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DEHN'S CONSTRUCTION AND THE POINCARÉ CONJECTURE

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M. Dehn used a type of homology preserving surgery on S^3 to produce an infinite family of irreducible homology 3-spheres. We apply Dehn's construction to arbitrary 3-manifolds, give groups invariant under Dehn's construction, give a reduction of the Poincaré conjecture, give a nontrivial example of links in S^3 with homeomorphic exteriors, and give a result concerning knots with property P.

1. Introduction and statement of main results.

Dehn's construction can be iterated, which gives rise to the notion of Dehn equivalence between 3-manifolds [see §2]. Dehn equivalent manifolds not only have the same homology, but we also have the following invariant groups.

THEOREM A. *If M and N are orientable and Dehn equivalent, then $\pi_1 M / (\pi_1 M)_3$ and $\pi_1 N / (\pi_1 N)_3$ are isomorphic, where $(\pi_1 M)_3$ and $(\pi_1 N)_3$ are the third lower central subgroups.*

It follows from Theorem A that manifolds with the same homology need not be Dehn equivalent, e.g., $S^1 \times S^1 \times S^1$ and the connected sum $(S^1 \times S^2) \# (S^1 \times S^2) \# (S^1 \times S^2)$.

The notion of Dehn equivalence can be stabilized as follows. Let M', N' be 3-manifolds. Attach finitely many 1-handles [thickend 1-cells] to $\partial M'$ and $\partial N'$ to obtain 3-manifolds M and N . We say that M' and N' are *stably Dehn equivalent* if there exist such manifolds M and N which are Dehn equivalent. This is an equivalence relation. Recall that a homotopy 3-disc is a compact, simply connected 3-manifold whose boundary is homeomorphic with S^2 . Our main result is the following.

THEOREM B. *Every homotopy 3-disc is stably Dehn equivalent with D^3 .*

It follows from Theorem B that every homotopy 3-sphere [i.e., simply connected, closed 3-manifold] is a connected sum factor of a manifold which is Dehn equivalent with S^3 .

Theorem B is a reduction of the Poincaré conjecture, i.e., the Poincaré conjecture can be proved by showing that D^3 is the only homotopy 3-disc which is stably Dehn equivalent with D^3 .

One can construct 3-manifolds as follows, (i) successively perform Dehn constructions on a solid torus T_g of genus $g \geq 0$ a finite number, $k \geq 1$, of times to produce a manifold T'_g , and (ii) remove 1-handles, g in number, from T'_g to produce a manifold M . According to Theorem B, if the Poincaré conjecture is false, then a counterexample M , i.e., a homotopy 3-disc different from D^3 , can be produced by this construction, which we call the *SD*-construction.

Attempting to construct a nontrivial homotopy 3-disc by the *SD*-construction in the case $g = 0, k = 1$ [or in the case $g = 1, k = 1$] is the same as trying to produce a nontrivial homotopy 3-sphere by doing a single Dehn construction on S^3 . The conjecture in knot theory that all knots have property *P* is the conjecture that a single nontrivial Dehn construction on S^3 never produces a simply connected manifold. [See § 2 for the concept of a trivial Dehn construction.] This conjecture has been studied by several authors, see (1), (2), (4), (6), (7) for example, and implies that two knots in S^3 with homeomorphic exteriors have the same type. [An exterior is the complement of a open tubular or regular neighborhood.]

We consider the case $g = 0, k = 2$ of the *SD*-construction. Let $L = (l_1, l_2)$ be a link in S^3 with two components. Successively perform Dehn constructions along l_1 and l_2 to obtain a manifold M . If either Dehn construction is trivial, then up to diffeomorphism, M can be produced by a single Dehn construction on S^3 , so assume both Dehn constructions to be nontrivial. If l_1 and l_2 have linking number zero and $l_1 \cup l_2$ is the boundary of an embedded copy of $S^1 \times I$ in S^3 , then again M can be produced by a single Dehn construction, so we eliminate this case as well and call the remaining case the nontrivial case. The obvious conjecture is that M is never simply connected in the nontrivial case. A counterexample to this conjecture is given as Example 3 of § 4.

The counterexample above has the following significance in the theory of links. It is well known that if a link in S^3 has an unknotted component, then the exterior of the link may be homeomorphic with the exterior of a link of different type [see Example 1 of § 4]. Let T_1, T_2 be closed tubular neighborhoods of two components, say l_1 and l_2 , of a link with no unknotted components. If there exists an embedding

$$(S^1 \times I, S^1 \times 0, S^1 \times 1) \longrightarrow (S^3 \setminus \text{int}(T_1 \cup T_2), \partial T_1, \partial T_2),$$

then again the exterior of L may be homeomorphic with the exterior of a link of different type [see Example 2 of § 4]. However, in this case one of the nontrivial knots l_1, l_2 must be a cable about the other, and consequently one of l_1, l_2 is a companion of the other. Example 3 of § 4 proves the following.

THEOREM C. *There exists a [tame] link in S^3 such that (i) the exterior is homeomorphic with the exterior of another link of different type, and (ii) no component is a companion of another component.*

The next four sections deal respectively with the definition of Dehn equivalence and the proofs of Theorems A, C, and B, together with related results. It is possible to obtain the same homology sphere from S^3 by Dehn constructions along different knots. In the final section we give an infinite class of such knots and verify that most of the homology spheres involved are not simply connected, which generalizes a result of (2) and (4).

In the following we work in the category of smooth manifolds and maps since this is convenient for the transversality argument in § 5. Everything we do has a *PL* analog however.

2. Dehn's construction. Let M be a 3-manifold and let $*$ be an element of S^1 . Let $f: S^1 \times D^2 \rightarrow \text{int } M = M \setminus \partial M$ be an embedding such that $f(S^1 \times *)$ is null-homologous in $M \setminus \text{int } f(S^1 \times D^2)$. Let $h: S^1 \times \partial D^2 \rightarrow f(S^1 \times \partial D^2)$ be a diffeomorphism such that $h(S^1 \times *) = f(S^1 \times *)$. Glue $S^1 \times D^2$ to $M \setminus \text{int } f(S^1 \times D^2)$ by h . The resulting manifold M_1 has been obtained from M by *Dehn's construction* along $f(S^1 \times 0)$. The construction is *trivial* if either $f(S^1 \times 0)$ bounds an embedded disc in M or $h(* \times \partial D^2)$ is null-homotopic in $f(S^1 \times D^2)$. A trivial Dehn construction does not change the diffeomorphism class of M . If $M_0, M_1, \dots, M_n, n \geq 0$ are such that M_{i+1} is obtained from M_i by Dehn's construction, then we say that any two manifolds diffeomorphic with M_0 and M_n respectively are *Dehn equivalent*. This is an equivalence relation. It is easily proved that there is a commutative square as follows.

$$\begin{array}{ccc} H_*(\partial M_0) & \xrightarrow{\cong} & H_*(\partial M_n) \\ \downarrow & & \downarrow \\ H_*(M_0) & \xrightarrow{\cong} & H_*(M_n) \end{array}$$

3. Proof of Theorem A and related results. Let M, M_1, f, h be as in § 2 and assume M to be orientable. Let F be a compact surface in $M \setminus \text{int } f(S^1 \times D^2)$ with $\partial F = f(S^1 \times *)$. Let P be a closed ε -neighborhood of $F \cup f(S^1 \times D^2)$ in M . Then $P \cong F \times I$ after smoothing, and $\pi_1 \partial P$ has generators $a_i, b_i, a'_i, b'_i, 1 \leq i \leq g$ and the defining relation

$$[a_1, b_1] \cdots [a_g, b_g] = [a'_1, b'_1] \cdots [a'_g, b'_g]$$

where g is the genus of F . [We use the definition $[x, y] = xyx^{-1}y^{-1}$.] The Dehn construction producing M_1 from M is simultaneously a Dehn construction producing a manifold P_1 from P . Presentations of $\pi_1 P$ and $\pi_1 P_1$ can be obtained from the presentation of $\pi_1 \partial P$ by adding the relations (i) $a_i = a'_i$, $b_i = b'_i$, and (ii) $d^n a_i d^{-n} = a'_i$, $d^n b_i d^{-n} = b'_i$ respectively, where d denotes $[a_1, b_1] \cdots [a_g, b_g]$ and the integer n is, up to sign, the intersection number of $f(* \times S^1)$ and $h(* \times S^1)$ in $f(S^1 \times \partial D^2)$. Given any group G we denote G/G_{c+1} , $c \geq 0$ by $V_c(G)$. We have a commutative diagram

$$\begin{array}{ccc} & V_2(\pi_1 \partial P) & \\ \swarrow & & \searrow \\ V_2(\pi_1 P) & \xrightarrow{\cong} & V_2(\pi_1 P_1) \end{array}$$

in which the slanted arrows are inclusion induced. Note that V_c is a functor. The image of this functor is the group variety V_c of groups with nil- c . The image of a push out diagram under the functor V_c is a push out diagram in the category V_c . The Seifert-VanKampen theorem applied to $M = P \cup (M \setminus \text{int } P)$, $M_1 = P_1 \cup (M \setminus \text{int } P)$ describes $\pi_1 M$ and $\pi_1 M_1$ as push outs. By the commutative triangle above these push out diagrams become isomorphic after the functor V_2 is applied. Thus $V_2(\pi_1 M) \cong V_2(\pi_1 M_1)$ and Theorem A is proved.

It follows from this proof that the diagram of § 2 also commutes when H_* is replaced by $V_2 \pi_1$.

We show that $V_3(\pi_1 M)$ is not necessarily invariant under Dehn equivalence by giving an example. Let P, P_1 be as in the proof above, but choose P, P_1 so that $g, n = 1$. Let N be the double of P , and glue ∂P to ∂P_1 by the identity map to obtain $N_1 = P \cup P_1$. Then N and N_1 are Dehn equivalent, but $V_3(\pi_1 N)$ is nilpotent of class three while $V_3(\pi_1 N_1)$ is nilpotent of class two.

If F , as in the proof of Theorem A, is chosen to have the smallest possible genus g , then we say that the Dehn construction has *genus* g . Let E be the group variety defined by the Engel law $[x, x, y] = 1$. [We define $[x, y, z]$ to be $[x, [y, z]]$.] The proof of Theorem A with slight modification proves the following.

THEOREM 1. *If N is obtained from M by a Dehn construction of genus one, then $E(\pi_1 M) \cong E(\pi_1 N)$.*

The groups in E have the following structure.

THEOREM 2. *Let S be a set which generates a group G . Then $E(G)$ is isomorphic with $V_3(G)$ modulo the relations $[a, b, c][b, a, c] = 1$, $a, b, c \in S$. Also $E(G)$ is an extension of an abelian group of period*

3, namely $E(G)_3$, by $V_2(G)$.

The proof of Theorem 2 has much in common with the proof of the structure theorem for Burnside groups of period three (5, pp. 320-324), so we omit the proof.

It follows from Theorem 2 that if G has a given finite presentation, then $E(G)$ has a solvable word problem.

The group $E(\pi_1 M)$ is not necessarily invariant under Dehn equivalence. This is shown by manifolds N, N_1 constructed as above, but in the case $g \geq 2, n = 1$ rather than the case $g, n = 1$.

4. **Proof of Theorem C.** Let M, M_1, f, h be as in § 2. Orient $S^1 \times \partial D^2$ and $f(S^1 \times \partial D^2)$ as the boundaries of $S^1 \times D^2$ and $f(S^1 \times D^2)$ respectively. Assume that M is oriented and that f, h are orientation preserving. Then M_1 has a canonical orientation, and we say that the Dehn construction has *type* (l, n) where l is the unoriented curve $f(S^1 \times 0)$ and n is the intersection number of $f(* \times \partial D^2)$ with $h(* \times \partial D^2)$ in $f(S^1 \times \partial D^2)$. [Thus if $n = 1$, the oriented curves $h(* \times \partial D^2)$ and $f(S^1 \times 0)$ are isotopic in $f(S^1 \times D^2)$.] Up to orientation preserving diffeomorphism M_1 is determined by n and the isotopy class of l in M .

Let K be the link of Figure 1. Let k_1 be the small unknotted component, let k_2 be the component which is a figure eight knot, and let k_3 be the remaining unknotted component. The example which proves Theorem C will be constructed in three steps each of which is an example.

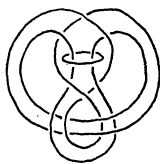


FIGURE 1

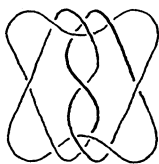


FIGURE 2

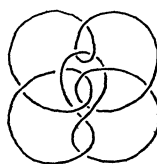


FIGURE 3

EXAMPLE 1. A Dehn construction of type $(k_1, 1)$ on S^3 has genus zero and changes K into a link K' whose components k'_2 and k'_3 are parallel figure eight knots. Thus K and K' have different types and diffeomorphic exteriors. Note that the Dehn construction $(k_1, 1)$ has genus 1 when considered to be a Dehn construction on the exterior of k_3 , and it changes this exterior into the exterior of k'_3 .

EXAMPLE 2. Let $n \neq 0$ be an integer. Do Dehn constructions of types (k'_2, n) and $(k'_3, -n)$ simultaneously to change K' into a link K'' . [These two constructions can also be performed successively,

but it is conceptually and notationally convenient to perform such constructions simultaneously.] The components k_2'' and k_3'' are figure eight knots while k_1'' has Alexander polynomial $nt^2 + (1 - 2n)t + n$. Thus K, K', K'' are of different types, but their exteriors are diffeomorphic. [Note that if we tie a small knot in k_1' before doing the construction of K'' , then we obtain links with different types, no unknotted components, and diffeomorphic exteriors.]

EXAMPLE 3. A Dehn construction of type $(k_3, -n)$ changes K into a link with components l_1, l_2, l_3 . Delete l_3 to obtain L . The components of L have polynomials $nt^2 + (1 - 2n)t + n$ and $t^2 - 3t + 1$ respectively. If $n \neq -1$, then neither polynomial divides the other, so neither component of L is a companion knot of the other component. Delete k_3'' from K'' to obtain L' whose components have polynomials $nt^2 + (1 - 2n)t + n$ and $t^2 - 3t + 1$ respectively. Neither component of L' is a companion of the other if $n \neq -1$. By the construction of L and L' we can transform L into L' by successively doing Dehn constructions of types (l_3, n) and $(k_1, 1)$ followed by simultaneously doing the constructions (k_2', n) and $(k_3', -n)$. The Dehn constructions (l_3, n) and $(k_3', -n)$ cancel each other out, so the simultaneous constructions $(l_1, 1)$ and (l_2, n) transform L into L' . Consequently L and L' have diffeomorphic exteriors. It remains to show that L and L' are of different types. This is not easy [even though l_1, l_2, l_1', l_2' all seem to have different types], so we give a proof only in the case $n = 1$. In the case $n = 1$ the components of L are a trefoil knot and the knot of Figure 2 while l_1', l_2' are respectively the knot of Figure 3 and a figure eight knot. We can distinguish l_1 from l_1' as follows. Let $G_i, i = 0, 1$ be the knot groups of l_i, l_1' respectively. Let $m_i \in G_i$ be a meridian element. Let H_i be G_i modulo the relation $m_i^2 = 1$. Note that H_i does not depend on the choice of m_i . Let N_i be the kernel of the canonical epimorphism $H_i \rightarrow Z_2$. Let A_i be the abelianization of the commutator subgroup of N_i . Present A_i by the Reidemeister-Schreier algorithm to show that A_0 is trivial while $A_1 \cong Z_8 \oplus Z_8$. Thus l_1, l_1' have different types and Theorem C is proved.

5. Proof of Theorem B. We prove the following Theorem 3 which clearly implies Theorem B.

THEOREM 3. *Let M be a compact, connected, and orientable 3-manifold. If ∂M is connected and $\pi_1(M, \partial M)$ is trivial, then M is stably Dehn equivalent with D^3 . The stable equivalence can be taken to only use Dehn constructions of genus one.*

Theorem 2 is proved by using link homotopy in a relative case. A *relative link* is an embedding $(U, \partial U) \rightarrow (M, \partial M)$ where U is a finite disjoint union of closed intervals. A *link homotopy* is a homotopy $H: (U, \partial U) \times I \rightarrow (M, \partial M)$ between relative links such that distinct components of U do not intersect each other during the homotopy, but self-intersections of components are allowed. By transversality theory we can approximate H by a link homotopy H' such that (i) $h'_0 = h_0$, $h'_1 = h_1$, (ii) each h'_t is an immersion [meeting ∂M transversely] with at most one self-intersection, and (iii) the induced map

$$(U \times U \setminus \partial U) \times I \longrightarrow M \times M$$

is transverse to the diagonal δM . In the following we will only use link homotopies with the properties (ii) and (iii), and we assume M to be as in Theorem 3. Let H be a link homotopy. We claim that the exteriors of h_0 and h_1 are Dehn equivalent. Note that h_t has a self-intersection for only finitely many values of t . The exterior of the link just before a self-intersection can be changed into the exterior just after the self-intersection by a Dehn construction of genus one which is simultaneously a Dehn construction of genus zero on M . [The situation is essentially that of the Dehn construction of Example 1 of § 4 which changes the exterior of k_s into the exterior of k'_s .] The claim is proved. Now we claim that any relative link in M is link homotopic with an isotopy trivial relative link. [A relative link is isotopy trivial if (i) the components lie in disjoint open sets each of which is diffeomorphic with $\mathbb{R}^2 \times [0, \infty)$, and (ii) in each copy of $\mathbb{R}^2 \times [0, \infty)$ the link component is given by $x = 0$, $y^2 + z^2 = 1$, $z \geq 0$.] In order to prove the claim by induction we temporarily drop the assumption that M be compact. Clearly the claim is true if U is connected. Assume the claim to be true when U has $n \geq 1$ components and consider the case $n + 1$. We can link homotop one component so that it becomes isotopy trivial. Extend this link homotopy to become a link homotopy of the entire link. Remove the isotopy trivial component from M to produce a manifold, say $M \setminus l_1$. By inductive hypothesis we can link homotop the remaining components in $M \setminus l_1$ to become an isotopy trivial link. The entire link is now isotopy trivial in M and the claim follows. The two claims prove Theorem 2 as follows. The compact manifold M has some handlebody decomposition

$$M \cong T_g \cup h_1 \cup \cdots \cup h_t, \quad 0 \leq t \leq g, \quad g \geq 0$$

where T_g is a solid torus of genus g and the h_i 's are disjoint 2-handles, i.e., disjoint thickened 2-cells attached to ∂T_g . [Sometimes

2-handles on a 3-manifold are called pillboxes.] The union $h_1 \cup \dots \cup h_t$ is a closed tubular neighborhood of a relative link. This relative link can be link homotoped to become isotopy trivial. Hence $M \setminus \text{int } T$, where T is a closed tubular neighborhood of an isotopy trivial link with t components, is Dehn equivalent with T_g . Since M can be obtained from $M \setminus \text{int } T$ by removing 1-handles, the theorem is proved.

6. The manifolds $B(p, q, r)$. Many examples of diffeomorphic manifolds obtained by Dehn constructions along different knots can be given as follows. Select an order l_1, l_2, l_3 for the components of a Borromean link. Let $B(p, q, r)$, p, q, r any integers, be the manifold obtained from S^3 by simultaneously doing Dehn constructions of type (l_1, p) , (l_2, q) , and (l_3, r) . Diffeomorphisms of S^3 which map the Borromean link onto itself induce orientation preserving diffeomorphisms

$$(*) \quad \begin{aligned} B(\pi(p), \pi(q), \pi(r)) &\cong B(p, q, r) \\ B(-p, -q, -r) &\cong -B(p, q, r) \end{aligned}$$

where π is a permutation of p, q, r and $-B(p, q, r)$ is $B(p, q, r)$ with reversed orientation. We can obtain $B(p, q, r)$ by first doing the (l_1, p) and (l_2, q) constructions to change l_3 into a knot $k(p, q)$ in S^3 and then doing the Dehn construction of type $(k(p, q), r)$. By (*) the Dehn constructions $(k(p, q), r)$, $(k(q, r), p)$ and $(k(r, p), q)$ all produce the same manifold from S^3 . The knots $k(p, q)$, $k(q, r)$, $k(r, p)$ are different in general since $k(p, q)$ has Alexander polynomial $pqt^2 + (1 - 2pq)t + pq$.

The question of whether any of the $B(p, q, r)$'s with $p, q, r \neq 0$ are simply connected naturally arises. The following theorem is a partial answer.

THEOREM 4. *The group $\pi_1 B(p, q, r)$, $p, q, r \neq 0$ is nontrivial if none of the absolute values $|p|, |q|, |r|$ is 2 or 3.*

Proof. The knot $k(1, q)$, $q \neq 0$ has property P according to (2) and (4). The theorem follows if any of $|p|, |q|, |r|$ is 1. The remaining case is $|p|, |q|, |r| > 3$, and we can assume $r > 0$. Present $\pi_1 B(p, q, r)$ as

$$\langle a, b, c; a[b^{-1}, c]^p, b[c^{-1}, a]^q, c[a^{-1}, b]^r \rangle.$$

Consider the group

$$G = \langle a, b, c; a = [b^{-1}, c], c = [a^{-1}, b] \rangle.$$

After the substitutions $b = a^{-1}d^{-1}$ and $a = e^{-1}d$ we obtain

$$G = \langle d, e; d^3 = e^3 \rangle .$$

Add the relation $d^3 = 1$ to G to obtain a group in which the relations of $\pi_1 B(p, q, r)$ become $(de^{-1})^{3p+1}$, $(de)^{3q+1}$, and $[d, e]^{r+1}$. Thus $\pi_1 B(p, q, r)$ has the group

$$\langle d, e; d^3, e^3, (de^{-1})^{3p+1}, (de)^{3q+1}, [d, e]^{r+1} \rangle$$

as a quotient. By Theorem 1 of [3] this group is nontrivial when $|p|, |q|, r > 3$, so we are done.

Finally it may be noted that the $B(1, 1, r)$'s with $r \neq 0$ are the irreducible homology 3-spheres constructed by Dehn. It might be asked if all of the $B(p, q, r)$'s are irreducible.

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