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COMPUTATION OF THE SURGERY OBSTRUCTION GROUPS
 $L_{4k}(\mathbf{1}; \mathbb{Z}_p)$

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The $4k$ -dimensional simply connected surgery obstruction group with coefficients \mathbf{Z}_P (i.e., the group of nonsingular even quadratic forms over \mathbf{Z}_P) is computed in terms of the classical Witt group and a Gauss sum invariant.

1. Introduction. Let $L_{4k}(1; \mathbf{Z}_P)$ be the simply connected surgery obstruction group, with coefficient $\mathbf{Z}_P = \mathbf{Z}[1/p: p \in P]$, in dimension $4k$, of [1]. By definition, this is the Witt group of even, nonsingular quadratic forms over the ring \mathbf{Z}_P . We compute $L_{4k}(1; \mathbf{Z}_P)$ in terms of the classical Witt group $W(\mathbf{Z}_P)$ ([4]).

Let $\gamma_p: W(\mathbf{Q}_p) \rightarrow \mathcal{U}$ denote the " p -primary Gauss sum" character of [4], Appendix 4, where $\mathcal{U} \subset \mathbf{C}^*$ is the multiplicative group of roots of unity. Define $\Phi_P: W(\mathbf{Z}_P) \rightarrow \mathbf{Z}/8\mathbf{Z}$ by

$$\exp(2\pi i \Phi_P(q)/8) = \exp(2\pi i \sigma(q)/8) \cdot \prod_{p \in P} (\gamma_p(q \otimes \mathbf{Q}_p))^{-1},$$

where σ is the signature.

THEOREM 1.1. (i) If $2 \in P$, then $L_{4k}(1; \mathbf{Z}_P) = W(\mathbf{Z}_P)$.
 (ii) If $2 \notin P$, then $L_{4k}(1; \mathbf{Z}_P) \cong \ker(\Phi_P)$.

(i) is obvious and the proof of (ii) occupies §2. An explicit description of $\ker(\Phi_P)$, necessary to obtain the ring structure, is given in §3.

The author would like to thank the referee for suggesting the brief statement and proof of Theorem 1.1 found here.

2. The proof of Theorem 1.1. For p an odd prime, let $\beta_p: W(\mathbf{Q}) \rightarrow W(\mathbf{F}_p)$ be the second residue homomorphism (called ∂_p in [4]), and $\beta_2: W(\mathbf{Q}) \rightarrow W(\mathbf{F}_2)$ the 2-adic value of the determinant. Let $\beta = \bigoplus_p \beta_p$. According to [4], $\sigma \oplus \beta: W(\mathbf{Q}) \rightarrow \mathbf{Z} \oplus \bigoplus_p W(\mathbf{F}_p)$ is an isomorphism.

Recall that $W(\mathbf{F}_2) \cong \mathbf{Z}/2\mathbf{Z}$, $W(\mathbf{F}_p) \cong \mathbf{Z}/4\mathbf{Z}$ if $p \equiv 3 \pmod{4}$, generated by $\langle 1 \rangle$, and $W(\mathbf{F}_p) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ if $p \equiv 1 \pmod{4}$, generated by $\langle 1 \rangle$ and $\langle s_p \rangle$, where s_p is some quadratic nonresidue \pmod{p} . Let $\pi_1, \pi_2: W(\mathbf{F}_p) \rightarrow \mathbf{Z}/2\mathbf{Z}$ be the projections, $p \equiv 1 \pmod{4}$. The invariants β_p and γ_p are related by the following lemma.

LEMMA 2.1. Let $[q] \in W(\mathbf{Q})$. Then:

(i) $\gamma_p(q \otimes \mathbf{Q}_p) = (i\varepsilon)^{\beta_p(q)}$, where $\varepsilon = (-1)^{(p+1)/4}$, if $p \equiv 3 \pmod{4}$.

$$(ii) \quad \gamma_p(q \otimes \mathbf{Q}_p) = \begin{cases} (-1)^{\pi_1 \beta(q)} & \text{if } p \equiv 5 \pmod{8} \\ (-1)^{\pi_2 \beta(q)} & \text{if } p \equiv 1 \pmod{8}. \end{cases}$$

Proof. (i) We have $q \otimes \mathbf{Q}_p = n\langle p \rangle + m\langle 1 \rangle$ in $W(\mathbf{Q}_p)$ and $\beta_p(q) = n \pmod{4}$. Therefore $\gamma_p(q \otimes \mathbf{Q}_p) = \gamma_p(\langle p \rangle)^{\beta_p(q)}$. By [4], $\gamma_p(\langle 4p \rangle) = \exp(\pi i(1-p)/4) = i\varepsilon$. (ii) is similar.

Let $\beta_P = \bigoplus_{p \in P} \beta_p: W(\mathbf{Z}_P) \rightarrow \bigoplus_{p \in P} W(\mathbf{F}_p)$. Then we have the following well-known result:

$$\text{LEMMA 2.2.} \quad \sigma \oplus \beta_P: W(\mathbf{Z}_P) \cong \mathbf{Z} \oplus \bigoplus_{p \in P} W(\mathbf{F}_p).$$

The proof is immediate from the localization sequence

$$0 \longrightarrow W(\mathbf{Z}_P) \longrightarrow W(\mathbf{Q}) \longrightarrow \bigoplus_{p \notin P} W(\mathbf{F}_p) \longrightarrow 0$$

of [4], Corollary IV. 3.3.

Proof of Theorem 1.1.(ii). Using the notation of [3], $L_{4k}(1; \mathbf{Z}_P) = \bar{W}(\mathbf{Z}_P)$ and we have the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bar{W}(\mathbf{Z}) & \longrightarrow & \bar{W}(\mathbf{Z}_P) & \xrightarrow{\bar{\beta}_P} & \bar{W}(\mathbf{Z}_P, \mathbf{Z}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow i_* \\ 0 & \longrightarrow & W(\mathbf{Z}) & \longrightarrow & W(\mathbf{Z}_P) & \xrightarrow{\beta_P} & \bigoplus_{p \in P} W(\mathbf{F}_p) \longrightarrow 0 \\ & & \downarrow \sigma_* & & \downarrow \Phi_P & & \\ & & \mathbf{Z}/8\mathbf{Z} & \xrightarrow{=} & \mathbf{Z}/8\mathbf{Z} & & \end{array}$$

Here σ_* is the signature mod(8). The left vertical sequence is exact by [4], the top horizontal sequence by [3] or [5], and the middle horizontal sequence by Lemma 2.2. Furthermore, by [3], i_* is an isomorphism.

We claim that $\bar{W}(\mathbf{Z}_P) = \ker(\Phi_P)$. Clearly $\bar{W}(\mathbf{Z}_P) \subset \ker(\Phi_P)$ by the reciprocity formula of [4]. Suppose $\Phi_P(x) = 0$. Choose $y \in \bar{\beta}_P^{-1} i_*^{-1} \beta_P(x)$. By a diagram chase, $x - y \in W(\mathbf{Z})$ and $\sigma_*(x - y) = 0$. Since $\bar{W}(\mathbf{Z}) = \ker(\sigma_*)$, $x \in \bar{W}(\mathbf{Z}_P)$.

3. The ring structure. The tensor product of even quadratic forms is again even, so $L_{4k}(1; \mathbf{Z}_P)$ has the structure of a commutative ring. Since $\sigma \oplus \beta_P: L_{4k}(1; \mathbf{Z}_P) \rightarrow \mathbf{Z} \oplus \bigoplus_{p \in P} W(\mathbf{F}_p)$ is injective, and $\sigma(q \otimes q') = \sigma(q)\sigma(q')$, it suffices to consider $\beta_p(q \otimes q')$.

Let $\alpha_p: W(\mathbf{Q}) \rightarrow W(\mathbf{F}_p)$ be the first residue homomorphism if $p \neq 2$, and the signature mod(2) if $p = 2$. We have:

PROPOSITION 3.1. $\beta_p(q \otimes q') = \alpha_p(q)\beta_p(q') + \alpha_p(q')\beta_p(q)$.

Proof. First assume $p \neq 2$. Diagonalize q over \mathbf{Q} as $q_0 \otimes \langle p \rangle + q_1$, where q_0, q_1 are diagonal forms with entries prime to p . Similarly write $q' \cong q'_0 \otimes \langle p \rangle + q'_1$. Then $\beta_p(q) = \bar{q}_0$, $\alpha_p(q) = \bar{q}_1$, $\beta_p(q') = \bar{q}'_0$, $\alpha_p(q') = \bar{q}'_1$, where “ $\bar{}$ ” denotes passing to the residue class field of \mathbf{Q}_p , and

$$\begin{aligned} \beta_p(q \otimes q') &= \beta_p(q_0 \otimes q'_0 \otimes \langle p^2 \rangle + q_0 \otimes q'_1 \otimes \langle p \rangle \\ &\quad + q_1 \otimes q'_0 \otimes \langle p \rangle + q_1 \otimes q'_1) \\ &= \bar{q}_0 \otimes \bar{q}'_1 + \bar{q}_1 \otimes \bar{q}'_0. \end{aligned}$$

The case $p = 2$ is an easy determinant argument and left to the reader.

The ring $L_{4k}(1; \mathbf{Z}_p)$ can now be completely determined by the values of the first residues of a set of generators, which we now describe.

Let $(n; x_1(p_1), \dots, x_k(p_k))$ denote the element $y \in W(\mathbf{Z}_p)$ with $\sigma(y) = n$, $\beta_{p_i}(y) = x_i$, $i = 1, \dots, k$, and $\beta_p(y) = 0$ otherwise. By Theorem 1.1 and Lemma 2.1, we have

LEMMA 3.2. *Let $2 \notin P$. Then: $(n; x_1(p_1), \dots, x_k(p_k)) \in L_{4k}(1; \mathbf{Z}_p)$ if and only if*

$$n + \sum_{p_i \equiv 3(4)} (-1)^{(p_i-3)/4} 2x_i + \sum_{p_i \equiv 5(8)} 4\pi_1(x_i) + \sum_{p_i \equiv 1(8)} 4\pi_2(x_i) \equiv 0 \pmod{8}.$$

Generators of $L_{4k}(1; \mathbf{Z}_p)$ are given by the following matrices:

(1) $p = 4k + 3$: (2; $(-1)^{k+1}(p)$) is obtained from the weighted graph

$$\begin{array}{c} \cdot \\ \text{---} \\ \cdot \\ -2 \quad \quad \quad -2(k+1) \end{array}$$

(2) $p = 8k + 5$: (0; $s(p)$) is obtained

$$\begin{array}{c} \cdot \\ \text{---} \\ \cdot \\ -2 \quad \quad \quad 2(2k+1) \end{array};$$

(4; $1(p)$) is obtained from

$$\begin{array}{cccc} \cdot & \text{---} & \cdot & \text{---} & \cdot & \text{---} & \cdot \\ -2 & & -2 & & -2 & & -2(k+1) \end{array}$$

(3) $p = 8k + 1$: (0; $1(p)$) is obtained from

$$\begin{array}{c} \cdot \\ \text{---} \\ \cdot \\ -2 \quad \quad \quad 4k \end{array}$$

In general, it is hard to write down an explicit matrix realizing

$(4; s(p))$. However, by the proof of Theorem IV. 2.1 of [4], a diagonalization can be obtained in a specific case. For example, $(4; s(17))$ is represented by $\langle 51, 3, 1, 1 \rangle$.

Finally, we include the following result on signatures of even forms over \mathbb{Z}_P . Let $a_P = \text{g.c.d.}\{\sigma(x) : x \in L_{4k}(1; \mathbb{Z}_P)\}$

COROLLARY 3.3. $a_P = 1$ (resp. 8) if and only if $2 \in P$ (resp. $P = \phi$). Otherwise, $a_P = 2$ if some $p \in P$ is $3 \pmod{4}$, and $a_P = 4$ if not.

The proof is immediate from Lemma 3.2. This shows that Proposition 2.2. of [6] is incorrect.

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Gerald Arthur Anderson, <i>Computation of the surgery obstruction groups</i> $L_{4k}(1; Z_p)$	1
R. K. Beatson, <i>The degree of monotone approximation</i>	5
Sterling K. Berberian, <i>The character space of the algebra of regulated functions</i>	15
Douglas Michael Campbell and Jack Wayne Lamoreaux, <i>Continua in the plane with limit directions</i>	37
R. J. Duffin, <i>Algorithms for localizing roots of a polynomial and the Pisot Vijayaraghavan numbers</i>	47
Alessandro Figà-Talamanca and Massimo A. Picardello, <i>Functions that operate on the algebra $B_0(G)$</i>	57
John Erik Fornæss, <i>Biholomorphic mappings between weakly pseudoconvex domains</i>	63
Andrzej Granas, Ronald Bernard Guenther and John Walter Lee, <i>On a theorem of S. Bernstein</i>	67
Jerry Grossman, <i>On groups with specified lower central series quotients</i>	83
William H. Julian, Ray Mines, III and Fred Richman, <i>Algebraic numbers, a constructive development</i>	91
Surjit Singh Khurana, <i>A note on Radon-Nikodým theorem for finitely additive measures</i>	103
Garo K. Kiremidjian, <i>A Nash-Moser-type implicit function theorem and nonlinear boundary value problems</i>	105
Ronald Jacob Leach, <i>Coefficient estimates for certain multivalent functions</i>	133
John Alan MacBain, <i>Local and global bifurcation from normal eigenvalues. II</i>	143
James A. MacDougall and Lowell G. Sweet, <i>Three dimensional homogeneous algebras</i>	153
John Rowlay Martin, <i>Fixed point sets of Peano continua</i>	163
R. Daniel Mauldin, <i>The boundedness of the Cantor-Bendixson order of some analytic sets</i>	167
Richard C. Metzler, <i>Uniqueness of extensions of positive linear functions</i>	179
Rodney V. Nillsen, <i>Moment sequences obtained from restricted powers</i>	183
Keiji Nishioka, <i>Transcendental constants over the coefficient fields in differential elliptic function fields</i>	191
Gabriel Michael Miller Obi, <i>An algebraic closed graph theorem</i>	199
Richard Cranston Randell, <i>Quotients of complete intersections by C^* actions</i>	209
Bruce Reznick, <i>Banach spaces which satisfy linear identities</i>	221
Bennett Setzer, <i>Elliptic curves over complex quadratic fields</i>	235
Arne Stray, <i>A scheme for approximating bounded analytic functions on certain subsets of the unit disc</i>	251
Nicholas Th. Varopoulos, <i>A remark on functions of bounded mean oscillation and bounded harmonic functions. Addendum to: "BMO functions and the $\bar{\partial}$-equation"</i>	257
Charles Irvin Vinsonhaler, <i>Torsion free abelian groups quasi-projective over their endomorphism rings. II</i>	261
Thomas R. Wolf, <i>Characters of p'-degree in solvable groups</i>	267
Toshihiko Yamada, <i>Schur indices over the 2-adic field</i>	273