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THE R -BOREL STRUCTURE ON A CHOQUET SIMPLEX

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The R -Borel structure on a Choquet simplex K is studied. It is shown that the central decomposition and maximal measures coincide, and this is used to improve the well-known theorem that maximal measures are pseudo-concentrated on the extreme boundary.

1. Introduction. Let K denote a compact convex subset of a locally convex Hausdorff topological vector space, and denote by $A^b(K)$ the Banach space of bounded real valued affine functions on K . The symbols $A(K)$, $A(K)^m$, and $A(K)_m$ denote respectively the sets of continuous, lower semi-continuous and upper semi-continuous functions in $A^b(K)$. Set $S(K) = A(K)^m + A(K)_m$, and let $S(K)^\mu$ be the smallest subset of $A^b(K)$ containing $S(K)$ and closed under the formation of pointwise limits of uniformly bounded monotone sequences. $S(K)^\mu$ is a Banach space, the following properties of which were obtained in [6].

THEOREM 1.1. *Consider $a \in S(K)^\mu$.*

- (i) $\|a\| = \|a|_{\partial_e K}\|$.
- (ii) $a \geq 0$ if and only if $a|_{\partial_e K} \geq 0$.

$S(K)^\mu$ is an order unit space and thus possesses a *centre* $Z(S(K)^\mu)$ defined in terms of order bounded operators [2]. However a more convenient formulation was obtained in [6]: $z \in S(K)^\mu$ is said to be a *central element* if and only if to each $a \in S(K)^\mu$ there corresponds $b \in S(K)^\mu$ satisfying $b(x) = a(x)z(x)$ for all $x \in \partial_e K$. $Z(S(K)^\mu)$ is then seen to be an algebra and a lattice with operations defined pointwise on $\partial_e K$.

Let π^s be the map which restricts elements of $S(K)^\mu$ to functions on $\partial_e K$. The following representation of $Z(S(K)^\mu)$ as an algebra of measurable functions on $\partial_e K$ was proved in [6]. The statement has been modified slightly to suit the purpose of this note.

THEOREM 1.2. *There exists a σ -algebra \mathcal{R} of subsets of $\partial_e K$ such that π^s is an isometric algebraic isomorphism from $Z(S(K)^\mu)$ onto the algebra $F(\partial_e K, \mathcal{R})$ of bounded \mathcal{R} -measurable functions on $\partial_e K$. There exists a unique affine map $x \rightarrow \nu_x$ from K into the set of probability measures on \mathcal{R} satisfying, for $z \in Z(S(K)^\mu)$,*

$$z(x) = \int_{\partial_e K} \pi^s(z) d\nu_x .$$

The σ -algebra \mathcal{R} is termed the *R-Borel structure* on $\partial_e K$, while the measures ν_x constitute the *central representation* of points of K with respect to $Z(S(K)^\mu)$.

When K is a simplex, $S(K)^\mu$ is a lattice [5], and hence equal to its centre [2]. In this case \mathcal{R} is large, and it is the purpose of this note to investigate further the *R-Borel structure* in this special situation. In particular the central decomposition measures ν_x are related to the unique maximal representing measures μ_x , and an extension is obtained of the well-known theorem that maximal representing measures vanish on every Baire set disjoint from $\partial_e K$.

[4] contains further information on Borel structures on compact convex sets, while [2] is the standard reference for convexity theory.

2. The main theorems. For the remainder of this note K is assumed to be a simplex. $S(K)^\mu$ is a lattice [5], and it follows, by the methods of [1], that the lattice operations are given, for $f, g \in S(K)^\mu$, $x \in K$, by

$$f \vee g(x) = \int_K f \vee g d\mu_x$$

$$f \wedge g(x) = \int_K f \wedge g d\mu_x.$$

Let \mathcal{B}_0 and \mathcal{B} denote the Baire and Borel structures on K respectively, and let their restrictions to $\partial_e K$ be denoted by $\overline{\mathcal{B}}_0$ and $\overline{\mathcal{B}}$.

THEOREM 2.1. $\overline{\mathcal{B}}_0 \subset \mathcal{R} \subset \overline{\mathcal{B}}$.

Proof. The second inclusion is clear since every function in $S(K)^\mu$ is Borel measurable.

Let E be an arbitrary compact G_δ subset of K . Then there exists a uniformly bounded decreasing sequence of continuous functions $(f_n)_{n=1}^\infty$ with pointwise limit χ_E . By [2, II.3.14] and Theorem 1.1, there exists a uniformly bounded decreasing sequence $(g_n)_{n=1}^\infty$ from $S(K)^\mu$ such that f_n and g_n agree on $\partial_e K$. This sequence has pointwise limit $g \in S(K)^\mu$, and clearly χ_E and g agree on $\partial_e K$. Hence $E \cap \partial_e K = g^{-1}(1) \cap \partial_e K \in \mathcal{R}$.

It is now clear that \mathcal{R} contains $\overline{\mathcal{B}}_0$.

LEMMA 2.2. *Let f and g be nonnegative functions in $S(K)^\mu$ such that $f^{-1}(0) \cap \partial_e K = g^{-1}(0) \cap \partial_e K$. Then $f^{-1}(0) = g^{-1}(0)$.*

Proof. Let $E = f^{-1}(0) \cap \partial_e K$. Then $E \in \mathcal{R}$, and so let h be the

unique element of $S(K)^\mu$ with the property that

$$E = h^{-1}(0) \cap \partial_e K, \quad E^c = h^{-1}(1) \cap \partial_e K.$$

For $n \geq 1$, $E_n = \{x \in \partial_e K : f(x) \geq 1/n\}$ is an element of \mathcal{A} . Let $h_n \in S(K)^\mu$ be the corresponding function for which

$$E_n = h_n^{-1}(1) \cap \partial_e K, \quad E_n^c = h_n^{-1}(0) \cap \partial_e K.$$

By Theorem 1.1, $(h_n)_{n=1}^\infty$ is a uniformly bounded increasing sequence from $S(K)^\mu$ with pointwise limit h . Hence $h^{-1}(0) = \bigcap_{n=1}^\infty h_n^{-1}(0)$. Now for each $n \geq 1$, $nf \geq h_n$, and thus $f^{-1}(0)$ is contained in $h_n^{-1}(0)$. It follows that $f^{-1}(0)$ is contained in $h^{-1}(0)$. Conversely, $f \leq \|f\|h$ and so $h^{-1}(0)$ is contained in $f^{-1}(0)$.

In conclusion $f^{-1}(0) = h^{-1}(0)$, and the fact that $g^{-1}(0) = h^{-1}(0)$ is established by the same reasoning.

COROLLARY 2.3. *If $f \in S(K)^\mu$ attains its lower bound then it does so at an extreme point.*

Proof. It may be assumed that $f \geq 0$ and that $f^{-1}(0)$ is nonempty. To derive a contradiction, suppose that $f^{-1}(0)$ does not contain an extreme point. Now apply Lemma 2.2 to the functions f and 1_K .

Let \mathcal{S} denote the smallest σ -algebra of subsets of K with respect to which every function in $S(K)^\mu$ is measurable. [2, I.1.1, I.1.3] together imply that every continuous function is \mathcal{S} -measurable. Since every function in $S(K)^\mu$ is Borel measurable, it follows that

$$\mathcal{B}_0 \subset \mathcal{S} \subset \mathcal{B}.$$

The following theorem relates the maximal representing measures to the central decomposition measures.

THEOREM 2.4. *For each $x \in K$ the maximal representing measure μ_x may be restricted to a measure $\bar{\mu}_x$ on $\mathcal{S} \cap \partial_e K$. $\mathcal{S} \cap \partial_e K = \mathcal{A}$ and $\bar{\mu}_x = \nu_x$.*

Proof. Define an equivalence relation on the algebra of bounded Borel measurable functions on K by setting $f \sim g$ if and only if, for all $x \in K$,

$$\int_K |f - g| d\mu_x = 0.$$

If $f, g \in S(K)^\mu$ then the relations

$$f \vee g \sim f \vee g, \quad f \wedge g \sim f \wedge g$$

are an easy consequence of the fact that functions in $S(K)^\mu$ satisfy the barycentric calculus (see [2]).

Let \mathcal{H} be the set of Borel sets E for which there exists $h \in S(K)^\mu$ such that $\chi_E \sim h$. The proof now proceeds in several stages.

(i) Suppose that $E, F \in \mathcal{H}$ with associated functions $f, g \in S(K)^\mu$ respectively. Then

$$\chi_{E \cap F} = \chi_E \wedge \chi_F \sim f \wedge g \sim f \wedge g \in S(K)^\mu$$

and

$$\chi_{E^c} = 1_E - \chi_E \sim 1_K - f \in S(K)^\mu .$$

Hence $E \cap F$ and E^c are members of \mathcal{H} .

Suppose that $(E_n)_{n=1}^\infty$ is an increasing sequence from \mathcal{H} with associated sequence $(h_n)_{n=1}^\infty$ from $S(K)^\mu$. Theorem 1.1 implies that the latter sequence is uniformly bounded and increasing with pointwise limit $h \in S(K)^\mu$. Let $E = \bigcup_{n=1}^\infty E_n$. Then the dominated convergence theorem implies that $\chi_E \sim h$. Since $K \in \mathcal{H}$, it follows that \mathcal{H} is a σ -algebra.

(ii) Suppose that f is a nonnegative element of $S(K)^\mu$, and write $E = f^{-1}(0)$. Then $E \cap \partial_e K \in \mathcal{B}$, and there exists a unique element $g \in S(K)^\mu$ such that

$$g^{-1}(1) \cap \partial_e K = E \cap \partial_e K , \quad g^{-1}(0) \cap \partial_e K = E^c \cap \partial_e K .$$

By Lemma 2.2, $E = g^{-1}(1)$ and hence $g \geq \chi_E$. If $x \in E$ then

$$\int_K (1_K - g) d\mu_x = 1 - g(x) = 0 .$$

$g \leq 1_K$ and thus μ_x is supported by $g^{-1}(1)$. Hence $\mu_x(E) = 1$. Similar arguments applied to 1_K and $1_K - g$ yield $\mu_x(E) = 0$ for $x \in g^{-1}(0)$. $g^{-1}(0)$ and $g^{-1}(1)$ are complementary split faces [1, 3]. Each $x \in K$ then has decomposition $g(x)y + (1 - g(x))z$ where $y \in g^{-1}(1)$, and $z \in g^{-1}(0)$, and

$$\int_K \chi_E d\mu_x = \mu_x(E) = g(x)\mu_y(E) + (1 - g(x))\mu_z(E) = g(x) = \int_K g(x) d\mu_x .$$

Since $g \geq \chi_E$ it follows that $\chi_E \sim g$ and $E \in \mathcal{H}$.

(iii) Suppose $f \in S(K)^\mu$ and $\alpha \in \mathbf{R}$. Write $g = f \wedge \alpha 1_K$ and $h = f \wedge \alpha 1_K$, and denote $g^{-1}(\alpha)$ and $h^{-1}(\alpha)$ by G and H respectively. For $x \in K$,

$$\int_K (g - h) d\mu_x = 0 .$$

$g \geq h$ and thus the set on which $g > h$ has μ_x -measure zero. This

set contains $G \setminus H$, hence $\mu_x(G) = \mu_x(H)$, and it follows that $\chi_H \sim \chi_G$. However by (ii), $H \in \mathcal{H}$, and therefore $G \in \mathcal{H}$.

$G = \{x \in K: f(x) \geq \alpha\}$ and, since f and α were arbitrary, every function in $S(K)^\mu$ is \mathcal{H} -measurable.

(iv) Suppose that $E \in \mathcal{H}$ and that E is disjoint from $\partial_e K$. If $\chi_E \sim h \in S(K)^\mu$ then $h|_{\partial_e K} = 0$. By Theorem 1.1 $h = 0$, and, for all $x \in K$, $\mu_x(E) = 0$.

(v) From (iii), \mathcal{H} contains \mathcal{S} , and it is not difficult to show that $\mathcal{S} \cap \partial_e K = \mathcal{B}$. Thus, by the methods of [2, I.4.13, I.4.14], each μ_x may be restricted to a measure $\bar{\mu}_x$ on \mathcal{B} satisfying, for $f \in S(K)^\mu$,

$$f(x) = \int_{\partial_e K} f d\bar{\mu}_x .$$

The uniqueness of the central decomposition measures (Theorem 1.2) now implies that, for each $x \in K$, $\bar{\mu}_x = \nu_x$. The proof is complete.

REMARKS. (i) Since $\mathcal{B}_0 \subset \mathcal{S} \subset \mathcal{B}$, it is clear that Theorem 2.4 extends the result that maximal measures are pseudo-concentrated on $\partial_e K$. For a nonmetrizable Bauer simplex it is clear that \mathcal{S} strictly contains \mathcal{B}_0 . On the other hand there exists a simplex K and a maximal measure μ such that $\partial_e K \in \mathcal{B}$ and $\mu(\partial_e K) = 0$. In this case \mathcal{S} and \mathcal{B} cannot be equal. This example is discussed in [2, II.3.17].

(ii) The set of step functions is dense in $F(K, \mathcal{S})$ and hence any \mathcal{S} -measurable function is equivalent to an element of $S(K)^\mu$. In particular if $f, g \in S(K)^\mu$ then fg is \mathcal{S} -measurable, and there exists $h \in S(K)^\mu$ such that $fg \sim h$. Denote the product of f and g in $S(K)^\mu$ by $f \circ g$. Then, for $x \in \partial_e K$, $f \circ g(x) = f(x)g(x)$, and thus h and $f \circ g$ agree on $\partial_e K$. It follows from Theorem 1.1 that $h = f \circ g$, and thus, for $x \in K$,

$$f \circ g(x) = \int_K fg d\mu_x .$$

Direct approaches do not seem to yield this formula.

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Pacific Journal of Mathematics

Vol. 73, No. 1

March, 1977

Thomas Robert Berger, <i>Hall-Higman type theorems. V</i>	1
Frank Peter Anthony Cass and Billy E. Rhoades, <i>Mercerian theorems via spectral theory</i>	63
Morris Leroy Eaton and Michael David Perlman, <i>Generating $O(n)$ with reflections</i>	73
Frank John Forelli, Jr., <i>A necessary condition on the extreme points of a class of holomorphic functions</i>	81
Melvin F. Janowitz, <i>Complemented congruences on complemented lattices</i>	87
Maria M. Klawe, <i>Semidirect product of semigroups in relation to amenability, cancellation properties, and strong $F\phi$ lner conditions</i>	91
Theodore Willis Laetsch, <i>Normal cones, barrier cones, and the "spherical image" of convex surfaces in locally convex spaces</i>	107
Chao-Chu Liang, <i>Involutions fixing codimension two knots</i>	125
Joyce Longman, <i>On generalizations of alternative algebras</i>	131
Giancarlo Mauceri, <i>Square integrable representations and the Fourier algebra of a unimodular group</i>	143
J. Marshall Osborn, <i>Lie algebras with descending chain condition</i>	155
John Robert Quine, Jr., <i>Tangent winding numbers and branched mappings</i>	161
Louis Jackson Ratliff, Jr. and David Eugene Rush, <i>Notes on ideal covers and associated primes</i>	169
H. B. Reiter and N. Stavrakas, <i>On the compactness of the hyperspace of faces</i>	193
Walter Roth, <i>A general Rudin-Carlson theorem in Banach-spaces</i>	197
Mark Andrew Smith, <i>Products of Banach spaces that are uniformly rotund in every direction</i>	215
Roger R. Smith, <i>The R-Borel structure on a Choquet simplex</i>	221
Gerald Stoller, <i>The convergence-preserving rearrangements of real infinite series</i>	227
Graham H. Toomer, <i>Generalized homotopy excision theorems modulo a Serre class of nilpotent groups</i>	233
Norris Freeman Weaver, <i>Dehn's construction and the Poincaré conjecture</i>	247
Steven Howard Weintraub, <i>Topological realization of equivariant intersection forms</i>	257