TANGENT WINDING NUMBERS AND BRANCHED MAPPINGS

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The notion of tangent winding number of a regular closed curve on a compact 2-manifold $M$ is investigated, and related to the notion of obstruction to regular homotopy. The approach is via oriented intersection theory. For $N$, a 2-manifold with boundary and $F: N \rightarrow M$ a smooth branched mapping, a theorem is proved relating the total branch point multiplicity of $F$ and the tangent winding number of $F|_{\partial N}$. The theorem is a generalization of the classical Riemann-Hurwitz theorem.

1. Introduction. Let $M$ be a smooth, connected, oriented 2-manifold and let $f$ and $g$ be regular closed curves on $M$ with the same initial point and tangent direction. An integer obstruction to regular homotopy $\gamma(f, g)$ is derived which is uniquely defined if $M \neq S^2$ and defined mod 2 if $M = S^2$. Let $F(t, \theta)$ be any homotopy such that $F(0, \theta) = f(\theta)$ and $F(1, \theta) = g(\theta)$ and $F$ is smooth on the interior of the unit square. It is shown that $\gamma(f, g) = I(\partial F/\partial \theta, M_0)$, where $M_0$ is the zero section as a sub-manifold of $TM$, and $I$ denotes the total number of oriented intersections. This is given interpretation as the number of loops acquired by curves $F(t, \theta) = f_t$ in homotopy.

If $M$ is compact and $y$ is not on the image of $f$, then we define $\text{twn}(f; y)$, a generalization of the tangent winding number. We show that $\gamma(f, g) = \text{twn}(g; y) - \text{twn}(f; y) + I(F, y)\chi(M)$, where is the Euler characteristic. If $N$ is a 2-manifold with boundary and $F: N \rightarrow M$ is a smooth branched mapping and $\partial F = F|_{\partial N}$, we show that $\text{twn}(\partial F; y) + I(F, y)\chi(M) = \chi(N) + r$, where $r$ is the total branchpoint multiplicity and $y$ is not in $F(\partial N)$. We show that the Riemann-Hurwitz theorem follows as a corollary.

2. The obstruction to regular homotopy. Let $M$ be a smooth, connected 2-manifold with Riemannian metric. Let $TM$ be the tangent bundle and $\tilde{TM}$ the unit tangent or sphere bundle. Let $f: R \rightarrow M$ with $f(\theta) = f(\theta + 1)$ for all $\theta \in R$ be a regular closed curve on $M$, that is, $f$ has continuously turning, nonzero tangent vector at each point. Given $F: [0, 1] \times R \rightarrow M$ continuous with $F(t, \theta) = F(t, \theta + 1)$ for all $\theta \in R$, then $F$ is said to be a regular homotopy if each closed curve $F(t, \theta)$ is regular for $0 \leq t \leq 1$. We say the curves $f(\theta) = F(0, \theta)$ and $g(\theta) = F(1, \theta)$ are regularly homo-
Suppose now that \( f \) and \( g \) are regular closed curves with \( f(0) = g(0) = y_0 \). Let \( \tilde{f} \) and \( \tilde{g} \) be the closed curves on \( TM \) obtained by taking the unit tangent vector at each point of \( f \) and \( g \) respectively. Suppose that \( \tilde{f}(0) = \tilde{g}(0) = \tilde{y}_0 \). Smale [9] has shown that \( f \) and \( g \) are regularly homotopic iff \( \tilde{f} \) and \( \tilde{g} \) are homotopic. Using this result we define the obstruction to regular homotopy, \( \gamma(f, g) \), as follows.

Let \( S^1 \) be the fiber of \( TM \) over \( y_0 \). Since \( \Pi_2(TM) = 0 \) for any 2-manifold \( M \) (this is clear if \( \Pi_2(M) = 0 \) and can be verified directly if \( M \) is \( S^2 \) or the projective plane), we have the following portion of the exact homotopy sequence of the bundle \( TM \) over \( M \)

\[
0 \longrightarrow \Pi_2(M) \xrightarrow{\phi} \Pi_1(S^1) \xrightarrow{\mu} \Pi_1(TM) \xrightarrow{\psi} \Pi_1(M) \,.
\]

The sequence (1) induces an isomorphism

\[
j : \ker \psi \longrightarrow \Pi_1(S^1)/\text{im} \phi \,.
\]

If \( f \) and \( g \) are homotopic, then the product \([\tilde{g}][\tilde{f}]^{-1}\) is in \( \ker \psi \). Writing \( \alpha = j([\tilde{g}][\tilde{f}]^{-1}) \), Smale’s theorem says that \( f \) and \( g \) are regularly homotopic iff \( \alpha = 0 \).

Now in what follows suppose \( M \) is oriented. This gives us a natural choice of orientation on \( S^1 \) as the fiber of \( TM \) at \( y_0 \), which in turn determines a “positively oriented” generator of \( \Pi_1(S^1) \). This generator determines an isomorphism of \( \Pi_1(S^1) \) with the integers \( Z \). Now \( \Pi_2(M) = 0 \) unless \( M = S^2 \). Identifying \( \Pi_1(S^1) \) with \( Z \), we see that \( \text{im} \phi = 2Z \) in case \( M = S^2 \). (Since the Euler characteristic of \( S^2 \) is 2, the fundamental 2-cycle is mapped into 2 by \( \phi \).) Thus for \( M \neq S^2 \), \( \alpha \) is an integer which we denote \( \gamma(f, g) \). If \( M = S^2 \), \( \alpha \) is an element of \( Z_2 \). In this case we write \( n = \gamma(f, g) \) if the integer \( n \) determines the class \( \alpha \) in \( Z_2 \). We will refer to \( \gamma(f, g) \) as the obstruction to regular homotopy. We remark that \( \gamma(f, g) \) is only defined if \( f \) and \( g \) are homotopic. In the next section we will show how to characterize \( \gamma(f, g) \) using intersection theory and in a later section we explain its relationship to tangent winding numbers on surfaces as in Reinhart [8] and Chillingworth [1].

3. A characterization of \( \gamma(f, g) \). Let \( f, g \) and \( M \) be as in the previous section. We will continue to assume that \( M \) is oriented. Suppose \( F(\theta, t) \) is a homotopy, not necessarily regular, with \( F(0, \theta) = f(\theta) \) and \( F(1, \theta) = g(\theta) \) for all \( \theta \). Let \( K \) be the square \([0, 1] \times [0, 1] \) and write \( F : K \to M \). Now the pullback bundle \( F^*(TM) \) is trivial over \( K \), so we can find vector valued functions \( v_1, v_2 : K \to TM \) such that the ordered pair \((v_1(x), v_2(x))\) is positively oriented in \( TM_{F(x)} \).
for all \( x \in K \). Now consider \( \partial/\partial \theta \) as a section of \( TK \) and write \( F_\ast \circ (\partial/\partial \theta) = (\partial F/\partial \theta): K \to TM \). Write \( (\partial F/\partial \theta)(x) = p_1(x)v_1(x) + p_2(x)v_2(x) \) where \( p = (p_1, p_2): K \to \mathbb{R}^2 \). By the definition of the map \( \mu \) in the exact sequence (1), we see that the preimage of \( [\tilde{g}][\tilde{f}]^{-1} \) under is just \( \text{deg } (p/|p|)|_{\partial K} \), where \( \partial K \) is the positively oriented boundary of \( K \), \(| | \) is the usual Euclidean norm in \( \mathbb{R}^2 \), and \( \text{deg} \) is topological degree. Thus \( \gamma(f, g) = \text{deg } (p/|p|)|_{\partial K} \). If \( M = \mathbb{R}^2 \) and \( v_1 = (1, 0) \), \( v_2 = (0, 1) \) then \( \gamma(f, g) = \text{twn } g - \text{twn } f \), where \( \text{twn} \) denotes tangent winding number.

Now suppose \( x \) is an isolated zero of \( \partial F/\partial \theta \) and \( D \) is a closed coordinate disc containing \( x \), but no other zeros of \( \partial F/\partial \theta \). We define

\[
\text{ind}_x \frac{\partial F}{\partial \theta} = \text{deg } \frac{p}{|p|} \bigg|_{\partial D}.
\]

This is easily verified to be independent of the choice of \( v_1 \) and \( v_2 \). Thus if all the zeros of \( \partial F/\partial \theta \) are isolated, then

\[
\gamma(f, g) = \sum_{x \in S} \text{ind}_x \frac{\partial F}{\partial \theta}
\]

where \( S \) is the set of zeros of \( \partial F/\partial \theta \).

Now suppose that \( F \) is smooth on \( \text{int } K \), and let \( M_0 \) be the zero section of \( TM \) considered as a smooth, oriented 2-submanifold of \( TM \). If \( \partial F/\partial \theta \) intersects \( M_0 \) transversely at \( x \in K \), then \( \text{ind}_x \partial F/\partial \theta \) is the same as the oriented intersection number of \( \partial F/\partial \theta \) with \( M_0 \) at \( x \). (For an explanation of intersection numbers see Guillemin and Pollack [3].) Thus \( \gamma(f, g) = I(\partial F/\partial \theta, M_0) \), the total number of oriented intersections of \( \partial F/\partial \theta \) with \( M_0 \). We remark that \( I(\partial F/\partial \theta, M_0) \) is defined even if \( \partial F/\partial \theta \) does not intersect \( M_0 \) transversely: we simply count the transverse intersections for a "nearby" map. Since \( \partial F/\partial \theta(\partial K) \cap M_0 = \emptyset \), the total number of intersections is the same for every "nearby" map. We summarize our results in

**Theorem 1.** Let \( f \) and \( g \) be regular closed curves on \( M \) with the same initial points and initial tangent directions. Suppose \( f \) and \( g \) are homotopic and \( F: K \to M \) is a homotopy, smooth on \( \text{int } K \), with \( F(0, \theta) = f(\theta) \) and \( F(1, \theta) = g(\theta) \), then the obstruction to regular homotopy \( \gamma(f, g) \) is equal to \( I(\partial F/\partial \theta, M_0) \), the total number of oriented intersections of \( \partial F/\partial \theta \) with the zero section \( M_0 \).

We give the following interpretation of Theorem 1. Suppose \( \partial F/\partial \theta(x) = 0 \) where \( x = (t_0, \theta_0) \) and suppose \( \partial F/\partial \theta \) intersects \( M_0 \) transversely at \( x \). The curve \( F(t_0, \theta) \) has a cusp at \( \theta = \theta_0 \). As \( t \) increases, if this cusp represents the appearance of a positively oriented
loop or the disappearance of a negatively oriented loop, then the intersection number at $x$ is 1. If it represents the appearance of a negatively oriented loop or the disappearance of a positively oriented loop, then the intersection number is $-1$. Thus $I(\partial F/\partial \theta, M_0)$ counts the null homotopic loops lost or gained in the homotopy.

4. Tangent winding numbers. We now wish to show the relationship between $\gamma(f, g)$ as defined in the previous section and the notion of tangent winding number of a regular curve with respect to a vector field $v$ on a compact 2-manifold $M$ as in Reinhart [8] and Chillingworth [1]. Suppose $f$ is a regular closed curve on $M$ and $v$ is a vector field on $M$ which vanishes at a single point $y$ not on the image of $f$. The order that $v$ vanishes at $y$ is clearly $\chi(M)$. We define $\text{twn}_v f$ to be the number of times the tangent of $f$ rotates in relation to $v$. More specifically, suppose $v = v_1$ and choose vector field $v_2$ such that $(v_1, v_2)$ is a positively oriented basis except at $y$, where both vanish to the order $\chi(M)$. Write $dF/d\theta = p_1 v_1 + p_2 v_2$ where $p = (p_1, p_2): S^1 \rightarrow R^2$. We then define $\text{twn}_v f$ to be $\deg p/|p|$. It is straightforward to show that $\text{twn}_v f$ depends only upon the choice of $y$, in fact, it depends only upon the component of $M - f(R)$ in which $y$ lies. Thus, we write $\text{twn}(f; y)$ in place of $\text{twn}_v f$.

**Theorem 2.** Suppose $M$ is compact and let $f, g,$ and $F$ be as in Theorem 1. Let $y \in M - f(R) \cup g(R)$, then $\gamma(f, g) = I(\partial F/\partial \theta, M_0) = \text{twn}(g; y) - \text{twn}(f; y) + I(F, y) \chi(M)$.

**Proof.** Let $v_1$ and $v_2$ be as in the definition of $\text{twn}(f; y)$. Without loss of generality, suppose $y$ is a regular value of $F$, $(\partial F/\partial \theta) \neq 0$ on $F^{-1}(y)$, and $\partial F/\partial \theta$ has only isolated zeros. Let $x_1, \ldots, x_m$ be the zeros of $\partial F/\partial \theta$ and $\{x_{m+1}, \ldots, x_l\} = F^{-1}(y)$. Write $(\partial F/\partial \theta)(x) = q_1(x)v_1(F(x)) + q_2(x)v_2(F(x))$ for $x \in F^{-1}(y)$. Let $T_1, \ldots, T_l$ be closed disjoint coordinate discs on $M$ such that $x_k \in T_k$ for $k = 1, \ldots, l$. Since $v_1$ and $v_2$ vanish of order $\chi(M)$ at $y$, we have

(a) For $k = m + 1, \ldots$, $\deg(p/|p|)|_{x_k} = \pm \chi(M)$ where the sign is negative if $F$ preserves orientation at $x_k$, and positive if $F$ reverses orientation at $x_k$.

(b) For $k = 1, \ldots, m$, $\deg(p/|p|)|_{x_k} = \text{ind}_{x_k} (\partial F/\partial \theta)$.

Now since $p: K - \bigcup_{k=1}^m T_k \rightarrow R^2$, we have that

$$\deg(p/|p|)|_{x_K} = \sum_{k=1}^m \deg(p/|p|)|_{x_k}.$$

Since by definition $\deg(p/|p|)|_{x_K} = \text{twn}(g; y) - \text{twn}(f; y)$, the theorem follows from Remarks (a) and (b).
Thus we see that \( \text{twn}(g; y) - \text{twn}(f; y) \) determines \( \mod \chi(M) \) the obstruction to regular homotopy.

5. Branched mappings. Let \( \bar{N} \) be a compact oriented 2-manifold and let \( D_1, \ldots, D_n \) be \( n \) disjoint copies of the closed unit disc on \( \bar{N} \). Let

\[
N = \bar{N} - \bigcup_{k=1}^{n} \text{int } D_k.
\]

Let \( M \) be a compact oriented 2-manifold. Let \( F: \bar{N} \to M \) be smooth. Say \( F \) is a branched mapping if \( F \) is nonsingular and orientation preserving except at a finite number of points in \( \text{int } \bar{N} \) where \( F \) behaves locally like the complex analytic mapping \( z^l \), for \( l \) an integer \( \geq 2 \). The multiplicity of this branch point is defined to be \( l - 1 \).

If \( F: \bar{N} \to M \) is smooth, we define \( \partial F = F|_{\partial \bar{N}} \). We say \( \partial F \) is regular if \( F|_{\partial D_k} \) is regular for \( k = 1, \ldots, n \). If \( y \in M \) is not on the image of \( \partial F \), we define \( \text{twn}(\partial F; y) = \sum_{k=1}^{n} \text{twn}(F|_{\partial D_k}; y) \). We wish to investigate the relationship between \( \text{twn}(\partial F; y) \) and the total branchpoint multiplicity at branchpoints of \( F \), if \( F \) is a branched mapping.

**Lemma 1.** Let \( F: \mathbb{C} \to \mathbb{C} \) be the complex map \( z^l \), \( l \geq 2 \) and let \( v \) be a nonzero vector field on \( \mathbb{C} \), then \( \text{ind}_0 F_* v = l - 1 \).

**Proof.** Let \( \tau = \tau(z) \) be a complex valued function giving the vector field \( v \). Identifying \( TC \) with \( \mathbb{C} \times \mathbb{C} \), the map \( F_* v \) is given by \( z \to (z', l z^{l-1} \tau) \). Now \( \text{ind}_0 F_* v = (1/2\pi) \int_{|z|=1} d \arg lz^{l-1} \tau \). Since \( \tau(z) \neq 0 \) for \( z \in \mathbb{C}, \int_{|z|=1} d \arg \tau = 0 \). Therefore

\[
\text{ind}_0 F_* v = (1/2\pi) \int_{|z|=1} d \arg lz^{l-1} = l - 1,
\]

which completes the proof of the lemma.

**Theorem 3.** Suppose \( F: \bar{N} \to M \) is a branched mapping, \( \partial F \) is regular, and \( y \in M - F(\partial N) \), then

\[
\text{twn}(\partial F; y) + I(F, y)\chi(M) = \chi(N) + \tau
\]

where \( \tau \) is the total branchpoint multiplicity at branchpoints of \( F \).

**Proof.** Let \( \{x_1, \ldots, x_m\} = B \) be the set of branchpoints of \( F \). Let \( \{x_{m+1}, \ldots, x_l\} = F^{-1}(y) \). Note that \( l - m = I(F, y) \).

Without loss of generality, assume that \( y \) is a regular value
of $F$ and $B \cap F^{-1}(y) = \emptyset$. Let $v_1$ and $v_2$ be vector fields on $M$ such that $(v_1, v_2)$ is positively oriented on $M$ except at $y$, where both vector fields vanish to the order $\chi(M)$. Let $w$ be a vector field on $N$ which defines positive orientation on $\partial N$. Suppose that $w$ vanishes only at $x_0 \in B \cup F^{-1}(y)$. Write

$$F_*w(x) = p_1(x)v_1(f(x)) + p_2(x)v_2(f(x))$$

where $p = (p_1, p_2): N - F^{-1}(y) \to R^2$. Choose disjoint closed coordinate discs $T_0, \ldots, T_l$ with $x_k \in T_k$ for $k = 0, 1, \ldots, l$.

Since $F$ is regular and preserves orientation except at $x_1, \ldots, x_m$, we have

(a) $\deg (p/|p|)|_{\partial T_0} = \chi(N)$.
(b) For $k = m + 1, \ldots, l$, $\deg (p/|p|)|_{\partial T_k} = -\chi(M)$.

Also by Lemma 1 we have

(c) For $k = 1, \ldots, m$, $\deg (p/|p|)|_{\partial T_k} = r_k - 1$ where $r_k$ is the branchpoint multiplicity at $x_k$.

Finally, by definition

(d) $\deg (p/|p|)|_{\partial N} = \text{twn} (\partial f; y)$.

Since $p$ is a smooth map from $N - \bigcup_{k=0}^l T_k$ into $R^2$, we have also $\deg (p/|p|)|_{\partial N} = \sum_{k=0}^l \deg (p/|p|)|_{\partial T_k}$. The theorem now follows from Remarks (a), (b), (c), and (d).

Theorem 3 is intended to be a generalization of results of the type stated by Titus [10], Haefliger [4], and Francis [2]. This is illustrated by the following corollaries.

**Corollary 1.** If $F: N \to R^2$ is a branched mapping and $\partial F$ is regular, then $\text{twn} \partial F = \chi(N) + r$ where $r$ is the total multiplicity at branchpoints of $F$, and $\text{twn}$ is the usual tangent winding number for regular curves in the plane.

**Proof.** Let $M = S^2$ in Theorem 3 and identify $R^2$ with $S^2 - \{y\}$. Then $I(F, y) = 0$, $\text{twn} \partial F = \text{twn} (\partial F; y)$, and the theorem follows.

**Corollary 2.** If $F: N \to R^2$ is a sense-preserving immersion and $\partial F$ is regular, then $\text{twn} \partial F = \chi(N)$.

For information on assembling branched mappings see Francis [2] and Marx [5].

To show how the classical Riemann-Hurwitz theorem follows from Theorem 3, we prove

**Corollary 3 (Riemann-Hurwitz).** If $\tilde{F}: \tilde{N} \to M$ is a branched
mapping, where $\hat{N}$ and $M$ are compact oriented 2-manifolds, then $\chi(\hat{N}) + r = (\deg \hat{F})\chi(M)$.

**Proof.** Let $y$ be a regular value of $\hat{F}$ and $D$ a sufficiently small open disc containing $y$ such that $\hat{F}^{-1}(D)$ consists of $\deg \hat{F}$ disjoint discs $D_j$. Let $N = \hat{N} - \bigcup D_j$ and $F = \hat{F}|_{\hat{N}}$. Now $\text{twn}(F|_{\partial D_j}; y) = \chi(M) - 1$ for $j = 1, \ldots, \deg \hat{F}$ and $I(F; y) = 0$. Therefore Theorem 3 gives

$$(\deg \hat{F})(\chi(M) - 1) = \chi(N) + r = \chi(\hat{N}) - \deg \hat{F} + r$$

and the conclusion follows.

**References**


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