

# Pacific Journal of Mathematics

**PRODUCTS OF BANACH SPACES THAT ARE UNIFORMLY  
ROTUND IN EVERY DIRECTION**

MARK ANDREW SMITH

## PRODUCTS OF BANACH SPACES THAT ARE UNIFORMLY ROTUND IN EVERY DIRECTION

MARK A. SMITH

**It is shown that the product of a collection of Banach spaces that are uniformly rotund in every direction (URED) over a URED Banach space need not be URED; this answers a question raised by M. M. Day. A positive result under an additional hypothesis is also proved.**

**Introduction.** A Banach space  $B$  is *uniformly rotund in every direction* (URED) if and only if, for every nonzero member  $z$  of  $B$  and  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $\|(1/2)(x + y)\| \leq 1 - \delta$  whenever  $\|x\| = \|y\| = 1$ ,  $x - y = \alpha z$  and

$$\|x - y\| \geq \varepsilon.$$

This generalization of uniform rotundity was introduced by Garkavi [3] to characterize Banach spaces in which every bounded subset has at most one Čebyšev center. Zizler [6] and Day, James, and Swaminathan [2] have investigated this geometrical notion more fully. The purpose of this note is to answer negatively the following question raised by M. M. Day [1, p. 148]: Is the product of a collection of URED Banach spaces over a URED Banach space still URED? In §1, a positive result is proved under an additional hypothesis; the counterexample, §2, is present exactly when this hypothesis fails.

Let  $S$  be an index set. A *full function space*  $X$  on  $S$  is a Banach space of real valued functions  $f$  on  $S$  such that for each  $f$  in  $X$ , each function  $g$  for which  $|g(s)| \leq |f(s)|$  for each  $s$  in  $S$  is again in  $X$  and  $\|g\| \leq \|f\|$ .

Note that  $X$  has a natural Banach lattice structure with positive cone  $\{f \in X: f(s) \geq 0 \text{ for all } s \in S\}$  and that  $X$  is order complete by its fullness. It follows easily from theorems of Lotz [4, p. 121] and McArthur [5, p. 5] that the following are equivalent:

- (1)  $X$  contains no closed sublattice order isomorphic to  $\mathcal{L}^\infty$ .
- (2) Each order interval in  $X$  is compact.

If for each  $s$  in  $S$ , a Banach space  $B_s$  is given, let  $P_X B_s$ , the *product of the  $B_s$  over  $X$* , be the space of all those functions  $x$  on  $S$  such that (i)  $x(s)$  is in  $B_s$  for each  $s$  in  $S$ , and (ii) if  $f$  is defined by  $f(s) = \|x(s)\|$  for all  $s$  in  $S$ , then  $f$  is in  $X$ . For each  $x$  in  $P_X B_s$ , define  $\|x\| = \|f\|_X$ . With the above definitions,  $(P_X B_s; \|\cdot\|)$  is a Banach space.

1. **A positive result.** The question of whether the product of a collection of URED spaces is isomorphic to a URED space was considered in [2, p. 1056]. There, it was shown that  $P_X B_s$  is isomorphic to a URED space if each  $B_s$  is URED, and if either (i)  $S$  is countable or (ii)  $X = \ell_p(S)$  for  $1 \leq p < \infty$ . Here, the isometric question raised by Day is considered.

**THEOREM.** *The product space  $P_X B_s$  is uniformly rotund in the direction  $z$  if each  $B_s$  and  $X$  is URED and the order interval  $[0, \{\|z(s)\|\}]$  is compact.*

*Proof.* Let  $z$  be a nonzero member of  $P_X B_s$  for which the order interval  $[0, \{\|z(s)\|\}]$  is compact. Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $P_X B_s$  such that  $\|x_n\| = \|y_n\| = 1$ ,  $\|x_n + y_n\| \rightarrow 2$  and  $x_n - y_n = \alpha_n z$ . Then

$$\|x_n - \eta \alpha_n z\| \longrightarrow 1 \quad \text{if } 0 \leq \eta \leq 1.$$

Define sequences  $\{f_n\}$  and  $\{g_n^\theta\}$ , for  $\theta = (1/2), 1$ , by letting

$$f_n(s) = \|x_n(s)\| \quad \text{and} \quad g_n^\theta(s) = \|x_n(s) - \theta \alpha_n z(s)\|$$

for  $s$  in  $S$ . Then  $\|f_n\| = 1$  and  $\|g_n^\theta\| \rightarrow 1$ . Since  $\|2x_n(s) - \theta \alpha_n z(s)\| \leq f_n(s) + g_n^\theta(s)$  for each  $s$  and  $\|2x_n - \theta \alpha_n z\| \rightarrow 2$ , we have

$$\|f_n + g_n^\theta\| \longrightarrow 2.$$

For each  $n$  and  $s$ , note that  $|f_n(s) - g_n^\theta(s)| \leq \|\theta \alpha_n z(s)\|$ . By the compactness hypothesis, there exist  $h^\theta$  in  $X$  and a sequence  $\{n_k\}$  such that

$$f_{n_k} - g_{n_k}^\theta \longrightarrow h^\theta.$$

Since  $X$  is URED, it follows by Theorem 1 of [2] that  $h^\theta = 0$ . Thus  $\|x_n(s)\| - \|x_n(s) - \theta \alpha_n z(s)\| \rightarrow 0$  for each  $s$  in  $S$  and  $\theta = (1/2), 1$ . Choosing  $s$  such that  $z(s) \neq 0$  and using the fact that  $B_s$  is URED, we conclude that  $\alpha_n \rightarrow 0$ . This completes the proof.

The following result is an immediate consequence of the theorem and the above remarks concerning full function spaces.

**COROLLARY.** *The product space  $P_X B_s$  is URED if each  $B_s$  and  $X$  is URED and  $X$  contains no closed sublattice order isomorphic to  $\ell^\infty$ .*

2. **The counterexample.** An equivalent full function space norm  $\|\cdot\|$  on  $\ell^\infty$  that is URED and a sequence  $\{B_i\}$  of URED

Banach spaces are defined such that, for  $X = (\mathcal{L}^\infty; \|\cdot\|)$ , the product space  $P_X B_i$  is not URED.

Let  $\{\alpha_j\}_{j=2}^\infty$  be a sequence of positive real numbers such that  $\sum_2^\infty \alpha_j^2 = 1$ . For  $x = (x_j)_{j=1}^\infty$  an element of  $\mathcal{L}^\infty$ , define

$$\| \|x\| \| = [ \|x\|_\infty^2 + \sum_2^\infty \alpha_j^2 (|x_1| + |x_j|)^2 ]^{1/2} .$$

It is straightforward to verify that  $\| \cdot \|$  is a norm on  $\mathcal{L}^\infty$  and that  $\| \cdot \|_\infty \leq \| \cdot \| \leq \sqrt{5} \| \cdot \|_\infty$ . Also note that  $\| \|x\| \| = \| \|x\| \|$  and that  $0 \leq x \leq y$  implies  $\| \|x\| \| \leq \| \|y\| \|$  for all  $x$  and  $y$  in  $\mathcal{L}^\infty$ . Therefore  $\| \cdot \|$  is an equivalent full function space norm on  $\mathcal{L}^\infty$ .

To show  $(\mathcal{L}^\infty; \| \cdot \|)$  is URED, let  $z$  be a member of  $\mathcal{L}^\infty$  such that  $\| \|z\| \| = 1$ . If  $\| \|x\| \| = \| \|y\| \| = 1$ , where  $y = x + \alpha z$ , then  $x + y = 2x + \alpha z$  and

$$\begin{aligned} \| \|2x + \alpha z\| \| &= \| 2x + \alpha z \|_\infty^2 + \sum_2^\infty \alpha_j^2 (|2x_1 + \alpha z_1| + |2x_j + \alpha z_j|)^2 \\ &\leq (\|x\|_\infty + \|x + \alpha z\|_\infty)^2 + \sum_2^\infty \alpha_j^2 (|x_1| + |x_1 + \alpha z_1| + |x_j| + |x_j + \alpha z_j|)^2 \\ &= 4 - [(\|x\|_\infty - \|x + \alpha z\|_\infty)^2 \\ &\quad + \sum_2^\infty \alpha_j^2 (|x_1 + \alpha z_1| + |x_j + \alpha z_j| - |x_1| - |x_j|)^2] , \end{aligned}$$

and hence

$$\begin{aligned} (1) \quad & \left[ 1 + \left\| \left\| x + \frac{1}{2} \alpha z \right\| \right\| \right]^{1/2} \geq \frac{1}{2} [(\|x\|_\infty - \|x + \alpha z\|_\infty)^2 \\ & \quad + \sum_2^\infty \alpha_j^2 (|x_1 + \alpha z_1| + |x_j + \alpha z_j| - |x_1| - |x_j|)^2]^{1/2} . \end{aligned}$$

Similarly, using  $2(\| \|x\| \|^2 + \| \|x + (1/2)\alpha z\| \|^2) \leq 4$ , we obtain

$$\begin{aligned} (2) \quad & \left[ 1 - \left\| \left\| x + \frac{1}{4} \alpha z \right\| \right\| \right]^{1/2} \geq \frac{1}{2} \left[ \left( \|x\|_\infty - \left\| \left\| x + \frac{1}{2} \alpha z \right\| \right\| \right)^2 \right. \\ & \quad \left. + \sum_2^\infty \alpha_j^2 \left( \left| x_1 + \frac{1}{2} \alpha z_1 \right| + \left| x_j + \frac{1}{2} \alpha z_j \right| - |x_1| - |x_j| \right)^2 \right]^{1/2} . \end{aligned}$$

It is sufficient to show that for each  $\varepsilon > 0$  the sum of the right members of (1) and (2) is bounded from zero, uniformly for all  $x$  such that  $\| \|x\| \| = \| \|x + \alpha z\| \| = 1$  with  $|\alpha| > \varepsilon$ .

(i) If  $z_1 = 0$ , choose any  $k$  with  $z_k \neq 0$ . Then at least one of  $|(|x_k + \alpha z_k| - |x_k|)|$  or  $|(|x_k + (1/2)\alpha z_k| - |x_k|)|$  is as great as  $2^{-2}|\alpha z_k|$ , so either the right member of (1) or the right member of (2) is greater than  $2^{-3}\alpha_k \varepsilon |z_k|$ .

(ii) If  $z_1 \neq 0$  and  $|z_k| < 2^{-3}|z_1|$  for some  $k$ , then either  $|(|x_1 + \alpha z_1| - |x_1|)|$  or  $|(|x_1 + (1/2)\alpha z_1| - |x_1|)|$  is as great as  $2^{-2}|\alpha z_1|$ , but

$|(|x_k + \alpha z_k| - |x_k|)| < 2^{-3}|\alpha z_1|$  and  $|(|x_k + (1/2)\alpha z_k| - |x_k|)| < 2^{-4}|\alpha z_1|$ , so either the right member of (1) or the right member of (2) is greater than  $2^{-4}\alpha_k \varepsilon |z_1|$ .

(iii) If  $z_1 \neq 0$  and  $|z_j| \geq 2^{-3}|z_1|$  for all  $j$ , then either

$$\left. \begin{aligned} & (||x||_\infty - ||x + \alpha z||_\infty) > 2^{-5}\varepsilon|z_1| \\ \text{or} & \left( \left| ||x||_\infty - \left\| x + \frac{1}{2}\alpha z \right\|_\infty \right| > 2^{-5}\varepsilon|z_1|, \right) \end{aligned} \right\} \quad (3)$$

and so either the right member of (1) or the right member of (2) is greater than  $2^{-6}\varepsilon|z_1|$ . To prove (3), we need only observe that if  $|(|x||_\infty - ||x + (1/2)\alpha z||_\infty)| < 2^{-5}|\alpha z_1|$  and  $j$  is chosen so that  $|(|x||_\infty - |x_j + (1/2)\alpha z_j|)| < 2^{-5}|\alpha z_1|$ , then  $|x_j + (1/2)\alpha z_j| > |x_j| - 2^{-2}|\alpha z_j|$  and hence  $|x_j + \alpha z_j| = |x_j + (1/2)\alpha z_j| + (1/2)|\alpha z_j|$ . Thus

$$||x + \alpha z||_\infty > ||x||_\infty - 2^{-5}|\alpha z_1| + \frac{1}{2}|\alpha z_j| > ||x||_\infty + 2^{-5}\varepsilon|z_1|.$$

This shows that  $|||\cdot|||$  is URED.

Now, let  $X = (\mathcal{L}^\infty; |||\cdot|||)$  and for each positive integer  $i$ , let  $B_i$  be the two dimensional  $\mathcal{L}^{i+1}$  space. Note that each  $B_i$  is URED. Let  $z$  in  $P_X B_i$  be defined by  $z(i) = (1, 0)$  in  $B_i$  for each  $i$ . For each  $n \geq 2$ , let  $x_n$  and  $y_n$  in  $P_X B_i$  be defined by

$$x_n(i) = \begin{cases} (0, 0) & \text{if } i = 1 \\ \left(\frac{1}{2}, b_n\right) & \text{if } i = n \\ (1, 0) & \text{if } i \neq 1, n \end{cases}$$

and

$$y_n(i) = \begin{cases} (-1, 0) & \text{if } i = 1 \\ \left(-\frac{1}{2}, b_n\right) & \text{if } i = n \\ (0, 0) & \text{if } i \neq 1, n \end{cases}$$

where  $b_n$  is chosen such that  $b_n > 0$  and  $(1/2)^{n+1} + (b_n)^{n+1} = 1$ . Then  $||x_n|| = \sqrt{2}$ ,  $||y_n|| = (2 + 3a_n^2)^{1/2}$ ,

$$||x_n + y_n|| = [4b_n^2 + 4 + (4b_n^2 + 4b_n - 3)a_n^2]^{1/2},$$

and  $x_n - y_n = z$  for each  $n \geq 2$ . Since  $b_n \rightarrow 1$  and  $a_n \rightarrow 0$ , it follows that  $||y_n|| \rightarrow \sqrt{2}$  and  $||x_n + y_n|| \rightarrow 2\sqrt{2}$ , and hence  $P_X B_i$  is not URED.

The author thanks the referee for helpful suggestions.

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Received April 7, 1976 and in revised form April 11, 1977. This paper is a revised version of part of the author's Ph. D. thesis written at the University of Illinois under the supervision of Professor M. M. Day.

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