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**TANGENT WINDING NUMBERS AND BRANCHED MAPPINGS**

JOHN ROBERT QUINE, JR.

## TANGENT WINDING NUMBERS AND BRANCHED MAPPINGS

J. R. QUINE

The notion of tangent winding number of a regular closed curve on a compact 2-manifold  $M$  is investigated, and related to the notion of obstruction to regular homotopy. The approach is via oriented intersection theory. For  $N$ , a 2-manifold with boundary and  $F: N \rightarrow M$  a smooth branched mapping, a theorem is proved relating the total branch point multiplicity of  $F$  and the tangent winding number of  $F|_{\partial N}$ . The theorem is a generalization of the classical Riemann-Hurwitz theorem.

1. Introduction. Let  $M$  be a smooth, connected, oriented 2-manifold and let  $f$  and  $g$  be regular closed curves on  $M$  with the same initial point and tangent direction. An integer obstruction to regular homotopy  $\gamma(f, g)$  is derived which is uniquely defined if  $M \neq S^2$  and defined mod 2 if  $M = S^2$ . Let  $F(t, \theta)$  be any homotopy such that  $F(0, \theta) = f(\theta)$  and  $F(1, \theta) = g(\theta)$  and  $F$  is smooth on the interior of the unit square. It is shown that  $\gamma(f, g) = I(\partial F / \partial \theta, M_0)$ , where  $M_0$  is the zero section as a sub-manifold of  $TM$ , and  $I$  denotes the total number of oriented intersections. This is given interpretation as the number of loops acquired by curves  $F(t, \cdot) = f_t$  in homotopy.

If  $M$  is compact and  $y$  is not on the image of  $f$ , then we define  $\text{twn}(f; y)$ , a generalization of the tangent winding number. We show that  $\gamma(f, g) = \text{twn}(g; y) - \text{twn}(f; y) + I(F, y)\chi(M)$ , where  $\chi$  is the Euler characteristic. If  $N$  is a 2-manifold with boundary and  $F: N \rightarrow M$  is a smooth branched mapping and  $\partial F = F|_{\partial N}$ , we show that  $\text{twn}(\partial F; y) + I(F, y)\chi(M) = \chi(N) + r$ , where  $r$  is the total branchpoint multiplicity and  $y$  is not in  $F(\partial N)$ . We show that the Riemann-Hurwitz theorem follows as a corollary.

2. The obstruction to regular homotopy. Let  $M$  be a smooth, connected 2-manifold with Riemannian metric. Let  $TM$  be the tangent bundle and  $\hat{T}M$  the unit tangent or sphere bundle. Let  $f: R \rightarrow M$  with  $f(\theta) = f(\theta + 1)$  for all  $\theta \in R$  be a regular closed curve on  $M$ , that is,  $f$  has continuously turning, nonzero tangent vector at each point. Given  $F: [0, 1] \times R \rightarrow M$  continuous with  $F(t, \theta) = F(t, \theta + 1)$  for all  $\theta \in R$ , then  $F$  is said to be a regular homotopy if each closed curve  $F(t, \cdot)$  is regular for  $0 \leq t \leq 1$ . We say the curves  $f(\theta) = F(0, \theta)$  and  $g(\theta) = F(1, \theta)$  are regularly homo-

topic.

Suppose now that  $f$  and  $g$  are regular closed curves with  $f(0) = g(0) = y_0$ . Let  $\tilde{f}$  and  $\tilde{g}$  be the closed curves on  $\tilde{T}M$  obtained by taking the unit tangent vector at each point of  $f$  and  $g$  respectively. Suppose that  $\tilde{f}(0) = \tilde{g}(0) = \tilde{y}_0$ . Smale [9] has shown that  $f$  and  $g$  are regularly homotopic iff  $\tilde{f}$  and  $\tilde{g}$  are homotopic. Using this result we define the obstruction to regular homotopy,  $\gamma(f, g)$ , as follows.

Let  $S^1$  be the fiber of  $\tilde{T}M$  over  $y_0$ . Since  $\Pi_2(\tilde{T}M) = 0$  for any 2-manifold  $M$  (this is clear if  $\Pi_2(M) = 0$  and can be verified directly if  $M$  is  $S^2$  or the projective plane), we have the following portion of the exact homotopy sequence of the bundle  $\tilde{T}M$  over  $M$

$$(1) \quad 0 \longrightarrow \Pi_2(M) \xrightarrow{\phi} \Pi_1(S^1) \xrightarrow{\mu} \Pi_1(\tilde{T}M) \xrightarrow{\psi} \Pi_1(M).$$

The sequence (1) induces an isomorphism

$$j: \ker \psi \longrightarrow \Pi_1(S^1)/\text{im } \phi.$$

If  $f$  and  $g$  are homotopic, then the product  $[\tilde{g}][\tilde{f}]^{-1}$  is in  $\ker \psi$ . Writing  $\alpha = j([\tilde{g}][\tilde{f}]^{-1})$ , Smale's theorem says that  $f$  and  $g$  are regularly homotopic iff  $\alpha = 0$ .

Now in what follows suppose  $M$  is oriented. This gives us a natural choice of orientation on  $S^1$  as the fiber of  $\tilde{T}M$  at  $y_0$ , which in turn determines a "positively oriented" generator of  $\Pi_1(S^1)$ . This generator determines an isomorphism of  $\Pi_1(S^1)$  with the integers  $Z$ . Now  $\Pi_2(M) = 0$  unless  $M = S^2$ . Identifying  $\Pi_1(S^1)$  with  $Z$ , we see that  $\text{im } \phi = 2Z$  in case  $M = S^2$ . (Since the Euler characteristic of  $S^2$  is 2, the fundamental 2-cycle is mapped into 2 by  $\phi$ .) Thus for  $M \neq S^2$ ,  $\alpha$  is an integer which we denote  $\gamma(f, g)$ . If  $M = S^2$ ,  $\alpha$  is an element of  $Z_2$ . In this case we write  $n = \gamma(f, g)$  if the integer  $n$  determines the class  $\alpha$  in  $Z_2$ . We will refer to  $\gamma(f, g)$  as the obstruction to regular homotopy. We remark that  $\gamma(f, g)$  is only defined if  $f$  and  $g$  are homotopic. In the next section we will show how to characterize  $\gamma(f, g)$  using intersection theory and in a later section we explain its relationship to tangent winding numbers on surfaces as in Reinhart [8] and Chillingworth [1].

**3. A characterization of  $\gamma(f, g)$ .** Let  $f, g$  and  $M$  be as in the previous section. We will continue to assume that  $M$  is oriented. Suppose  $F(\theta, t)$  is a homotopy, not necessarily regular, with  $F(0, \theta) = f(\theta)$  and  $F(1, \theta) = g(\theta)$  for all  $\theta$ . Let  $K$  be the square  $[0, 1] \times [0, 1]$  and write  $F: K \rightarrow M$ . Now the pullback bundle  $F^*(TM)$  is trivial over  $K$ , so we can find vector valued functions  $v_1, v_2: K \rightarrow TM$  such that the ordered pair  $(v_1(x), v_2(x))$  is positively oriented in  $TM_{F(x)}$

for all  $x \in K$ . Now consider  $\partial/\partial\theta$  as a section of  $TK$  and write  $F_* \circ (\partial/\partial\theta) = (\partial F/\partial\theta): K \rightarrow TM$ . Write  $(\partial F/\partial\theta)(x) = p_1(x)v_1(x) + p_2(x)v_2(x)$  where  $p = (p_1, p_2): K \rightarrow R^2$ . By the definition of the map  $\mu$  in the exact sequence (1), we see that the preimage of  $[\tilde{g}][\tilde{f}]^{-1}$  under is just  $\deg(p/|p|)|_{\partial K}$ , where  $\partial K$  is the positively oriented boundary of  $K$ ,  $|p|$  is the usual Euclidean norm in  $R^2$ , and  $\deg$  is topological degree. Thus  $\gamma(f, g) = \deg(p/|p|)|_{\partial K}$ . If  $M = R^2$  and  $v_1 = (1, 0)$ ,  $v_2 = (0, 1)$  then  $\gamma(f, g) = \text{twn } g - \text{twn } f$ , where  $\text{twn}$  denotes tangent winding number.

Now suppose  $x$  is an isolated zero of  $\partial F/\partial\theta$  and  $D$  is a closed coordinate disc containing  $x$ , but no other zeros of  $\partial F/\partial\theta$ . We define

$$\text{ind}_x \frac{\partial F}{\partial\theta} = \deg \frac{p}{|p|} \Big|_{\partial D}.$$

This is easily verified to be independent of the choice of  $v_1$  and  $v_2$ . Thus if all the zeros of  $\partial F/\partial\theta$  are isolated, then

$$\gamma(f, g) = \sum_{x \in S} \text{ind}_x \frac{\partial F}{\partial\theta}$$

where  $S$  is the set of zeros of  $\partial F/\partial\theta$ .

Now suppose that  $F$  is smooth on  $\text{int } K$ , and let  $M_0$  be the zero section of  $TM$  considered as a smooth, oriented 2-submanifold of  $TM$ . If  $\partial F/\partial\theta$  intersects  $M_0$  transversely at  $x \in K$ , then  $\text{ind}_x \partial F/\partial\theta$  is the same as the oriented intersection number of  $\partial F/\partial\theta$  with  $M_0$  at  $x$ . (For an explanation of intersection numbers see Guillemin and Pollack [3].) Thus  $\gamma(f, g) = I(\partial F/\partial\theta, M_0)$ , the total number of oriented intersections of  $\partial F/\partial\theta$  with  $M_0$ . We remark that  $I(\partial F/\partial\theta, M_0)$  is defined even if  $\partial F/\partial\theta$  does not intersect  $M_0$  transversely: we simply count the transverse intersections for a "nearby" map. Since  $\partial F/\partial\theta(\partial K) \cap M_0 = \emptyset$ , the total number of intersections is the same for every "nearby" map. We summarize our results in

**THEOREM 1.** *Let  $f$  and  $g$  be regular closed curves on  $M$  with the same initial points and initial tangent directions. Suppose  $f$  and  $g$  are homotopic and  $F: K \rightarrow M$  is a homotopy, smooth on  $\text{int } K$ , with  $F(0, \theta) = f(\theta)$  and  $F(1, \theta) = g(\theta)$ , then the obstruction to regular homotopy  $\gamma(f, g)$  is equal to  $I(\partial F/\partial\theta, M_0)$ , the total number of oriented intersections of  $\partial F/\partial\theta$  with the zero section  $M_0$ .*

We give the following interpretation of Theorem 1. Suppose  $\partial F/\partial\theta(x) = 0$  where  $x = (t_0, \theta_0)$  and suppose  $\partial F/\partial\theta$  intersects  $M_0$  transversely at  $x$ . The curve  $F(t_0, \theta)$  has a cusp at  $\theta = \theta_0$ . As  $t$  increases, if this cusp represents the appearance of a positively oriented

loop or the disappearance of a negatively oriented loop, then the intersection number at  $x$  is 1. If it represents the appearance of a negatively oriented loop or the disappearance of a positively oriented loop, then the intersection number is  $-1$ . Thus  $I(\partial F/\partial\theta, M_0)$  counts the null homotopic loops lost or gained in the homotopy.

4. **Tangent winding numbers.** We now wish to show the relationship between  $\gamma(f, g)$  as defined in the previous section and the notion of tangent winding number of a regular curve with respect to a vector field  $v$  on a compact 2-manifold  $M$  as in Reinhart [8] and Chillingworth [1]. Suppose  $f$  is a regular closed curve on  $M$  and  $v$  is a vector field on  $M$  which vanishes at a single point  $y$  not on the image of  $f$ . The order that  $v$  vanishes at  $y$  is clearly  $\chi(M)$ . We define  $\text{twn}_v f$  to be the number of times the tangent of  $f$  rotates in relation to  $v$ . More specifically, suppose  $v = v_1$  and choose vector field  $v_2$  such that  $(v_1, v_2)$  is a positively oriented basis except at  $y$ , where both vanish to the order  $\chi(M)$ . Write  $df/d\theta = p_1 v_1 + p_2 v_2$  where  $p = (p_1, p_2): S^1 \rightarrow R^2$ . We then define  $\text{twn}_v f$  to be  $\deg p/|p|$ . It is straightforward to show that  $\text{twn}_v f$  depends only upon the choice of  $y$ , in fact, it depends only upon the component of  $M - f(R)$  in which  $y$  lies. Thus, we write  $\text{twn}(f; y)$  in place of  $\text{twn}_v f$ .

**THEOREM 2.** *Suppose  $M$  is compact and let  $f, g$ , and  $F$  be as in Theorem 1. Let  $y \in M - f(R) \cup g(R)$ , then  $\gamma(f, g) = I(\partial F/\partial\theta, M_0) = \text{twn}(g; y) - \text{twn}(f; y) + I(F, y)\chi(M)$ .*

*Proof.* Let  $v_1$  and  $v_2$  be as in the definition of  $\text{twn}(f; y)$ . Without loss of generality, suppose  $y$  is a regular value of  $F$ ,  $(\partial F/\partial\theta) \neq 0$  on  $F^{-1}(y)$ , and  $\partial F/\partial\theta$  has only isolated zeros. Let  $x_1, \dots, x_m$  be the zeros of  $\partial F/\partial\theta$  and  $\{x_{m+1}, \dots, x_l\} = F^{-1}(y)$ . Write  $(\partial F/\partial\theta)(x) = q_1(x)v_1(F(x)) + q_2(x)v_2(F(x))$  for  $x \notin F^{-1}(y)$ . Let  $T_1, \dots, T_l$  be closed disjoint coordinate discs on  $M$  such that  $x_k \in T_k$  for  $k = 1, \dots, l$ . Since  $v_1$  and  $v_2$  vanish of order  $\chi(M)$  at  $y$ , we have

(a) For  $k = m + 1, \dots, l$ ,  $\deg(p/|p|)|_{\partial T_k} = \pm \chi(M)$  where the sign is negative if  $F$  preserves orientation at  $x_k$ , and positive if  $F$  reverses orientation at  $x_k$ .

(b) For  $k = 1, \dots, m$ ,  $\deg(p/|p|)|_{\partial T_k} = \text{ind}_{x_k}(\partial F/\partial\theta)$ .

Now since  $p: K - \bigcup_{k=1}^m T_k \rightarrow R^2$ , we have that

$$\deg(p/|p|)|_{\partial K} = \sum_{k=1}^m \deg(p/|p|)|_{\partial T_k}.$$

Since by definition  $\deg(p/|p|)|_{\partial K} = \text{twn}(g; y) - \text{twn}(f; y)$ , the theorem follows from Remarks (a) and (b).

Thus we see that  $\text{twn}(g; y) - \text{twn}(f; y)$  determines mod  $\chi(M)$  the obstruction to regular homotopy.

5. **Branched mappings.** Let  $\tilde{N}$  be a compact oriented 2-manifold and let  $D_1, \dots, D_n$  be  $n$  disjoint copies of the closed unit disc on  $\tilde{N}$ . Let

$$N = \tilde{N} - \bigcup_{k=1}^n \text{int } D_k.$$

Let  $M$  be a compact oriented 2-manifold. Let  $F: N \rightarrow M$  be smooth. Say  $F$  is a branched mapping if  $F$  is nonsingular and orientation preserving except at a finite number of points in  $\text{int } N$  where  $F$  behaves locally like the complex analytic mapping  $z^l$ , for  $l$  an integer  $\geq 2$ . The multiplicity of this branch point is defined to be  $l-1$ .

If  $F: N \rightarrow M$  is smooth, we define  $\partial F = F|_{\partial N}$ . We say  $\partial F$  is regular if  $F|_{\partial D_k}$  is regular for  $k = 1, \dots, n$ . If  $y \in M$  is not on the image of  $\partial F$ , we define  $\text{twn}(\partial F; y) = \sum_{k=1}^n \text{twn}(F|_{\partial D_k}; y)$ . We wish to investigate the relationship between  $\text{twn}(\partial F; y)$  and the total branchpoint multiplicity at branchpoints of  $F$ , if  $F$  is a branched mapping.

LEMMA 1. *Let  $F: C \rightarrow C$  be the complex map  $z^l$ ,  $l \geq 2$  and let  $v$  be a nonzero vector field on  $C$ , then  $\text{ind}_0 F_* v = l - 1$ .*

*Proof.* Let  $\tau = \tau(z)$  be a complex valued function giving the vector field  $v$ . Identifying  $TC$  with  $C \times C$ , the map  $F_* v$  is given by  $z \rightarrow (z^l, lz^{l-1}\tau)$ . Now  $\text{ind}_0 F_* v = (1/2\pi) \int_{|z|=1} d \arg lz^{l-1}\tau$ . Since  $\tau(z) \neq 0$  for  $z \in C$ ,  $\int_{|z|=1} d \arg \tau = 0$ . Therefore

$$\text{ind}_0 F_* v = (1/2\pi) \int_{|z|=1} d \arg lz^{l-1} = l - 1,$$

which completes the proof of the lemma.

THEOREM 3. *Suppose  $F: N \rightarrow M$  is a branched mapping,  $\partial F$  is regular, and  $y \in M - F(\partial N)$ , then*

$$\text{twn}(\partial F; y) + I(F, y)\chi(M) = \chi(N) + r$$

where  $r$  is the total branchpoint multiplicity at branchpoints of  $F$ .

*Proof.* Let  $\{x_1, \dots, x_m\} = B$  be the set of branchpoints of  $F$ . Let  $\{x_{m+1}, \dots, x_l\} = F^{-1}(y)$ . Note that  $l - m = I(F, y)$ .

Without loss of generality, assume that  $y$  is a regular value

of  $F$  and  $B \cap F^{-1}(y) = \emptyset$ . Let  $v_1$  and  $v_2$  be vector fields on  $M$  such that  $(v_1, v_2)$  is positively oriented on  $M$  except at  $y$ , where both vector fields vanish to the order  $\chi(M)$ . Let  $w$  be a vector field on  $N$  which defines positive orientation on  $\partial N$ . Suppose that  $w$  vanishes only at  $x_0 \notin B \cup F^{-1}(y)$ . Write

$$F_* w(x) = p_1(x)v_1(f(x)) + p_2(x)v_2(f(x))$$

where  $p = (p_1, p_2): N - F^{-1}(y) \rightarrow R^2$ . Choose disjoint closed coordinate discs  $T_0, \dots, T_l$  with  $x_k \in T_k$  for  $k = 0, 1, \dots, l$ .

Since  $F$  is regular and preserves orientation except at  $x_1, \dots, x_m$ , we have

(a)  $\deg(p|p)|_{\partial T_0} = \chi(N)$ .

(b) For  $k = m + 1, \dots, l$ ,  $\deg(p|p)|_{\partial T_k} = -\chi(M)$ .

Also by Lemma 1 we have

(c) For  $k = 1, \dots, m$ ,  $\deg(p|p)|_{\partial T_k} = r_k - 1$  where  $r_k$  is the branchpoint multiplicity at  $x_k$ .

Finally, by definition

(d)  $\deg(p|p)|_{\partial N} = \text{twn}(\partial f; y)$ .

Since  $p$  is a smooth map from  $N - \bigcup_{k=0}^l T_k$  into  $R^2$ , we have also  $\deg(p|p)|_{\partial N} = \sum_{k=0}^l \deg(p|p)|_{\partial T_k}$ . The theorem now follows from Remarks (a), (b), (c), and (d).

Theorem 3 is intended to be a generalization of results of the type stated by Titus [10], Haefliger [4], and Francis [2]. This is illustrated by the following corollaries.

**COROLLARY 1.** *If  $F: N \rightarrow R^2$  is a branched mapping and  $\partial F$  is regular, then  $\text{twn } \partial F = \chi(N) + r$  where  $r$  is the total multiplicity at branchpoints of  $F$ , and  $\text{twn}$  is the usual tangent winding number for regular curves in the plane.*

*Proof.* Let  $M = S^2$  in Theorem 3 and identify  $R^2$  with  $S^2 - \{y\}$ . Then  $I(F, y) = 0$ ,  $\text{twn } \partial F = \text{twn}(\partial F; y)$ , and the theorem follows.

**COROLLARY 2.** *If  $F: N \rightarrow R^2$  is a sense-preserving immersion and  $\partial F$  is regular, then  $\text{twn } \partial F = \chi(N)$ .*

For information on assembling branched mappings see Francis [2] and Marx [5].

To show how the classical Riemann-Hurwitz theorem follows from Theorem 3, we prove

**COROLLARY 3 (Riemann-Hurwitz).** *If  $\tilde{F}: \tilde{N} \rightarrow M$  is a branched*

mapping, where  $\tilde{N}$  and  $M$  are compact oriented 2-manifolds, then  $\chi(\tilde{N}) + r = (\deg \tilde{F})\chi(M)$ .

*Proof.* Let  $y$  be a regular value of  $\tilde{F}$  and  $D$  a sufficiently small open disc containing  $y$  such that  $\tilde{F}^{-1}(D)$  consists of  $\deg \tilde{F}$  disjoint discs  $D_j$ . Let  $N = \tilde{N} - \cup D_j$  and  $F = \tilde{F}|_N$ . Now  $\text{twn}(F|_{\partial D_j}; y) = \chi(M) - 1$  for  $j = 1, \dots, \deg \tilde{F}$  and  $I(F; y) = 0$ . Therefore Theorem 3 gives

$$(\deg \tilde{F})(\chi(M) - 1) = \chi(N) + r = \chi(\tilde{N}) - \deg \tilde{F} + r$$

and the conclusion follows.

#### REFERENCES

1. D. R. J. Chillingworth, *Winding numbers on surfaces I*, Math. Ann., **196** (1972), 218-249.
2. G. K. Francis, *Assembling compact Riemann surfaces with given boundary curves and branch points on the sphere*, Illinois J. Math., **20** (1976), 198-217.
3. V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974.
4. A. Haefliger, *Quelques remarques sur les applications différentiable d'une surface dans le plan*, Ann. Inst. Fourier, **10** (1960), 47-60.
5. M. L. Marx, *Extensions of normal immersions of  $S^1$  into  $R^2$* , Trans. Amer. Math. Soc., **187** (1974), 309-326.
6. J. R. Quine, *Homotopies and intersection sequences*, Pacific J. Math., (to appear).
7. ———, *A global theorem for singularities of maps between oriented 2-manifolds*, (to appear), Trans. Amer. Math. Soc.
8. B. L. Reinhart, *The winding number on two manifolds*, Ann. Inst. Fourier, **10** (1960), 271-283.
9. S. Smale, *Regular curves on Riemannian manifolds*, Trans. Amer. Math. Soc., **87** (1958), 492-512.
10. C. J. Titus, *Extensions through codimension one to sense preserving mappings*, Ann. Inst. Fourier, **23** (1973), 215-227.
11. H. Whitney, *On regular closed curves in the plane*, Compositio Math., **4** (1937), 276-286.
12. ———, *On singularities of mappings of Euclidean spaces, I, mappings of the plane into the plane*, Ann. of Math., **62** (1955), 374-410.

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