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Let K be a closed subspace in a real or complex normed linear space L. The "Main Interpolation Problem" as formulated by L. Asimow reads as follows: Given a bounded convex neighborhood V of 0 in L and a bounded closed convex U containing 0, their polars V^0 and U^0 in the dual L' of L, define the functionals on L $p_{V_K}(x) = \sup(x, V^0 \cap K^0)$ and $p_U(x) = \sup(x, U^0)$. For $x_0 \in L$ we are looking for an element $x \in L$ satisfying

(1) $x - x_0 \in K$ $(x|_{K^0} = x_0|_{K^0})$ and

(2) $p_U(x) = p_{V_K}(x_0)$ (exact solution), respectively

(2′) $p_U(x) \leqq p_{V_K}(x_0) + \varepsilon$ for given $\varepsilon > 0$ (approximate solution).

The problem is formulated in a different but equivalent way in this paper using the canonical projection p from L to L/K. For a real linear subspace M of L, a convex cone N in M and bounded closed convex neighborhoods U and V we prove conditions in terms of the dual space of L which are necessary and sufficient for the inclusions

$$p(N \cap U) \supset p(M) \cap \underline{p(V)} \text{ resp. } p(N \cap U) \supset p(M) \cap \overline{p(V)}$$

($\{\underline{\cdots}\}$ means the topological interior, $\{\overline{\cdots}\}$, the closure).

Theorem 1 shows the equivalence of the first inclusion to the existence of a not necessarily linear map with certain properties form the dual L' to K^0, the second inclusion is shown to be valid if the first one holds for a certain family of 0-neighborhoods U and V. Theorems 2 and 3 are applications of the first one and in the case $L = C(X)$, where X is a compact Hausdorff space give generalizations of several well-known results: Gamelin's extended Rudin-Carleson theorem [12], theorems by Björk [10] and Alfsen [1] and T.B. Andersen's split-face theorem [3]. Some of the following results are closely related to Ando's paper [4] on closed range theorems, which gives conditions for the validity of the second inclusion if there exists a projection in the dual of L with range K^0. The notation of "splitability" there coincides with restrictions on neighborhoods ("strongly admissible") in this paper.

I am grateful to L. Asimow for some useful suggestions on the subject.

1. A basic theorem. Let L be a real or complex normed linear space, L' its dual. The polar S^0 of a subset S in L is defined as the set of all $\mu \in L'$ such that $\operatorname{Re} \mu(f) \leqq 1$ for every $f \in S$. The following well-known facts on polars are used in this paper (for proofs, for instance see [17]): The bipolar of S in L is the $\sigma(L, L')$ closed convex hull of $S \cup \{0\}$. If S_1, S_2 are subsets of L, we have $(S_1 \cup S_2)^0 = S_1^0 \cap S_2^0$. If both S_1 and S_2 are closed and convex $(S_1 \cup S_2)^0$ coinsides with the $\sigma(L', L)$ closure of the convex hull of $S_1^0 \cup S_2^0$ in L', and if in addition S_1 and S_2 are 0-neighborhoods in L this convex hull is $\sigma(L', L)$ compact, hence $(S_1 \cap S_2)^0 = \operatorname{conv}(S_1^0 \cup S_2^0)$. We state now our first theorem.

THEOREM 1. *Let K be a closed subspace of the real or complex normed linear space L, M a real linear subspace of L, N a norm complete convex cone in M, V a bounded convex, U a bounded convex and closed neighborhood of* 0 *in L. $p: L \rightarrow L/K$ is the canonical projection. For the following assertions*

(a) $p(N \cap U) \supset p(M) \cap \underline{p(V)}$.

(b) $p(N \cap U) \supset p(M) \cap \overline{p(V)}$.

(c) $\{p(N \cap U)\}^0 \subset \{p(M) \cap p(V)\}^0$.

(d) *There is a map $\varphi: L' \rightarrow K^0$ with the properties*:

 (d1) *For every $\mu \in K^0$ $(\varphi(\mu) - \mu) \in M^0$.*

 (d2) *For every $\mu \in U^0$ and every $f \in M$ such that $p(f) \in p(V)$ we have* $\operatorname{Re} \varphi(\mu)(f) \leqq 1$.

 (d3) *For all $\mu, \nu \in L'$ such that $(\mu - \nu) \in N^0$ we have $(\varphi(\mu) - \varphi(\nu)) \in M^0$.*

(e) *For every $h \in N \cap \underline{U}$ such that the Minkowski functional of $K + V$ $q_{K+V}(h) < 1$ define*

$$U_h = U \cap \frac{1}{\lambda(h)}(U - h)\ , \qquad \lambda(h) = 1 - q_{K+V}(h)$$

and we have

$$p(N \cap U_h) \supset p(M) \cap \underline{p(V)}\ .$$

(a) *and* (c) *are equivalent,* (d) *implies* (a), (a) *implies* (d) *if N is a real linear space,* (e) *implies* (b), *and* (b) *implies* (a).

Proof. The implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a) are trivial. To prove (c) $\Rightarrow$ (a) and (e) $\Rightarrow$ (b) we need Lemma 1.

(c) $\Rightarrow$ (a): Taking the polars on both sides of inclusion (c) shows that $\overline{p(N \cap U)} \supset p(M) \cap p(V)$. Applying Lemma 1, part (1), with $A = L$, $B = p(M) \subset L/K$, $C = N \cap U$ and $D = p(M) \cap p(V)$ we conclude (a).

(e) $\Rightarrow$ (b) is a consequence of Lemma 1, part (2) with the same insertion for A, B, C and D. Then $U \cap 1/\lambda(C - h) = U \cap 1/\lambda(N \cap U - h) =$

$U \cap 1/\lambda(N - h \cap U - h) \supset U \cap N \cap 1/\lambda(U - h) = N \cap U_h$. Obviously for $h \in N \cap U$ both Minkowski-functionals in (e) and in Lemma 1 are equal:

$$\begin{aligned} q_0(p(h)) &= \inf\{\rho \mid p(h) \in \rho(p(M) \cap p(V))\} \\ &= \inf\{\rho \mid h \in \rho(M + K \cap V + K)\} \\ &= \inf\{\rho \mid h \in \rho(V + K)\} = q_{V+K}(h) . \end{aligned}$$

So (e) implies the assumptions of Lemma 1, part (2), and we derive (b).

(d) $\Rightarrow$ (c): This argument makes use of an extended Hahn-Banach theorem by Kaufmann [15] which states the following:

> Let L be a real linear space, N a convex cone in L, q a subadditive, positive-homogeneous functional on L, and let μ be an additive positive-homogeneous functional on N such that $\mu \leqq q$ on N. Then there is a linear functional θ on L such that $\theta \leqq q$ and $\mu \leqq \theta$ on N.

Now suppose (d) holds and let μ be an element of $(p(N \cap U))^0$. Then $\mu \in K^0$ (K^0 is the dual of L/K) and $\operatorname{Re} \mu(f) \leqq 1$ for every $f \in N \cap U$. Let q be the positive-homogeneous subadditive functional on L generated by U:

$$q(f) = \inf\{\lambda \in \boldsymbol{R}_+ \mid f \in \lambda U\} .$$

There is a constant $r > 0$ such that $q(f) \leqq r\|f\|$ for every f in L, because U is a neighborhood of 0.

Let μ_1 be the real functional on L: $\mu_1 = \operatorname{Re} \mu$. Then $\mu_1(f) \leqq q(f)$ for every f in N and applying Kaufmann's theorem we find a real valued functional μ_2 on L such that

$$\mu_2(f) \leqq q(f) \text{ for } f \in L \text{ and } \mu_1(f) \leqq \mu_2(f) \text{ on } N .$$

Clearly μ_2 is continuous, hence the real part of an element $\mu_3 \in L'$. So we have for every f in M

$$\operatorname{Re} \varphi(\mu_3)(f) = \operatorname{Re} \varphi(\mu)(f) = \operatorname{Re} \mu(f) .$$

(This is a consequence of assumption (d3) because $\mu - \mu_3 \in N^0$, hence $\varphi(\mu) - \varphi(\mu_3) \in M^0$, and of (d1) because $\mu \in K^0$, hence $\varphi(\mu) - \varphi \in M^0$.) Now suppose $f \in M$ such that $p(f) \in p(V)$, then (d2) implies $\operatorname{Re} \varphi(\mu_3)(f) \leqq 1$, because $\mu_3 \in U^0$. Therefore $\operatorname{Re} \mu(f) \leqq 1$, and μ belongs to the polar of $p(M) \cap p(V)$.

If N is a real linear space too we prove the implication

(a) $\Rightarrow$ (d): Suppose $p(N \cap U) \supset p(M) \cap \underline{p(V)}$ and define the map $\varphi: L' \longrightarrow K^0$ using the axiom of choice as follows:

$$\varphi(\mu) = \begin{cases} \bar{\mu} \text{ if there is } \bar{\mu} \in K^0 \text{ such that } \mu - \bar{\mu} \in N^0 \\ 0, \text{ else .} \end{cases}$$

Thus φ is well-defined and meets the requirements (d1), (d2), (d3):

(d1) Suppose $\mu \in K^0$ and $\bar{\mu} \in K^0$ such that $\mu - \bar{\mu} \in N^0$.
For every $f \in M$ there exists by assumption $g \in N$ such that $p(f) = p(g)$, hence $\operatorname{Re} \mu(f) = \operatorname{Re} \mu(g) = \operatorname{Re} \bar{\mu}(g) = \operatorname{Re} \varphi(\mu)(f)$.

(d2) Suppose $\mu \in U^0$, $\bar{\mu} = \varphi(\mu)$, $f \in M$ such that $p(f) \in p(V)$. Then for every $\gamma \in (0, 1)$ $\gamma f \in p(M) \cap \overline{p(V)}$, therefore we find $g \in N \cap U$ with $p(g) = p(\gamma f)$, hence $\operatorname{Re} \varphi(\mu)(f) = \operatorname{Re} \varphi(\mu)((1/\gamma)g) = \operatorname{Re} \bar{\mu}((1/\gamma)g) = \operatorname{Re} \mu((1/\gamma)g) \leqq (1/\gamma)$. Thus $\operatorname{Re} \varphi(\mu)(f) \leqq 1$.

(d3) Suppose $\mu, \nu \in L'$ such that $(\mu - \nu) \in N^0$.
Then in case there is no proper $\bar{\mu}$ in K^0, we have $\varphi(\mu) = \varphi(\nu) = 0$. Else let be $\bar{\mu} = \varphi(\mu)$, $\bar{\nu} = \varphi(\nu)$. Then $\bar{\mu} - \mu \in N^0$, $\bar{\nu} - \nu \in N^0$, hence $\bar{\mu} - \bar{\nu} \in N^0$, and $\bar{\mu} - \bar{\nu} \in M^0$ as well because $\bar{\mu} - \bar{\nu} \in K^0$.

To complete the proof of Theorem 1 we need the following lemma:

LEMMA 1. *Let A and B be normed real linear spaces, $p: A \to B$ a continuous linear map, C a complete bounded convex subset in A containing* 0, *D a bounded convex neighborhood of* 0 *in B. Then*

(1) *$\overline{p(C)} \supset D$ implies $p(C) \cap \underline{D}$.*

(2) *If there is a bounded neighborhood U of* 0 *in A containing C, such that for every h in the algebraic interior of C for which*

$$\lambda(h) = \sup \{\rho \in R_+ \mid p(h) \in (1 - \rho)D\} = 1 - q_D(p(h)) > 0$$

(where q_D denotes the Minkowski-functional of D on B)

$$\overline{p\Big(U \cap \frac{1}{\lambda(h)}(C - h)\Big)} \supset D ,$$

then $p(C) \supset \bar{D}$.

Proof. (1) Suppose $\overline{p(C)} \supset D$ and let $f \in D$. Given $\varepsilon > 0$ there is $g_0 \in C$ such that $\|f - p(g_0)\| < \varepsilon$. Suppose $g_1, \cdots, g_n \in C$ have been selected such that

$$\Big\|f - p\Big(\sum_{i=0}^{k} \Big(\frac{\varepsilon}{r}\Big)^i g_i\Big)\Big\| \leqq \varepsilon\Big(\frac{\varepsilon}{r}\Big)^k , \qquad \text{for every} \quad k = 1, \cdots, n$$

where $r > 0$ is a constant, such that $rE_B \subset D$. (E_B denotes the closed unit ball in B.) Then $(r/\varepsilon)^{n+1}(f - p(\sum_{i=0}^{n} (\varepsilon/r)^i g_i)) \in D$ and we find $g_{n+1} \in C$ such that

$$\Big\|\Big(\frac{r}{\varepsilon}\Big)^{n+1}\Big(f - p\Big(\sum_{i=0}^{n} \Big(\frac{\varepsilon}{r}\Big)^i g_i\Big)\Big) - p(g_{n+1})\Big\| \leqq \varepsilon ,$$

hence

$$\left\| f - p\left(\sum_{i=0}^{n+1}\left(\frac{\varepsilon}{r}\right)^i g_i\right)\right\| \leqq \varepsilon\left(\frac{\varepsilon}{r}\right)^{n+1} .$$

Set $g = \sum_{i=0}^{\infty}(\varepsilon/r)^i g_i$. Then $p(g) = f$, and $g \in (1/(1-(\varepsilon/r)))C$, hence for every $\gamma > 1$, $p(\gamma C) \supset D$, $p(C) \supset (1/\gamma)D$, which proves part (1).

(2) U is bounded, so $U \subset RE_A$, where E_A denotes the unit ball in A. Let $f \in \bar{D}$. Then $(f/2) \in \underline{D}(1-(1/2)^2)$ and by hypothesis (set $h = 0$) and part (1) there is $g_1 \in C(1-(1/2)^2)$ such that $p(g_1) = (f/2)$ and $\|g_1\| \leqq (3/2^2)R$ (because $g_1 \in (1-(1/2)^2)U$). Suppose $g_1, g_2, \cdots, g_n$ have been selected such that $\sum_{i=1}^{n} g_i \in C(1-(1/2)^{n+1})$, $p(g_i) = (f_1/2^i)$, $\|g_i\| \leqq (3/2^{i+1})R$, $i = 1, \cdots, n$. Set

$$h = \frac{1}{\left(1-\left(\frac{1}{2}\right)^{n+2}\right)}\sum_{i=1}^{n} g_i .$$

Then $h \in \bigcup_{0 \leqq \gamma < 1} \gamma C$ and

$$p(h) = \frac{1}{\left(1-\left(\frac{1}{2}\right)^{n+2}\right)}\sum_{i=1}^{n}\frac{f}{2^i} = \frac{\left(1-\left(\frac{1}{2}\right)^{n}\right)}{\left(1-\left(\frac{1}{2}\right)^{n+2}\right)}f$$
$$= \frac{2^{n+2}-2^2}{2^{n+2}-1}f \in \frac{2^{n+2}-2^2}{2^{n+2}-1}D ,$$

hence

$$\lambda(h) \geqq 1 - \frac{2^{n+2}-2^2}{2^{n+2}-1} = \frac{3}{2^{n+2}-1} .$$

By hypothesis and part (1) of the lemma then

$$p\left(U \cap \frac{2^{n+2}-1}{3}(C-h)\right) \supset \underline{D} ,$$

and there is

$$g' \in U \cap \frac{2^{n+2}-1}{3}(C-h)$$

such that

$$p(g') = \frac{2}{3}f .$$

Now let

$$g_{n+1} = \frac{3}{2}\frac{1}{2^{n+1}}g' .$$

Then

$$g_{n+1} \in \frac{3}{2}\frac{1}{2^{n+1}} \cdot \frac{2^{n+2}-1}{3}(C-h) = \left(1-\left(\frac{1}{2}\right)^{n+2}\right)(C-h)$$
$$= \left(1-\left(\frac{1}{2}\right)\right)\left(C - \frac{1}{\left(1-\left(\frac{1}{2}\right)^{n+2}\right)}\sum_{i=1}^{n} g_i\right) = \left(1-\left(\frac{1}{2}\right)^{n+2}\right)C - \sum_{i=1}^{n} g_i ,$$

hence

$$\sum_{i=1}^{n+1} g_i \in C\left(1-\left(\frac{1}{2}\right)^{n+2}\right), \qquad p(g_{n+1}) = \frac{f}{2^{n+1}} , \qquad ||g_{n+1}|| \leqq \frac{3}{2^{n+2}}R .$$

Set $g = \sum_{i=1}^{\infty} g_i$. Then $p(g) = f$ and $g \in C$, which completes the proof.

2. A Rudin-Carleson theorem. Throughout this section we assume that K^0 is the range of a norm continuous linear projection π in the dual space L' of L. Applying the implications (d) $\Rightarrow$ (a) (setting $\varphi = \pi$) and (e) $\Rightarrow$ (b) in Theorem 1 we derive an extended Rudin-Carleson-type theorem in Banach spaces. Since the above assumption coincides with Ando's [4] some of the results are related to his.

Let K be a closed subspace of the Banach space L, $\pi\colon L' \to K^0$ a continuous linear projection. To apply Theorem 1 we need some requirements on "admissible" neighborhoods of the origin in L.

DEFINITION. Let U and V be closed convex bounded neighborhoods of 0 in L. (U, V) is called admissible, iff $\pi(U^0) \subset V^0$. U is called strongly admissible, iff $U^0 = \overline{\operatorname{conv}\{\pi(U^0) \cup (I-\pi)(U^0)\}}$ ($\overline{\{\ \}}$ denotes the closure in the norm topology of L'.)

REMARKS.

(2.1) $(E, (1/||\pi||)E)$, where E is the closed unit ball in L, is admissible.

(2.2) If L is an AM-space (Banach lattice with property $||f \vee g|| = ||f|| \vee ||g||$ for all positive elements f, g in L, cf. [19]), K an ideal in L, $\pi\colon L' \to K^0$ the band projection, then the closed unit ball E in L is strongly admissible: The inclusion $E^0 \supset \overline{\operatorname{conv}(\pi(E^0) \cup (I-\pi)(E^0))}$ is trivial. Conversely let $\mu \in E^0$, then $\mu = \pi(\mu) + (I-\pi)\mu$, and because $\pi(\mu)$ and $(I-\pi)\mu$ are orthogonal and L' is an AL-space $||\pi(\mu)|| + ||(I-\pi)\mu|| = ||\,|\pi(\mu)| + |(I-\pi)(\mu)|\,|| \leqq ||\pi(|\mu|) + (I-\pi)(|\mu|)|| \leqq 1$, hence $\mu \in \operatorname{conv}(\pi(E^0) \cup (I-\pi)(E^0))$.

(2.3) Let (U_1, V_1) and (U_2, V_2) be admissible 0-neighborhoods. Then $(U_1 \cap U_2, V_1 \cap V_2)$ is admissible.

This is an immediate consequence of the fact that $(U_1 \cap U_2)^0 = \operatorname{conv}(U_1^0 \cup U_2^0)$, hence $\pi((U_1 \cap U_2)^0) = \pi(\operatorname{conv}(U_1^0 \cup U_2^0) \subset \operatorname{conv}(\pi(U_1^0) \cup \pi(U_2^0)) \subset \operatorname{conv}(V_1^0 \cup V_2^0) = (V_1 \cap V_2)^0)$.

(2.4) If both U and V are strongly admissible, $U \cap V$ is strongly admissible, because

$$\begin{aligned}(U \cap V)^0 &= \operatorname{conv}(U^0 \cup V^0)\\ &= \operatorname{conv}\overline{(\operatorname{conv}(\pi(U^0)) \cup (I-\pi)(U^0)) \cup \operatorname{conv}(\pi(V^0) \cup (I-\pi)(V^0))}\\ &= \overline{\operatorname{conv}(\pi(U^0) \cup \pi(V^0) \cup (I-\pi)(U^0) \cup (I-\pi)(V^0))}\\ &= \overline{\operatorname{conv}(\pi(\operatorname{conv}(U^0 \cup V^0) \cup (I-\pi)(\operatorname{conv}(U^0 \cap V^0)))}\\ &= \overline{\operatorname{conv}(\pi(U \cap V)^0 \cup (I-\pi)(U \cap V)^0)}\,.\end{aligned}$$

(2.5) Let U be strongly admissible, $h \in \underline{U}$. Then (U_h, U) is admissible (U_h was defined: $U_h = U \cap (1/\lambda(h))(U - h)$). To prove (2.5) it is sufficient (because of (2.3)) to show that $((1/\lambda(h))(U - h), U)$ is admissible, i.e., $\pi((1/\lambda(h))(U-h))^0 \subset U^0$, i.e., $\pi(U-h)^0 \subset (1/\lambda(h))U^0$. Let $\mu \in (U-h)^0$, then $\operatorname{Re}\mu(f) \leqq 1 + \operatorname{Re}\mu(h)$ for every $f \in U$. From $0 \in \underline{U}$ we conclude that $\operatorname{Re}\mu(h) > -1$, hence $\mu \in (1 + \operatorname{Re}\mu(h))U^0$. By assumption U is strongly admissible, hence there is $\nu \in L'$ such that $\|\mu - \nu\| < \varepsilon$, $\nu = \lambda_1\nu_1 + \lambda_2\nu_2$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \geqq 0$, $\nu_1 \in (1 + \operatorname{Re}(h))\pi(U^0)$, $\nu_2 \in (1 + \operatorname{Re}\mu(h))(U^0)$. Then $\pi(\nu) = \lambda_1\nu_1$, $(I-\pi)(\nu) = \lambda_2\nu_2$.

From the definition of $\lambda(h) = \sup\{\rho \in R_+ \mid p(h) \in (1-\rho)p(U)\}$ and because $\pi(\nu) \in K^0$ for every $f \in U$ we conclude $\operatorname{Re}\pi(\nu)(h) + \lambda\operatorname{Re}\pi(\nu)(f) \leqq \sup\{\pi(\nu)(g) \mid g \in U\} \leqq \lambda_1(1 + \operatorname{Re}\mu(h))$, hence $\lambda\operatorname{Re}\pi(\nu)(f) \leqq \lambda_1(1+\operatorname{Re}\mu(h)) + \operatorname{Re}(I-\pi)(\nu)(h) - \operatorname{Re}\nu(h) \leqq \lambda_1(1 + \operatorname{Re}\mu(h)) + \lambda_2(1 + \operatorname{Re}\mu(h)) - \operatorname{Re}\nu(h) = 1 + \operatorname{Re}\mu(h) - \operatorname{Re}\nu(h) \leqq 1 + \varepsilon\|h\|$. (Note that $h \in \underline{U}$, $(I-\pi)(\nu) \in \lambda_2(1 + \operatorname{Re}\mu(h))(I-\pi)(U^0)$ imply $\operatorname{Re}(I-\pi)(\nu)(h) \leqq \lambda_9(1 + \operatorname{Re}\mu(h))$.)

Thus $\pi(\nu) \in (1 + \varepsilon\|h\|)(U^0/\lambda)$ for $\varepsilon > 0$. Because π is norm continuous from this we conclude $\pi(\mu) \in (U^0/\lambda) = (\overline{U^0}/\lambda)$.

Now Theorem 2 is at hand.

THEOREM 2. *Let K be a closed subspace of the real or complex Banach space L, K^0 be the range of a norm continuous linear projection π on L', $p\colon L \to L/K$ the canonical map. Suppose M is a real linear subspace of L, N a norm closed convex cone in M. For the following assertions*

(a) *For all closed convex bounded neighborhoods U and V of O in L such that (U, V) is admissible*

$$p(N \cap U) \supset p(M) \cap \underline{p(V)}\,.$$

(b) *For every strongly admissible closed convex bounded neigh-*

borhood U of 0 *in* L

$$p(N \cap U) \supset P(M) \cap \overline{p(U)}\,.$$

(c) $\pi(N^0) \subset M^0$.
(a) *implies* (b), *and* (c) *and* (a) *are equivalent.*

Proof.
(a) $\Rightarrow$ (b) is an immediate consequence of implication
(e) $\Rightarrow$ (b) in Theorem 1 and of Remark (2.5). To prove
(c) $\Rightarrow$ (a) we show Condition (d) in Theorem 1 holds with $\varphi = \pi$.
(d1) is trivial, and because of the linearity of π (d3) corresponds to assertion (c) of Theorem 2. To verify (d2) let $\mu \in U^0$, $f \in V$. Then $\pi(\mu) \in V^0$ because (U, V) is admissible, hence $\operatorname{Re}\pi(\mu)(f) \leqq 1$.

(a) $\Rightarrow$ (c). Assume (a) holds and let $\mu \in N^0$, $f \in M$. To prove $\operatorname{Re}\pi(\mu)(f) = 0$ we have to define proper 0-neighborhoods U and V. Let

$$V = (1/\|\pi\|)E \quad \text{and} \quad U_\varepsilon = E \cap \{h \in L \mid |(I - \pi)(\mu)(h)| \leqq \varepsilon\}$$

where E denotes the closed unit ball in L. Both U_ε and V are bounded convex and closed and (U_ε, V) is admissible: $U_\varepsilon^0 = \operatorname{conv}(E^0 \cup \{\cdots\}^0)$, hence $\pi(U_\varepsilon^0) \subset \operatorname{conv}(\pi(E^0) \cup \pi\{\cdots\}^0)$. So obviously it suffices to verify $\pi\{\cdots\}^0 \subset V^0$. But $\{\cdots\} = \varepsilon \cdot \{e^{i\alpha}(I - \pi)(\mu) \mid \alpha \in [0, 2\pi]\}^0$, therefore $\{\cdots\}^0 = (1/\varepsilon)\{\lambda(I - \pi)(\mu) \mid |\lambda| \leqq 1\}$ and $\pi\{\cdots\}^0 = \{0\}$.

Now select $\lambda > \|f\| \cdot \|\pi\|$. Then $(1/\lambda)f \in V$ and $p((1/\lambda)f) \in p(M) \cap \underline{p(V)}$ and by assumption there is $g_\varepsilon \in N \cap U_\varepsilon$ such that $p(g_\varepsilon) = p((1/\lambda)f)$, hence $|(I - \pi)(\mu)(g_\varepsilon)| \leqq \varepsilon$, i.e., $|\mu(g_\varepsilon) - \pi(\mu)(g_\varepsilon)| \leqq \varepsilon$.

On the other hand we know because $g_\varepsilon - (1/\lambda)f \in K$, $\mu \in N^0$ and $\pi(\mu) \in K^0$ that $\operatorname{Re}\pi(\mu)((1/\lambda)f) = \operatorname{Re}\pi(\mu)(g_\varepsilon) \leqq \operatorname{Re}\mu(g_\varepsilon) + \varepsilon \leqq \varepsilon$.

The argument holds for every $\varepsilon > 0$ independent of λ, hence $\operatorname{Re}\pi(\mu)(f/\lambda) \leqq 0$ and $\pi(\mu) \in M^0$.

3. **Applications in Banach lattices.** In this section we are going to take advantage of the fact that the map $\varphi: L' \to K^0$ in Theorem (1d) needs not necessarily be linear. For the following suppose L is a real or complex Banach lattice, i.e., in the complex case L is the complexification of a real Banach lattice L_0 (for details cf. [19]) $L = L_0 + iL_0$. Let K be an ideal in L, then K^0 is a band in the order complete dual $L' = L_0' + iL_0'$ of L. By π we denote the band projection from L' onto K^0. π is norm continuous and monotone (cf. ([19]). As before M is a real linear subspace of L, N a closed convex cone in M. For the construction of φ we introduce a new parameter:

Let R be a sup-stable (i.e., $f \vee g \in R$ for all $f, g \in R$) convex cone in L_0 such that

(3.1) $\operatorname{Re}(\operatorname{lin} N) \subset R$, i.e., R contains all real parts of the elements of $\operatorname{lin} N$, the complex linear hull of N.

(3.2) R is total in L_0, i.e., $\overline{R - R} = L_0$.

(3.3) $(E_{L_0} \cap R) + L_+$, where E_{L_0} denotes the unit ball, L_+ the positive cone in L_0, is a neighborhood of the origin in L_0.

In straightforward analogy to the concept of the Choquet ordering for measures on a convex compact Hausdorff space we define an order relation "$<_R$" on L'_+ (L'_+ denotes the positive cone in L'_0) by $\mu <_R \nu$ iff $\mu(f) \leqq \nu(f)$ for all $f \in R$. Here R takes the part of the continuous convex functions in the classical case (cf. Alfsen [1]). Like there we show that there are sufficiently many maximal elements in this ordering (Lemma 2) and then define φ using the axiom of choice of the composition of a map from L' in the set of maximal elements and the band projection onto K^0. According to the choice of the parameter R Theorem 3 yields a wide range of applications. For $R = L_0$ for instance, the ordering is trivial and it leads to the Rudin-Carleson-type theorem of §2. In §4 we shall apply it to the case $L = C(X)$ with different choices for R.

The proof of Lemma 4.1 in [16] can be adapted to derive the following lemma on the existence of maximal elements in L'_+. (Note that condition (3.3) for R guarantees the $\sigma(L', L)$ compactness of the set $\{\nu \in L'_+ \mid \nu >_R \mu\}$ for given $\mu \in L'_+$.)

LEMMA 2. *For every $\mu \in L'_+$ there is $\bar{\mu} \in L'_+$ such that $\bar{\mu} >_R \mu$ and $\bar{\mu}$ is maximal in the ordering "$>_R$".*

For every $f \in L_0$ define the upper respectively lower R-envelope (cf. [1], §5) in L''_0, the order complete bidual of L_0

$$\hat{f} = \inf\{h \in -R \mid h \geqq f\}, \qquad \check{f} = \sup\{h \in R \mid h \leqq f\}.$$

Then for $\mu, \nu \in L'_+$ $\mu <_R \nu$ implies $\mu(\check{f}) \leqq \nu(\check{f})$ and $\mu(\hat{f}) \geqq \nu(\hat{f})$. This is an immediate consequence of Propositions 4.2 and 4.5 in Schäfer's book [19], because $\mu(\hat{f}) = \inf\{\mu(h) \mid h \in -R, h \geqq f\}$ for positive μ.

Corresponding to the set of boundary measures in Choquet-theory we define

$$\partial L' = \{\mu \in L' \mid |\mu| \text{ is } R\text{-maximal}\}.$$

(Recall that $|\mu| = \sup_{\alpha \in [0, 2\pi]} |\cos \alpha \mu_1 + \sin \alpha \mu_2| \in L'_0$ where $\mu = \mu_1 + i\mu_2$.)

For every $\mu \in L'$ there is $\bar{\mu} \in \partial L'$ such that $\mu - \bar{\mu} \in (\operatorname{lin} N)^0$, because there is a decomposition of $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ such that

$\mu_i \in L'_+$. According to Lemma 2 select $\bar{\mu}_i \in L'_+$ such that the $\bar{\mu}_i$ are R-maximal and $\bar{\mu}_i \succ_R \mu_i$, hence $\bar{\mu}_i - \mu_i \in (\operatorname{lin} N)^0$. Set $\bar{\mu} = (\bar{\mu}_1 - \bar{\mu}_2) + i(\bar{\mu}_3 - \bar{\mu}_4)$.

A very useful characterization of the elements of $\partial L'$ is given by a reformulation of [1], Proposition I.4.5.

LEMMA 3. *$\mu \in \partial L'$ if and only if $|\mu|(f) = |\mu|(\hat{f})$ for every $f \in L_0$. $\partial L'$ is an order ideal in L'.*

The proof of the first assertion follows straightforward the proof of Proposition I.3.5 and the argument in Proposition I.4.5 in Alfsen's book [1]. To verify that $\partial L'$ is an order ideal in L' let $\mu, \nu \in \partial L'$. Then for $f \in L_0$ $\hat{f} - f$ is clearly positive in L'', $|\mu + \nu|(\hat{f} - f) \leqq |\mu|(\hat{f} - f) + |\nu|(\hat{f} - f) = 0$, hence $\mu + \nu \in \partial L'$. If $\mu \in \partial L'$ and $\nu \in L'$ such that $|\nu| \leqq |\mu|$. Then $|\nu|(\hat{f} - f) \leqq |\mu|(\hat{f} - f) = 0$, which completes the proof.

To formulate the main theorem we need some additional requirements on K and on 0-neighborhoods in L.

DEFINITION. Let U and V be subsets in L. (U, V) is called R-stable, iff for every $\mu \in U^0$ there is $\bar{\mu} \in V^0 \cap \partial L'$ such that $\bar{\mu} - \mu \in (\operatorname{lin} N)^0$. A subset U in L is called R-stable iff (U, U) is R-stable.

REMARKS.

(3.4) Let U_1, U_2, V_1, V_2 be closed convex 0-neighborhoods in L such that (U_1, V_1) and (U_2, V_2) are R-stable. Then $(U_1 \cap U_2, V_1 \cap V_2)$ is R-stable.

To prove (3.4) let $\mu \in (U_1 \cap U_2)^0 = \operatorname{conv}(U_1^0 \cup U_2^0)$. Then $\mu = \lambda_1\mu_1 + \lambda_2\mu_2$, $\mu_1 \in U_1^0$, $\mu_2 \in U_2^0$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \geqq 0$. By hypothesis there are $\bar{\mu}_1 \in V_1^0 \cap \partial L'$ and $\bar{\mu}_2 \in V_2^0 \cap \partial L'$ such that $\bar{\mu}_i - \mu_i \in (\operatorname{lin} N)^0$, $i = 1, 2$. Set $\bar{\mu} = \lambda_1\bar{\mu}_1 + \lambda_2\bar{\mu}_2$, then $\bar{\mu} \in \partial L'$, $\bar{\mu} - \mu \in (\operatorname{lin} N)^0$ and $\bar{\mu} \in \operatorname{conv}(V_1^0 \cup V_2^0) = (V_1 \cap V_2)^0$.

(3.5) Suppose U is an R-stable closed convex bounded 0-neighborhood in L, $h \in N \cap \underline{U}$. Then $U - h$ is R-stable.

Let $\mu \in (U - h)^0$. Then $\operatorname{Re}\mu(h) > -1$ and $\operatorname{Re}\mu(f) \leqq 1 + \operatorname{Re}\mu(h)$ for every $f \in U$, hence $\mu \in (1 + \operatorname{Re}\mu(h))U^0$. Because U is R-stable there is $\bar{\mu} \in \partial L' \cap (1 + \operatorname{Re}\mu(h))U^0$ such that $\bar{\mu} - \mu \in (\operatorname{lin} N)_0$. From $\operatorname{Re}\mu(h) = \operatorname{Re}\bar{\mu}(h)$ we conclude $\operatorname{Re}\bar{\mu}(f) \leqq 1 + \operatorname{Re}\bar{\mu}(h)$ for every $f \in U$, hence $\bar{\mu} \in (U - h)^0$.

(3.6) There is a handy characterization for R-stable admissible 0-neighborhoods in the case L is a real Banach lattice:

Suppose U is a 0-neighborhood such that $\mu_+(\hat{f}_+) + \mu_-(\hat{f}_-) \leqq 1$

for all $\mu \in U^0$, $f \in U$, then U is R-convex and (U, U) is admissible for every band projection π on L'.

To derive the first assertion, let $\mu = \mu_+ - \mu_- \in U^0$ and select $\mu_+, \mu_- \in \partial L' \cap L'_+$ such that $\bar{\mu}_+ \succ_R \mu_+$, $\bar{\mu}_- \succ_R \mu_-$. Then $\bar{\mu} = \bar{\mu}_+ - \bar{\mu}_- \in \partial L'$ and $\bar{\mu} - \mu \in (\mathrm{lin}\, N)^0$ and for every $f \in U$

$$\bar{\mu}(f) \leqq \bar{\mu}_+(f_+) + \bar{\mu}_-(f_-) \leqq \bar{\mu}_+(\hat{f}_+) + \bar{\mu}_-(\hat{f}_-) \leqq \mu_+(\hat{f}_+) + \bar{\mu}_-(\hat{f}_-) \leqq 1\,,$$

hence $\bar{\mu} \in U^0$. Secondly assume π is a band projection on L'. Then $\pi(\mu)(f) = \pi(\mu_+)(f) - \pi(\mu_-)(f) \leqq \pi(\mu_+)(f_+) + \pi(\mu_-)(f_-) \leqq \mu_+(\hat{f}_+) + \mu_-(\hat{f}_-) \leqq 1$, hence $\pi\mu \in U^0$.

Now we state

THEOREM 3. *Let L be a real or complex Banach lattice, M a real linear subspace of L, N a closed convex cone in M, R a sup-stable convex cone in L_0* (the underlying real Banach lattice of L) *such that* (3.1), (3.2), *and* (3.3) *hold. Suppose K is an R-stable ideal in L, $p: L \to L/K$ the canonical projection, $\pi: L' \to K^0$ the band projection from L' onto K^0. For the following assertions*

(a) *For each triple (U, W, V) of closed convex bounded 0-neighborhoods in L such that (U, W) is R-stable and (W, V) is admissible (with respect to π)*

$$p(N \cap U) \supset p(M) \cap \underline{p(V)}\,.$$

(b) *For every strongly admissible R-stable closed convex bounded 0-neighborhood U in L*

$$p(N \cap U) \supset p(M) \cap \overline{p(U)}\,.$$

(c) $\pi(\partial L' \cap N^0) \subset M^0$ *and* $K^0 \cap (\mathrm{lin}\, N)^0 \subset M^0$.

(a) *implies* (b), *and* (c) *and* (a) *are equivalent.*

Proof.

(a) $\Rightarrow$ (b). For every $h \in N \cap U$ $U_h = U \cap (1/\lambda(h))(U - h)$ is R-stable (Remarks (3.4) and (3.5)) and (U_h, U) is admissible, (b) then is a sequence of implication (e) $\Rightarrow$ (b) in Theorem 1.

(c) $\Rightarrow$ (a). To apply Theorem 1, (d) $\Rightarrow$ (a), we construct $\varphi: L' \to K^0$ as follows: Let $\mu \in L'$ and $\lambda = \inf\{\rho \in \boldsymbol{R}_+ \mid \mu \in \rho U^0\}$. There is $\bar{\mu} \in \lambda W^0 \cap \partial L'$, such that $\bar{\mu} - \mu \in (\mathrm{lin}\, N)^0$. Define $\varphi(\mu) = \pi(\bar{\mu}) \in \lambda V^0$. Conditions (d1), (d2), (d3) hold.

(d1): Let $\mu \in K^0$. Because of the R-stability of K there is $\bar{\nu} \in K^0 \cap \partial L'$ such that $\mu - \bar{\nu} \in (\mathrm{lin}\, N)^0$, hence $\mu - \bar{\nu} \in K^0 \cap (\mathrm{lin}\, N)^0 \subset M^0$ by assumption (c). On the other hand $\bar{\mu}$ in the construction of $\varphi(\mu)$

was selected such that $\bar{\mu} - \mu \in (\operatorname{lin} N)^0$, hence $\bar{\mu} - \bar{\nu} \in \partial L' \cap (\operatorname{lin} N)^0 \subset \partial L' \cap N^0$ and again by (c) we conclude $\pi(\bar{\mu} - \bar{\nu}) = \pi(\bar{\mu}) - \bar{\nu} = \varphi(\mu) - \bar{\nu} \in M^0$, hence, $\varphi(\mu) - \mu \in M^0$. (d2) is obvious, because $\mu \in U^0$ implies $\varphi(\mu) \in V^0$. To verify (d3) let $\mu, \nu \in L'$ such that $(\mu - \nu) \in N^0$. Then $\bar{\mu} - \bar{\nu} \in \partial L' \cap N^0$, hence by (c) $\pi(\bar{\mu} - \bar{\nu}) = \varphi(\mu) - \varphi(\nu) \in M^0$.

(a) $\Rightarrow$ (c). Clearly $K^0 \cap (\operatorname{lin} N)^0 \subset M^0$ is a necessary condition for (a) because (a) implies $p(N) = p(M)$. To prove the other inclusion let $\mu \in \partial L' \cap N^0$, $f \in M$. To show $\operatorname{Re} \pi(\mu)(f) \leqq 1$ we construct a proper triple of neighborhoods U_ε, W_ε, V: Define

$$U_\varepsilon = E \cap \{h \in L \mid |(I - \pi)(\mu)(h)| \leqq \varepsilon\}$$

(E denotes the unit ball in L). By assumption (3.3) for R there is a constant $r > 0$ such that $rE_{L_0} \subset (E_{L_0} \cap R)$. ($E_{L_0}$ is the unit ball in L_0.)

Let $W_\varepsilon = (r/4)E \cap \{h \in L \mid |(I - \pi)(\mu)(h)| \leqq \varepsilon\}$, $V = (r/4)E$. The pair (W_ε, V) is admissible (cf. the proof (a) $\Rightarrow$ (c) in Theorem 2). We shall prove now $(U_\varepsilon, W_\varepsilon)$ is R-stable. Let $\mu \in E^0$, then there is a decomposition $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ such that $\mu_i \in E^0 \cap L'_+$. Let $\bar{\mu}_i$ be R-maximal in L'_+ such that $\bar{\mu}_i \succ_R \mu_i$. Suppose $f \in rE_{L_0}$ then there is $h \in E_{L_0} \cap -R$ such that $h \geqq f$, hence $\bar{\mu}_i(f) \leqq \bar{\mu}_i(h) \leqq \mu_i(h) \leqq 1$, hence $\bar{\mu}_i \in (rE_{L_0})^0$, and for $f \in rE$ we conclude $f = f_1 + if_2$, $f_1, f_2 \in rE_{L_0}$, hence $\operatorname{Re} \bar{\mu}_i(f) = \bar{\mu}_i(f_1) \leqq 1$, and $\bar{\mu}_i \in (rE)^0$. Thus

$$\bar{\mu} = \bar{\mu}_1 - \bar{\mu}_2 + i(\bar{\mu}_3 - \bar{\mu}_4) \in 4(rE)^0 = \left(\frac{r}{4}E\right)^0,$$

and $(E, (r/4)E)$ is R-stable. Because of Remark (3.4) all left to show now is R-stability of the set $\{h \in L \mid |(I - \pi)(\mu)(h)| < \varepsilon\}$. But this is obvious because $\mu \in \partial L'$ implies $(I - \pi)(\mu) \in \partial L'$ (cf. Lemma 3) and $\{\cdots\}^0 = \{\lambda(I - \pi)(\mu) \mid |\lambda| \leqq 1\} \subset \partial L'$. Therefore $(U_\varepsilon, W_\varepsilon)$ is R-stable, and to complete the proof we adapt the conclusion in (a) $\Rightarrow$ (c) in Theorem 2.

4. The case $L = C(X)$. There are some interesting applications of Theorem 3 to the case $L = C(X)$, where X is a compact Hausdorff space. With proper choice of the parameter R then quite a few generalizations of well-known results about dominated extensions of continuous functions are at hand. We have to distinguish the cases $L = C_R(X)$ (real valued continuous functions on X) and the complex case $L = C_C(X)$. The latter one requires more sophisticated techniques to stady R-stable neighborhoods, corresponding to Hustad's [14] method to derive a norm preserving complex Choquet theorem. We apply a generalization of his result [18].

Throughout the chapter suppose X is a compact Hausdorff space,

$L = C_R(X)$ (resp. $L = C_C(X)$) provided with the supremum norm. Let K be the closed ideal in $C(X)$ of all functions vanishing on the compact subset $Y \subset X$. L' then is the space of all real (resp. complex) valued Borel measures on X, $\pi\colon L' \to K^0$ the usual restriction to the subset Y.

To define strongly admissible 0-neighborhoods in L let $\gamma = \{z \in \boldsymbol{C} \mid |z| = 1\}$, $\rho\colon X \times \gamma \to \boldsymbol{R}_+$ a lower semicontinuous bounded strictly positive function and

$$U = \{f \in C_C(X) \mid \operatorname{Re}(zf(x)) \leqq \rho(x, z) \quad \text{for all } x \in X,\ z \in \gamma\}\ .$$

To see that U is a strongly admissible 0-neighborhood in $L = C_C(X)$ with respect to the restriction map π, let $f, g \in U$, $\mu \in U^0$, χ_Y the characteristic function of Y. We shall prove first that

$$\operatorname{Re} \mu(f\chi_Y + g(1 - \chi_Y)) \leqq 1\ . \tag{4.1}$$

Given $\varepsilon > 0$ there is a compact subset $K \in X \backslash Y$ such that

$$|\mu|(X \backslash (Y \cup K)) < \varepsilon\ .$$

Let $x \in X \backslash (Y \cup K)$. Urysohn's lemma guanties the existence of continuous functions χ_x and ϕ_x such that $0 \leqq \phi_x,\ \psi_x \leqq 1$ and

$$\psi_x|_{K \cup \{x\}} = 0\ , \qquad \psi_x|_Y = 1$$
$$\phi_x|_{Y \cup \{x\}} = 0\ , \qquad \phi_x|_K = 1\ .$$

Let $\alpha < 1$ and $\bar{f} = \alpha f$, $\bar{g} = \alpha g$, and

$$G_x = \{(x, z) \in X \times \gamma \mid \operatorname{Re}(z(\phi_x \bar{f} + \psi_x \bar{g})) < \rho(x, z)\}$$

then $Y \times \gamma$, $K \times \gamma$, $\{x\} \times \gamma$ all are subsets of G_x, which is open in $X \times \gamma$, because ρ is lower semicontinuous. $\operatorname{Re} z(\phi_x \bar{f} + \psi_x \bar{g})$ is continuous on $X \times \gamma$. Thus $\bigcup_x G_x = X \times \gamma$, which is compact, and there are $x_1, x_2, \cdots, x_n \in X \backslash (Y \cup K)$ such that $\bigcup_{i=1}^n G_{x_i} = X \times \gamma$. Let $\psi = \inf \psi_{x_i}$, $\phi = \inf \phi_{x_i}$. Thus $h = \phi \bar{f} + \psi \bar{g} \in U$, $h|_Y = \bar{f}$, $h|_K = \bar{g}$, and

$$|\mu(h) - \mu(\bar{f}\chi_Y + \bar{g}(1 - \chi_Y))| \leqq \varepsilon(||\bar{f}|| + ||\bar{g}||)\ ,$$

and

$$|\mu(h) - \mu(f\chi_Y + g(1 - \chi_Y))| \leqq \varepsilon(||f|| + ||g||) + ||\mu||(1 - \alpha)\ .$$

Because $||f||$, $||g||$, $||\mu||$ are bounded, $\varepsilon > 0$ and $\alpha < 1$ arbitrary, and $h \in U$ we conclude (4.1).

Now set $\lambda = \sup\{\operatorname{Re} \pi(\mu)(f) \mid f \in U\}$, $\delta = \sup\{\operatorname{Re}(1 - \pi)(\mu)(f) \mid f \in U\}$. Then $\lambda, \delta \geqq 0$, $\lambda + \delta \leqq 1$ by (4.1), and $\pi(\mu) \in \lambda U^0$, $(1 - \pi)(\mu) \in \delta U^0$, hence $\pi(\mu) \in \lambda \pi(U^0)$, $(1 - \pi)(\mu) \in \delta(1 - \pi)(U^0)$, and

$$\mu = \pi(\mu) + (1 - \pi)(\mu) \in \operatorname{conv}(\pi(U^0) \cup (1 - \pi)(U^0))$$
$$U \subset \operatorname{conv}(\pi(U^0) \cup (1 - \pi)(U^0))\ .$$

The converse is obvious, since $\mu \in U^0$, $f \in U$ implies by (4.1) $\pi(\mu)(f) = \mu(f\chi_Y) \leqq 1$, hence $\pi\mu \in U^0$.

As a first corollary of Theorem 3 choosing $R = C_R(X)$ we now prove a Rudin-Carleson theorem, which generalizes Gamelin's [10] version by requiring N only to be a convex cone:

COROLLARY 3.1. *Let Y be a closed subset of the compact Hausdorff space X, M a real linear subspace of $C_C(X)$, N a closed convex cone in M. Then the following conditions are equivalent*:

(a) *For every 0-neighborhood U in $C_C(X)$ defined by a strictly positive bounded lower semicontinuous function $\rho: X \times \gamma \to \boldsymbol{R}_+$* (as above) *and every $f \in M$ such that $f_{|Y} \in U_{|Y}$* (restrictions to the subset Y) *there is a function $g \in N \cap U$ such that $g_{|Y} = f_{|Y}$.*

(b) *For every complex Borel measure μ on X $\mu \in N^0$ implies $\mu_{|Y} \in M^0$.*

Proof.

(b) $\Rightarrow$ (a) is an immediate consequence of (c) $\Rightarrow$ (b) in Theorem 3. To prove the converse suppose $\mu \in N^0$, $h \in M$ such that $h \neq 0$. Define U by $\rho(x, \varphi) = \|h\|$ if $x \in G$ and $\rho(x, \varphi) = \varepsilon$ else, where G is an open neighborhood of Y. Clearly $h_{|Y} \in U_{|Y}$ and by assumption there is $g \in N \cap U$ such that $h_{|Y} = g^{|}{}_Y$, hence

$$\begin{aligned} 0 \geqq \operatorname{Re} \mu(g) &= \operatorname{Re} \mu_{|Y}(g) + \operatorname{Re} \mu_{|G-Y}(g) + \operatorname{Re} \mu_{|X-G}(g) \\ &\geqq \operatorname{Re} \mu_{|Y}(h) - h\,|\mu|(G - Y) - \varepsilon\,|\mu|(X - G)\,, \end{aligned}$$

and because G and ε were arbitrary and μ is regular $0 \geqq \operatorname{Re} \mu_{|Y}(h)$, hence $\mu_{|Y} \in M^0$.

We are going to state now a corollary, which implies and generalizes results by Björk [10], Alfsen-Hirsberg [2], and T.B. Andersen [3]. Recall that the Choquet boundary of R $\partial_R X$ is defined to be the subset of all $x \in X$ such that the Dirac measure ε_x is maximal in the "$<_R$" ordering. Every "boundary measure" $\mu \in \partial L'$ on X is known to vanish on every Baire set disjoint from the Choquet boundary (cf. [1] or [14]). For a linear subspace N in $C_C(X)$, which separates the points of X and contains the constants, we say $\partial_N X = \partial_R X$, where R is the sup-stable cone in $C_R(X)$ generated by the real parts of N.

Note that in the real case the R-stability of a given neighborhood U is relatively easy to be checked, whereas in the complex case the arguments turn out to be much more complicated. Hustad [14] (along with Hirsberg's [13] interpretation) proves the R-stability of the unit ball in $C_C(X)$. We shall apply a generalization of his result given in [18]:

Suppose U is defined by a strictly positive l.s.c. function $\rho: X \times \gamma \to \boldsymbol{R} \cup \{\infty\}$

$$U = \{f \in C_C \mid \mathrm{Re}\,(zf(x)) \leqq \rho(x, z), \text{ for all } x \in X,\ z \in \gamma\}$$

and for every $z \in \gamma$, the function

$$\rho_z: X \to \boldsymbol{R} \cup \{\infty\}\,, \qquad \rho_z(x) = \rho(z, x)$$

is R-superharmonic, i.e., $\rho_z(x) \geqq \mu(\rho_z)$ for all $x \in X$ and $\mu \succ_R \varepsilon_x$ (Dirac measure in x). Then U is R-stable.

COROLLARY 3.2. *Let X be a compact Hausdorff space, M a real linear subspace in $C_C(X)$ (resp. $C_R(X)$), N a closed convex cone in M, which separates the points of X and contains the constant functions, R a sup-stable convex cone in $C_R(X)$ which contains the real parts of all functions in* lin N.

Suppose Y is a compact subset of X such that

(1) *for every measure μ supported by Y there is a boundary measure $\bar{\mu}$ supported by Y such that $\bar{\mu} - \mu \in (\mathrm{lin}\, N)^0$.*

(2) *for every complex boundary measure $\mu \in N^0$ implies $\mu_{|Y} \in M^0$.*

(3) lin $N_{|Y}$ *is dense in $M_{|Y}$.*

Suppose U is a 0-neighborhood in $C_C(X)$ defined by a strictly positive bounded l.s.c. *function $\rho: X \times \gamma \to R$, such that $\rho_z: X \to R$ is R-superharmonic for every $z \in \gamma$.*

Then for every $f \in M$ such that $f_{|Y} \in U_{|Y}$ there is $g \in N \cap U$ such that $f_{|Y} = g_{|Y}$.

With the above notations and remarks this follows directly from Theorem 3. If Y is a subset of $\partial_{\mathrm{lin}\, N} X$ condition (1) is obviously true, (2) implies (3), so Corollary (3.2) generalizes Björk's [10] result and the main theorem in the Alfsen-Hirsberg paper [2]. To derive a complex version of T.B. Andersen [3] extension theorem about continuous affine functions on split-faces let Y be a closed split-face in the compact convex set X, $N = A(X)$ the space of all continuous (complex) affine functions on X, M the subspace of $C_C(X)$ such that all function in $M_{|Y}$ are affine on X. Conditions (1) and (3) then are obvious, because Y is a face and because $A(X)_{|Y}$ is dense in $A(Y)$. (2) is known to be a characterization for split-faces (cf. [1], Theorem II.6.12).

Note that in the real case ρ reduces to two strictly positive bounded l.s.c. functions $f_0, f_U: X \to R_+$ defining U by

$$U = \{f \in C_R(X) \mid -f_U \leqq f \leqq f_0\}\,.$$

U is R-stable if both f_U and f_0 are R-superharmonic.

Another obvious consequence of our main result is Alfsen's Theorem II. 4.5 [1].

COROLLARY 3.3. *Let Y be the topological closure of the set of extreme points $\partial_e X$ of the compact convex set X, $f: \partial_e X \to R$ a continuous function. Then f can be extended to a function in $A(X)$ iff the following two conditions are satisfied*:

(1) $\hat{f}$ *and* $\check{f}$ *coincide on* $\overline{\partial_e K}$

($\hat{f}$ is defined to be $\inf\{g \in A(X) \mid g \geqq f\}$).

(2) *The common restriction of* $\hat{f}$ *and* $\check{f}$ *to* $\overline{\partial_e X}$ *is annihilated by every* $\mu \in \partial L' \cap A(X)^0$.

To prove this set $N = A(X)$, $M = A + \boldsymbol{R}\tilde{f}$, where $\tilde{f}$ is any continuous extension of f on X, U the unit ball in $C_R(X)$. With the same choice of K and R as before the assertion is obvious.

Finally we are going to derive a corollary of the type of Bauer's classical theorem on the abstract Dirichlet-problem (cf. [8], [1] Theorem II. 4.3, [17]).

COROLLARY 3.4. *Let N be a closed convex cone in $C(X)$ ($C_C(X)$ resp. $C_R(X)$), where X is a compact Hausdorff space, which separates the points of X and contains the constant functions. Set $Y = \overline{\partial_{\mathrm{lin}\, N} X}$ and R the sup-stable convex cone generated by* lin N. *Then $N_{|Y} = C(Y)$ if and only if $N^0 \cap \partial L' = \{0\}$ and $\partial_{\mathrm{lin}\, N} X = Y$.*

To prove this set $M = C(X)$. $K = \{f \in C(X) \mid f_{|Y} = 0\}$ is R-stable as well as the unit ball U in $C_R(X)$. All left to show is $K^0 \cap (\mathrm{lin}\, N)^0 \subset M^0$. But this is obvious because $\mu \in K^0 \cap (\mathrm{lin}\, N)^0$ implies $\mu \in \partial L'$ (K^0 is the set of all measures carried by $Y = \partial_{\mathrm{lin}\, N} X$, hence the set of all boundary measures), therefore $\mu \in \partial L' \cap N^0 = \{0\}$.

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