PRODUCTS OF BANACH SPACES THAT ARE UNIFORMLY ROTUND IN EVERY DIRECTION

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It is shown that the product of a collection of Banach spaces that are uniformly rotund in every direction (URED) over a URED Banach space need not be URED; this answers a question raised by M. M. Day. A positive result under an additional hypothesis is also proved.

Introduction. A Banach space $B$ is uniformly rotund in every direction (URED) if and only if, for every nonzero member $z$ of $B$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|(1/2)(x + y)\| \leq 1 - \delta$ whenever $\|x\| = \|y\| = 1$, $x - y = \alpha z$ and $\|x - y\| \geq \varepsilon$.

This generalization of uniform rotundity was introduced by Garkavi [3] to characterize Banach spaces in which every bounded subset has at most one Čebyšev center. Zizler [6] and Day, James, and Swaminathan [2] have investigated this geometrical notion more fully.

The purpose of this note is to answer negatively the following question raised by M. M. Day [1, p. 148]: Is the product of a collection of URED Banach spaces over a URED Banach space still URED? In §1, a positive result is proved under an additional hypothesis; the counterexample, §2, is present exactly when this hypothesis fails.

Let $S$ be an index set. A full function space $X$ on $S$ is a Banach space of real valued functions $f$ on $S$ such that for each $f$ in $X$, each function $g$ for which $|g(s)| \leq |f(s)|$ for each $s$ in $S$ is again in $X$ and $\|g\| \leq \|f\|$.

Note that $X$ has a natural Banach lattice structure with positive cone $\{f \in X : f(s) \geq 0 \text{ for all } s \in S\}$ and that $X$ is order complete by its fullness. It follows easily from theorems of Lotz [4, p. 121] and McArthur [5, p. 5] that the following are equivalent:

1. $X$ contains no closed sublattice order isomorphic to $\mathcal{J}^\omega$.
2. Each order interval in $X$ is compact.

If for each $s$ in $S$, a Banach space $B_s$ is given, let $P_xB_s$, the product of the $B_s$ over $X$, be the space of all those functions $x$ on $S$ such that (i) $x(s)$ is in $B_s$ for each $s$ in $S$, and (ii) if $f$ is defined by $f(s) = \|x(s)\|$ for all $s$ in $S$, then $f$ is in $X$. For each $x$ in $P_xB_s$, define $\|x\| = \|f\|_x$. With the above definitions, $(P_xB_s; \|\cdot\|)$ is a Banach space.
1. A positive result. The question of whether the product of a collection of URED spaces is isomorphic to a URED space was considered in [2, p. 1056]. There, it was shown that $P_xB_s$ is isomorphic to a URED space if each $B_s$ is URED, and if either (i) $S$ is countable or (ii) $X = \ell_p(S)$ for $1 \leq p < \infty$. Here, the isometric question raised by Day is considered.

**THEOREM.** The product space $P_xB_s$ is uniformly rotund in the direction $z$ if each $B_s$ and $X$ is URED and the order interval $[0, ||z(s)||]$ is compact.

**Proof.** Let $z$ be a nonzero member of $P_xB_s$ for which the order interval $[0, ||z(s)||]$ is compact. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $P_xB_s$ such that $||x_n|| = ||y_n|| = 1$, $||x_n + y_n|| \rightarrow 2$ and $x_n - y_n = \alpha_n z$. Then

$$||x_n - \eta \alpha_n z|| \rightarrow 1 \text{ if } 0 \leq \eta \leq 1.$$ Define sequences $\{f_n\}$ and $\{g_n\}$, for $\theta = (1/2), 1$, by letting

$$f_n(s) = ||x_n(s)|| \text{ and } g_n(s) = ||x_n(s) - \theta \alpha_n z(s)||$$

for $s$ in $S$. Then $||f_n|| \rightarrow 1$ and $||g_n|| \rightarrow 1$. Since $||2x_n(s) - \theta \alpha_n z(s)|| \leq f_n(s) + g_n(s)$ for each $s$ and $||2x_n - \theta \alpha_n z|| \rightarrow 2$, we have

$$||f_n + g_n|| \rightarrow 2.$$ For each $n$ and $s$, note that $|f_n(s) - g_n(s)| \leq ||\theta \alpha_n z(s)||$. By the compactness hypothesis, there exist $h^\theta$ in $X$ and a sequence $\{n_k\}$ such that

$$f_{n_k} - g_{n_k} \rightarrow h^\theta.$$ Since $X$ is URED, it follows by Theorem 1 of [2] that $h^\theta = 0$. Thus $||x_n(s)|| - ||x_n(s) - \theta \alpha_n z(s)|| \rightarrow 0$ for each $s$ in $S$ and $\theta = (1/2), 1$. Choosing $s$ such that $z(s) \neq 0$ and using the fact that $B_s$ is URED, we conclude that $\alpha_n \rightarrow 0$. This completes the proof.

The following result is an immediate consequence of the theorem and the above remarks concerning full function spaces.

**COROLLARY.** The product space $P_xB_s$ is URED if each $B_s$ and $X$ is URED and $X$ contains no closed sublattice order isomorphic to $\ell^\infty$.

2. The counterexample. An equivalent full function space norm $||| \cdot |||$ on $\ell^\infty$ that is URED and a sequence $\{B_s\}$ of URED
Banach spaces are defined such that, for \( X = (\ell^\infty; \|\cdot\|) \), the product space \( P_xB_i \) is not URED.

Let \( \{a_j\}_{j=1}^\infty \) be a sequence of positive real numbers such that \( \sum_{j=1}^\infty a_j^2 = 1 \). For \( x = (x_j)_{j=1}^\infty \) an element of \( \ell^\infty \), define

\[
\|x\| = \left( \|x\|_\infty + \sum_{j=1}^\infty a_j^2(|x_1| + |x_j|)^2 \right)^{1/2}.
\]

It is straightforward to verify that \( \|\cdot\| \) is a norm on \( \ell^\infty \) and that \( \|\cdot\|_\infty \leq \|\cdot\| \leq \sqrt{\|\cdot\|_\infty} \). Also note that \( \|x\| = \|x\|_\infty \) and that \( 0 \leq x \leq y \) implies \( \|x\| \leq \|y\| \) for all \( x \) and \( y \) in \( \ell^\infty \). Therefore \( \|\cdot\| \) is an equivalent full function space norm on \( \ell^\infty \).

To show \( (\ell^\infty; \|\cdot\|) \) is URED, let \( z \) be a member of \( \ell^\infty \) such that \( \|z\| = 1 \). If \( \|x\| = \|y\| = 1 \), where \( y = x + \alpha z \), then \( x + y = 2x + \alpha z \) and

\[
\|2x + \alpha z\|^2 = \|2x + \alpha z\|_\infty^2 + \sum_{j=1}^\infty a_j^2(|2x_1 + \alpha z_1| + |2x_j + \alpha z_j|)^2
\]

\[
\leq (\|x\|_\infty + \|x + \alpha z\|_\infty)^2 + \sum_{j=1}^\infty a_j^2(|x_1| + |x_1 + \alpha z_1| + |x_j| + |x_j + \alpha z_j|)^2
\]

\[
= 4 - [(\|x\|_\infty - \|x + \alpha z\|_\infty)^2
\]

\[
+ \sum_{j=1}^\infty a_j^2(|x_1| + |x_1 + \alpha z_1| + |x_j + \alpha z_j| - |x_1| - |x_j|)^2],
\]

and hence

\[
(1) \quad \left[ 1 + \|x + \frac{1}{2} \alpha z\|^2 \right]^{1/2} \geq \frac{1}{2} [ (\|x\|_\infty - \|x + \alpha z\|_\infty)^2
\]

\[
+ \sum_{j=1}^\infty a_j^2(|x_1 + \alpha z_1| + |x_j + \alpha z_j| - |x_1| - |x_j|)^2]^{1/2}.
\]

Similarly, using \( 2\|x\|^2 + \|x + (1/2)\alpha z\|^2 \leq 4 \), we obtain

\[
(2) \quad \left[ 1 - \|x + \frac{1}{4} \alpha z\|^2 \right]^{1/2} \geq \frac{1}{2} [ (\|x\|_\infty - \|x + \frac{1}{2} \alpha z\|_\infty)^2
\]

\[
+ \sum_{j=1}^\infty a_j^2\left(|x_1 + \frac{1}{2} \alpha z_1| + |x_j + \frac{1}{2} \alpha z_j| - |x_1| - |x_j|\right)^2]^{1/2}.
\]

It is sufficient to show that for each \( \varepsilon > 0 \) the sum of the right members of (1) and (2) is bounded from zero, uniformly for all \( x \) such that \( \|x\| = \|x + \alpha z\| = 1 \) with \( |\alpha| \geq \varepsilon \).

(i) If \( z_i = 0 \), choose any \( k \) with \( z_k \neq 0 \). Then at least one of \( |(|x_k + \alpha z_k| - |x_k|)| \) or \( |(|x_k + (1/2)\alpha z_k| - |x_k|)| \) is as great as \( 2^{-3}|\alpha z_k| \), so either the right member of (1) or the right member of (2) is greater than \( 2^{-3}a_z\varepsilon |z_k| \).

(ii) If \( z_i \neq 0 \) and \( |z_i| < 2^{-3}|z_k| \) for some \( k \), then either \( |(|x_i + \alpha z_i| - |x_i|)| \) or \( |(|x_i + (1/2)\alpha z_i| - |x_i|)| \) is as great as \( 2^{-3}|\alpha z_i| \), but
\[ |(x_k + az_k) - |x_k|) < 2^{-5}|az| \text{ and } |(|x_k + (1/2)az| - |x_k|) < 2^{-4}|az|, \]
so either the right member of (1) or the right member of (2) is greater than \(2^{-4}a_k\varepsilon|z_i|\).

(iii) If \(z_i \neq 0\) and \(|z_j| \geq 2^{-3}|z_i|\) for all \(j\), then either

\[
\begin{align*}
|(|x|| - |x + az||) > 2^{-5}\varepsilon|z_i| \\
\left(|(|x|| - |x + \frac{1}{2}az||) > 2^{-5}\varepsilon|z_i| \right)
\end{align*}
\]

and so either the right member of (1) or the right member of (2) is greater than \(2^{-5}\varepsilon|z_i|\). To prove (3), we need only observe that if

\[
|(|x|| - |x + (1/2)az||) < 2^{-5}|az| \text{ and } j \text{ is chosen so that } |(|x|| - |x + (1/2)az||) < 2^{-5}|az|, \]
then \(|x_j + (1/2)az| > |x_j| - 2^{-2}|az|\) and hence \(|x_j + az| = |x_j + (1/2)az| + (1/2)|az|\). Thus

\[ |x + az|| > |x|| - 2^{-2}|az| + \frac{1}{2}|az| > |x|| + 2^{-3}|z_i|. \]

This shows that \(||\cdot||\) is URED.

Now, let \(X = (\ell^\infty; ||\cdot||)\) and for each positive integer \(i\), let \(B_i\) be the two dimensional \(\ell_i^{+1}\) space. Note that each \(B_i\) is URED. Let \(z\) in \(P_xB_i\) be defined by \(z(i) = (1, 0)\) in \(B_i\) for each \(i\). For each \(n \geq 2\), let \(x_n\) and \(y_n\) in \(P_xB_i\) be defined by

\[ x_n(i) = \begin{cases} (0, 0) & \text{if } i = 1 \\ (1, 0) & \text{if } i = n \\ \left(\frac{1}{2}, b_n\right) & \text{if } i \neq 1, n \end{cases} \]

and

\[ y_n(i) = \begin{cases} (-1, 0) & \text{if } i = 1 \\ \left(\frac{1}{2}, b_n\right) & \text{if } i = n \\ (0, 0) & \text{if } i \neq 1, n \end{cases} \]

where \(b_n\) is chosen such that \(b_n > 0\) and \((1/2)^{n+1} + (b_n)^{n+1} = 1\). Then

\[ ||x_n|| = \sqrt{2}, ||y_n|| = (2 + 3a_n^{1/2}), \]
\[ ||x_n + y_n|| = [4b_n + 4 + (4b_n^2 + 4b_n - 3a_n^{1/2})^{1/2}, \]

and \(x_n - y_n = z\) for each \(n \geq 2\). Since \(b_n \to 1\) and \(a_n \to 0\), it follows that \(||y_n|| \to \sqrt{2}\) and \(||x_n + y_n|| \to 2\sqrt{2}\), and hence \(P_xB_i\) is not URED.

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REFERENCES


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