G)\), because \(G\) is noncompact [9].

**Remark 2.** Let \(G\) be as before and let \(K\) be a compact normal subgroup of \(G\). Then \(G/K\) is again a noncompact unimodular group whose regular representation is the direct sum of its irreducible components. Indeed \(G/K\) is clearly unimodular and noncompact and \(A(G/K)\) is isometric to the biinvariant closed selfadjoint subalgebra of \(A(G)\) of the functions which are constant on \(K\)-cosets [4]. It follows easily that \(A(G/K) = A_d(G/K)\).

We conclude this section with a result which is related to the contents of the last remark but not to the main theme of the paper. If \(G\) is any locally compact group and \(K\) is a compact normal subgroup, the functions of \(A(G)\) which are constant on the cosets of \(K\) form a closed biinvariant selfadjoint subalgebra of \(A(G)\). The fact that viceversa every closed biinvariant selfadjoint subalgebra of \(A(G)\) is of this type is a special case of a result of M. Takesaki and N. Tatsuuma [Duality and subgroups, Annals of Math., v. 93 (1971) 344–364, Theorem 9]. It can also be deduced from the following theorem which is a slight improvement of a result of [1]. We believe that our proof, shorter than that of [1] can also shed light on the result of Takesaki and Tatsuuma.

Let \(\mathcal{U}\) be a nonzero right invariant closed selfadjoint subalgebra of \(A(G)\). Then by [11] there exists a projection \(P \in VN(G)\) such that \(\mathcal{U} = PA(G)\). Let \(H_P = P(L^2(G))\) be the corresponding subspace of \(L^2(G)\).

**Theorem 1.2.** The space \(H_P\) is closed under multiplication by functions of \(\mathcal{U}\). If \(\mathcal{U}\) separates the points of \(G\) then \(\mathcal{U} = A(G)\).

**Proof.** First we claim that \(\mathcal{U} \cap H_P\) is dense in \(H_P\). Indeed let \(f\) be any function in \(H_P\) and let \(\phi_\alpha\) be an approximated identity for the convolution of continuous functions with compact support in \(G\). Then \(\phi_\alpha*f\) is in \(A(G) \cap L^2(G)\) and \(\lim \phi_\alpha*f = f\) in \(L^2(G)\). Since \(P(\phi_\alpha*f) \in \mathcal{U} \cap H_P\), the claim is proved. Now let \(u \in \mathcal{U}\), \(f \in H_P\) and let \(\{f_\alpha\}\) be a net in \(\mathcal{U} \cap H_P\) converging to \(f\) in \(L^2(G)\). Then \(uf_\alpha \in \mathcal{U} \cap H_P\), because \(u\) is a bounded function and \(\mathcal{U}\) is an algebra. Since \(\lim uf_\alpha = uf\), we have \(uf \in H_P\).

To prove the last assertion of the theorem, observe that if \(\mathcal{U}\) separates the points of \(G\), by the Stone-Weierstrass theorem, \(\mathcal{U}\) is uniformly dense in the space \(C_0(G)\) of continuous functions on \(G\) vanishing at infinity. Therefore \(H_P\) is also closed under the multiplication by
functions in $C_0(G)$. Let $g$ be any nonzero continuous function in $H_P$. Such function actually exists, because $𝒰 ∩ H_P$ is dense in $H_P$. Let $𝒰$ be any open set on which $g$ is bounded away from zero. Now let $f$ be any continuous function with compact support in $𝒰$ and denote by $h$ the function so defined: $h(x) = f(x)/g(x)$ for $x ∈ 𝒰$, $h(x) = 0$ for $x ∉ 𝒰$. Then $h ∈ C_0(G)$ and $f = hg$ is in $H_P$. Hence $H_P$ contains the space $C_0(𝒰)$ of the continuous functions with compact support in $𝒰$. Applying the translation invariance of $H_P$ and a simple partition of unity argument, it is easy to see that $C_0(𝒰) ⊂ H_P$. Therefore $H_P = L^2(G)$. Hence $P = I$ and $𝒰 = A(G)$.

2. Fell’s example. In [2] L. Baggett describes the following example, due to Fell of group $G$ such that $A(G) = A_d(G)$.

Let $p$ be a prime number, $N$ the $p$-adic numbers field, $K$ the subset of $p$-adic numbers $k$ whose valuation $|k|_p$ is one. $K$ is a compact abelian group w.r.t. multiplication. For $n ∈ N$, $k ∈ K$ set $k(n) = kn$. Then $K$ acts as group of automorphisms of the additive group $N$. The orbits of $N$ under the action of $K$ are $\{0\}$ and $N_j = \{n: n ∈ N, |n|_p = p^{-j}\}$, $j ∈ ℤ$. Let $G = K ⋊ N$ be the semidirect product of $K$ and $N$. Then $G$ is a regular semidirect product because $\hat{N} = N = (\bigcup_{j ∈ ℤ} N_j) ∪ \{0\}$. Therefore using the representation theory of group extensions [6], we can describe the irreducible representations of $G$.

One verifies that $\hat{G}$ is the union of two sets $\hat{G}_1 = \{π_j: j ∈ ℤ\}$, $\hat{G}_2 = \{π_θ: θ ∈ \hat{K}\}$. The representations in $\hat{G}_1$ can be realized on the Hilbert space $L^2(K)$, while the representations in $\hat{G}_2$ are one-dimensional.

If $π_j ∈ \hat{G}_1$ and $f ∈ L^2(K)$, then:

$$[π_j(l, m)f](k) = \exp(2πip^jkm)f(kl)$$

for $(l, m) ∈ G$, $k ∈ K$. Here the exponential of a $p$-adic number

$$n = p^j \sum_{i ≥ 0} n_ip^i, 0 ≤ n_i < p,$$

is defined as follows:

$$\exp(2πin) = \begin{cases} \exp(2πip^j \sum_{i+j<0} n_ip^i) & \text{for } j < 0 \\ 1 & \text{for } j ≥ 0 \end{cases}$$

If $π_θ ∈ \hat{G}_2$, $(l, m) ∈ G$, then $π_θ(l, m) = θ(l)$.

Figà-Talamanca in [5] proved that when $A(G) ≠ A_d(G)$ there exist positive definite continuous functions which vanish at infinity but are not in $A(G)$. He also asked whether or not unimodularity alone is sufficient to prove the existence of such functions for $G$ noncompact.

The following corollary answers in the negative to this question. Recall first that the Fourier-Stieltjes algebra $B(G)$ is the algebra,
under pointwise operations of all linear combinations of continuous positive definite functions on $G$ [4]. $B(G)$ is also the Banach involution algebra of the coefficients $u(x) = (\pi(x)\xi | \eta)$, $\xi, \eta \in H_\pi$ of all unitary continuous representations of $G$, normed thus:

$$\|u\|_B = \min \{|\xi| |\eta| : u(x) = (\pi(x)\xi | \eta)\}.$$ 

**Corollary 2.1.** $B(G)$ is the direct sum $A(G) \oplus AP(G)$, where $AP(G)$ is the Banach involution algebra of the almost periodic functions on $G$. In particular a function $u \in B(G)$ vanishes at infinity if and only if $u \in A(G)$.

**Proof.** Since $\hat{G}$ is countable an arbitrary unitary representation $\pi$ of $G$ is the direct sum (rather than the direct integral) $\pi = (\bigoplus_{x \in \mathbb{Z}} n_j\pi_j) \oplus (\bigoplus_{\omega \in \hat{K}} n_\omega\pi_\omega)$. Therefore if $u$ is any coefficient of $\pi$, $u$ decomposes into the sum $u = u_1 + u_2$, $u_1 \in A(G)$ and $u_2 = \sum_{\omega \in \hat{K}} n_\omega\theta$, where the series converges in $B(G)$, and hence uniformly. Thus $u_2$ is an almost periodic function [3, 16.2.1. (v)]. If $u$ vanishes at infinity $u_2$, $\bar{u}_2$ and hence $|u_2|^2$ vanish at infinity. Since $AP(G)$ is a Banach involution algebra with respect to pointwise operations and complex conjugation, $|u_2|^2$ is an almost periodic function whose mean is zero. Hence $u_2 = 0$ [3, 16.3].

We shall now evaluate the "diagonal" coefficients of the representation $\pi_j \in \hat{G}_d$, with respect to the orthonormal basis $K$ in $L^p(K)$. This will enable us to compute the formal degrees of the square integrable representations of $G$ and to study the convergence of the Fourier series for functions in $L^p(G), 1 \leq p < + \infty$.

**Lemma 2.2.** For $\pi_j \in \hat{G}_d$ denote by $\phi^{(j)}_{\theta}(l, m) = (\pi_j(l, m)|\theta)$ the coefficient of $\pi_j$ corresponding to $\theta \in \hat{K}$. Then $\phi^{(j)}_{\theta}(l, m) = \phi^{(j)}(m|\theta(l)$, where:

$$\phi^{(j)}(m) = \begin{cases} 1 & \text{for } |m|_p \leq p^j \\ \frac{1}{1 - p} & \text{for } |m|_p = p^{j+1} \\ 0 & \text{for } |m|_p > p^{j+1}. \end{cases}$$

The formal degree of $\pi_j$ is $d_j = p^{-j}[(p - 1)/p]^j$.

**Proof.** We have for $(l, m) \in G$:

$$\phi^{(j)}_{\theta}(l, m) = (\pi_j(l, m)|\theta) = \int_K \exp(2\pi ip^j km)\theta(k)\theta(k)d_h(k)$$

where $d_h(k)$ denotes the Haar measure on $K$ which coincides with the Haar measure $d_N(k)$ on $N$, since $d_N(m, k) = |m|_p d_N(k)$ for every $m \in N$.  

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Then:

\[ \phi^{ij}_{lm}(l, m) = \theta(l) \int_K \exp(2\pi ip^j km) d_N(k) \quad (l, m) \in G . \]

By a change of variable, setting \( p^j km = t \) and \( |m|^p = p^n \), the integral

\[ I = \int_K \exp(2i\pi p^j km) d_N(k) \]

becomes:

\[ I = p^{j-n} \int_{N_{j-n}} \exp(2\pi it) d_N(t) . \]

So we need to compute, for every relative integer \( s \), the integral:

\[ I_s = \int_{N_s} \exp(2\pi it) d_N(t) . \]

Let \( t \) be in \( N_s \); then \( t = p^s \sum_{n=0}^{+\infty} t_n p^n \) where \( 0 \leq t_n < p \) for \( n \in \mathbb{N} \), and \( t_0 \neq 0 \). Then:

\[ \exp(2\pi it) = \begin{cases} \exp(2\pi ip^s \sum_{0 \leq i < -s} t_i p^n) & \text{for } s < 0 \\ 1 & \text{for } s \geq 0 . \end{cases} \]

Therefore for \( s \geq 0 \), \( I_s \) is just the measure of \( N_s = p^s K \), i.e., \( I_s = p^{-s} \).

For \( s = -1 \) we have:

\[ I_{-1} = \sum_{j=1}^{p-1} \int_{N_{-1,j}} \exp(2\pi ij p^{-1}) d_N(t) = \sum_{j=1}^{p-1} \exp(2\pi ij p^{-1}) \int_{N_{-1,j}} d_N(t) \]

where \( N_{-1,j} = \{ t: t \in N_{-1}, t_0 = j \} \). Since \( N_{-1} \) is the disjoint union of the \( N_{-1,j} \) for \( j = 1, \ldots, p - 1 \) and \( N_{-1,j} = N_{-1,i} + j - i \) for \( i, j = 1, \ldots, p - 1 \):

\[ \int_{N_{-1,j}} d_N(t) = \frac{1}{p - 1} \int_{N_{-1}} d_N(t) = \frac{p}{p - 1} \]

for every \( j = 1, \ldots, p - 1 \).

So

\[ I_{-1} = \frac{p}{p - 1} \sum_{j=1}^{p-1} \exp(2\pi ij p^{-1}) = \frac{-p}{p - 1} . \]

Now for \( s \leq -2 \), let \( J \) be the set of multiindices \( j = (j_0, j_1, \ldots, j_{1-s}) \) s.t. \( 0 \leq j_t < p \) for \( 1 = 0, \ldots, 1 - s \) and \( j_0 \neq 0 \). Setting \( \langle j, p \rangle = \sum_{t=0}^{1-s} j_t p^t \) we have:

\[ I_s = \sum_{j \in J} \exp(2\pi ip^s \langle j, p \rangle) \int_{N_s,j} d_N(t) \]

where \( N_{s,j} = \{ t: t \in N_s, t_0 = j_0, t_1 = j_1, \ldots, t_{1-s} = j_{1-s} \} \). Since \( N_s \) is the
disjoint union of the $N_{s,j}$, $j \in J$ and $N_{s,j} = N_{s,j'} + \langle j, p \rangle - \langle j', p \rangle$, $j$, $j' \in J$, then

$$\int_{N_{s,j}} d_N(t) = \frac{p^{-s}}{p^{1-s}(p - 1)}$$

for every $j \in J$. Therefore:

$$I_s = \frac{1}{p(p - 1)} \sum_{j \in J} \exp (2\pi ip^s \langle j, p \rangle) = 0$$

because

$$\sum_{j \in J} \exp (2\pi ip^s \langle j, p \rangle) = \sum_{k=0}^{p^{-s-1}} \exp (2\pi ip^sk) - \sum_{k=0}^{p^{1-s-1}} \exp (2\pi ip^{s-1}k)$$

and

$$\sum_{k=0}^{\alpha-1} \exp (2\pi i\alpha^{-1}k) = 0$$

for every positive integer $\alpha$.

We have thus that:

$$\phi^{(j)}(l, m) = \theta(1)p^{i-m}I_{j-m} = \begin{cases} \theta(l) & \text{for } |m|_p \leq p^i \\ \frac{1}{1-p} \theta(l) & \text{for } |m|_p = p^{i+1} \\ 0 & \text{for } |m|_p > p^{i+1} \end{cases}$$

To compute the formal degree $d_j$ of $\pi_j$ it is sufficient to observe that, since $\phi^{(j)}$ is a positive definition function, whose value in the identity is one $d_j^{-1} = |\phi^{(j)}|^2 = p^{|p/p - 1|^2}$ [3, 14.4.3].

Let $\gamma_j$ be the positive definite central measure on $G$, defined by $\gamma_j(f \ast g^*) = \text{tr} (\pi_j(f)\pi_j(g^*))$, for $f, g \in C_c(G)$ and $\pi_j \in \hat{G}$. If $\delta_1$ denotes the Dirac measure at 1 on $K$ then $\gamma_j = \phi^{(j)} \otimes \delta_1$ (here we have identified $\phi^{(j)}$ with the measure $\phi^{(j)}(x) d_N(x)$). Indeed it is easy to verify by means of Lemma 2.2 that for every $\psi \in C_c(N)$ and $\theta \in \hat{K}$

$$\left(\phi^{(j)} \otimes \delta_1\right)(\psi \otimes \theta) = \gamma_j(\psi \otimes \theta).$$

The measure $\gamma_j$ is called the “character measure” of the representation $\pi_j$ [3, 17.2.4].

**DEFINITION 1.** We define the **Dirichlet Kernel** $\{D_n\}$ by:

$$D_n = \sum_{j=-n}^{n} d_j \gamma_j$$

for every positive integer $n$.

An easy but lengthy computation shows that, for every $n$, $D_n$
is a measure whose total variation \( \| A_{\gamma} \|_M \) is bounded by the constant 2. Therefore the convolution operator \( \lambda(D_n)f = D_n*f \) is a bounded operator on \( L^p(G) \) for \( 1 \leq p \leq +\infty \). Since \( \gamma_i \ast \gamma_j = d_{ij} \delta_{ij} \gamma_j \), for \( i, j \in \mathbb{Z}, \lambda(D_n) \) is a projection, which for \( p = 2 \) coincides with the sum \( \sum_{j=-n}^{n} P_{\pi_j} \) of the minimal central projections associated with the square integrable representations \( \pi_j, j = -n, \ldots, n \).

Therefore for \( f \in L^p(G) \):

\[
    f = \sum_{j \in \mathbb{Z}} d_j(\gamma_j \ast f)
\]

where the series converges in \( L^p(G) \).

**Definition 2.** For every function \( f \in L^p(G), 1 \leq p \leq +\infty \) we call \( \sum_{j \in \mathbb{Z}} d_j(\gamma_j \ast f) \) the formal Fourier series of \( f \). We say that a function \( f \in L^p(G) \) is a trigonometric polynomial in \( L^p(G) \) if

\[
    D_n \ast f = \sum_{j=-n}^{n} d_j(\gamma_j \ast f) = f
\]

for some \( n \). The following theorem shows that the Dirichlet kernel is a summability kernel for \( L^p(G) \) \( 1 < p < \infty \).

**Theorem 2.3.** If \( f \in L^p(G), 1 < p < \infty \), then:

\[
    f = \lim_{n \to +\infty} D_n \ast f = \lim_{n \to +\infty} \sum_{j=-n}^{n} d_j(\gamma_j \ast f)
\]

in the \( L^p(G) \) norm.

**Proof.** Assume that trigonometric polynomials are dense in \( L^p(G) \) for \( 1 < p < \infty \). Let \( f \in L^p(G) \) \( \varepsilon > 0 \) and let \( P \) be a trigonometric polynomial in \( L^p(G) \), satisfying \( |f - P|_p < \varepsilon/4 \). For \( n \) large enough we have \( D_n \ast P = P \) and hence:

\[
    |D_n f - f|_p \leq |D_n \ast (f - P)|_p + |P - f|_p < \varepsilon.
\]

Therefore it remains only to show that trigonometric polynomials are actually dense in \( L^p(G), 1 < p < \infty \). Let \( g \) be a continuous function with compact support in \( G \). Then \( g \in L^p(G) \) and \( \lim_{n \to +\infty} D_n \ast g = g \) in the \( L^p(G) \) norm. On the other hand, since \( |D_n \ast g|_p \leq 2|g|_p \) and \( 1 < p < \infty \), there exists a subsequence \( \{D_{n_k} \ast g\} \) which converges in the weak topology of \( L^p(G) \) to some limit \( h \). By the Banach-Saks theorem, there is a sequence of convex combinations of the \( D_{n_k} \ast g \), which converges to \( h \) in the norm of \( L^p(G) \). Taking a subsequence which converges almost everywhere, we have \( g = h \) a.e. Thus trigonometric polynomials are dense in \( L^p(G), 1 < p < \infty \).
Remark. The Dirichlet kernel \( \{D_n\} \) is not a summability kernel for \( L^1(G) \). In fact let \( f \in L^1(G) \) be a function such that
\[
\pi_{\theta_0}(f) = \int_N \int_K f(1, m) \theta_0(1) d_N(1) d_N(m) \neq 0
\]
for some \( \theta_0 \in \hat{K} \). For every function \( h \in L^1(G) \) and for every \( \theta \in \hat{K} \), we have \( \lim_{j \to \infty} (\pi_j(h) \theta | \theta) = \pi_\theta(h) \), by Lemma 2.2 and the dominated convergence theorem. Since \( \pi_j(D_n * h) = 0 \) for \( |j| > n \), then \( \pi_\theta(D_n * h) = 0 \) for every positive integer \( n \).

Now, for every \( n \in \mathbb{N} \):
\[
|D_n * f - f|_1 \geq \|\lambda(D_n * f - f)\| = \sup_{j \in \mathbb{Z}} \|\pi_j(D_n * f - f)\| \geq |\pi_{\theta_0}(f)|
\]
since \( \|\pi_j(D_n * f - f)\| \geq |(\pi_j(D_n * f - f) \theta_0 | \theta_0)\|, j \in \mathbb{Z} \) and
\[
\lim_{j \to \infty} |(\pi_j(D_n * f - f) \theta_0 | \theta_0)\| = |\pi_{\theta_0}(f)| .
\]
This proves that \( D_n * f \) cannot converge to \( f \) in \( L^1(G) \), if \( \pi_{\theta_0}(f) \neq 0 \).

We conclude now our study of the group \( G \) with some final remarks on the algebra \( M(G) \) of all complex measures of bounded variation on \( G \).

Remark. There exists a measure \( \mu \in M(G) \) such that the operator \( \lambda(\mu)f = \mu * f \) for \( f \in L^1(G) \) has inverse and yet \( \mu^{-1} \) does not exist, as an element of the algebra \( M(G) \). This phenomenon was first discovered in \( M(R) \) by Wiener and Pitt [8], who showed that there exists a measure \( \mu \in M(R) \) such that 0 is in the spectrum of \( \mu \) but the Fourier-Stieltjes transform \( \hat{\mu} \) of \( \mu \) is bounded away from zero. Since \( K \) is a compact abelian group by [10, Th. 6.4.1], there exists a measure \( \nu \in M(K) \) such that 0 \( \in \text{sp} \nu \) and \( |\lambda(\nu)| \leq 1 \) for every \( \theta \in \hat{K} \).

Then it is straightforward to check that if \( \mu = \delta_e - d_\phi(0) \otimes (\delta_1 - \nu) \), where \( \delta_e \) is the Dirac measure at the identity \( e = (1, 0) \) of \( G \), \( 0 \in \text{sp} \mu \). Moreover, for \( j \neq 0 \), \( \pi_j(\mu) \) is the identity on \( L^2(K) \) and \( \pi_\theta(\mu) \) is the operator whose matrix representation with respect to the basis \( \hat{K} \) in \( L^2(K) \) is given by the diagonal matrix whose eigenvalues are the \( \hat{\nu}(\theta) \), \( \theta \in \hat{K} \). Therefore \( \lambda(\mu)^{-1} \) exists.

Remark. The same construction as above, together with [10, Th. 6.4.1] can be used to show that the spectral radius of a measure \( \mu \) in \( M(G) \) is much larger than the spectral radius of the operator \( \lambda(\mu) \). Actually, given any complex number \( z_0 \) there is a measure \( \mu \in M(G) \) such that \( z_0 \in \text{sp}(\mu) \) and \( \|\lambda(\mu)\| \leq 1 \). (Take \( \nu \in M(K) \) such that \( z \in \text{sp} \nu \) and \( |\hat{\nu}(\theta)| \leq 1 \) for every \( \theta \in \hat{K} \), and set \( \mu = d_\phi(0) \otimes \nu \).)

After this paper was completed we learned that Corollary 2.1 was also proved independently by M. E. Walter [12].
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