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THE DEGREE OF MONOTONE APPROXIMATION

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Jackson type theorems are obtained for generalized monotone approximation. Let $E_{n,k}(f)$ be the degree of approximation of f by n th degree polynomials with k th derivative nonnegative on $[-1/4, 1/4]$. Then for each $k \geq 2$ there exists an absolute constant D_k , such that for all $f \in C[-1/4, 1/4]$ with k th difference nonnegative on $[-1/4, 1/4]$; $E_{n,k}(f) \leq D_k \omega(f, n^{-1})$. If in addition $f' \in C[-1/4, 1/4]$ then $E_{n,k}(f) \leq D_k n^{-1} \omega(f', n^{-1})$.

Given a function f with nonnegative k th difference on $[-1/4, 1/4]$ (equivalently any finite real interval) it is natural to ask whether Jackson type estimates hold for

$$E_{n,k}(f) \parallel \inf_{\{p \in \Pi_n: p^{(k)}(x) \geq 0, x \in [-1/4, 1/4]\}} \|f - p\| \parallel$$

where the norm is the uniform norm, and Π_n is the space of algebraic polynomials of degree not exceeding n . In the case $k = 1$, Lorentz and Zeller [4] and Lorentz [5] have shown that there exists a constant D_1 such that if f is increasing on $[-1/4, 1/4]$

$$(1) \quad E_{n,1}(f) \leq D_1 \omega(f, n^{-1}), \quad n = 1, 2, \dots,$$

where $\omega(f, \cdot)$ denotes the modulus of continuity of f . If, in addition, $f' \in C[1/4, 14]$ then

$$(2) \quad E_{n,1}(f) \leq D_1 n^{-1} \omega(f', n^{-1}), \quad n = 1, 2, \dots$$

DeVore [2, 3] has given a much simpler proof of the $k = 1$ results. The results of this paper are obtained with similar arguments.

NOTATION. Throughout C_1, C_2, \dots denote positive constants depending on k , but not depending on f, x or $n \geq k$. Whenever it causes no confusion, $\|\cdot\|_\beta$ denotes $\|\cdot\|_{[-\beta, \beta]}$ and $\omega(e, \cdot)$ denotes $\omega_{[-1/4, 1/4]}(e, \cdot)$.

A function with nonnegative k th difference on $[a, b]$ cannot, in general, be extended to a function with nonnegative k th difference on a larger interval. For example the piecewise linear and convex function, $f \in C[0, \sum_{n=1}^\infty n^{-3}]$, with slope n on the interval

$$\left[\sum_{i=1}^{n-1} i^{-3}, \sum_{i=1}^n i^{-3} \right],$$

cannot be extended to the right and remain convex. This motivates the construction of a preapproximation (see Lemma 1) to f , to which

we will apply appropriate polynomial convolution operators (see Lemma 2).

LEMMA 1. Suppose $k \geq 2$. Let

$$(3) \quad L_n(h, x) = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} h(x + t_1 + \cdots + t_k) dt_1 \cdots dt_k$$

where $h \in C[-1/4, 1/4]$ and

$$(4) \quad \lambda = 1/8n, \quad n = k, k + 1, \dots$$

Extend the definition of $L_n(h)$ from

$$[-\alpha, \alpha] = \left[-\frac{1}{4} + \frac{k}{8n}, \frac{1}{4} - \frac{k}{8n} \right]$$

to $[-1/2, 1/2]$ by adjoining, to the right and left the Taylor polynomials of degree k , corresponding to $L_n(h)$ at the points $\alpha, -\alpha$. Then there exists constants $E_k, F_k, G_k; \bar{E}_k, \bar{F}_k, \bar{G}_k$; such that; for all $f \in C[-1/4, 1/4]$ with $f(-1/4) = f(1/4) = 0$ and nonnegative k th difference on $[-1/4, 1/4]$; for $n = k, k + 1, \dots$;

$$(5) \quad L_n(f, x)^{(k)} \geq 0, \quad x \in R,$$

$$(6) \quad \|L_n(f)^{(j)}\|_{1/4} \leq E_k n^j \omega(f, n^{-1}) \quad (j = 1, \dots, k - 1),$$

$$(7) \quad \|L_n(f)^{(k)}\|_{1/2} \leq E_k n^k \omega(f, n^{-1}),$$

$$(8) \quad \|f - L_n(f)\|_{1/4} \leq F_k \omega(f, n^{-1}),$$

and

$$(9) \quad \|L_n(f)\|_{1/4} \leq G_k n \omega(f, n^{-1}).$$

If in addition $f' \in C[-1/4, 1/4]$ then

$$(6') \quad \|L_n(f)^{(j)}\|_{1/4} \leq \bar{E}_k n^{j-1} \omega(f', n^{-1}) \quad (j = 2, \dots, k - 1),$$

$$(7') \quad \|L_n(f)^{(k)}\|_{1/2} \leq \bar{E}_k n^{k-1} \omega(f', n^{-1}),$$

$$(8') \quad \|f - L_n(f)\|_{1/4} \leq \bar{F}_k n^{-1} \omega(f', n^{-1}),$$

and

$$(9') \quad \|L_n(f)^{(2-j)}\|_{1/4} \leq \bar{G}_k n \omega(f', n^{-1}). \quad (j = 1, 2).$$

Proof. For $x \in [-\alpha, \alpha]$

$$L_n(f, x) = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \int_{x+t_2+\dots+t_{k-1}}^{x+t_2+\dots+t_k+\lambda} f(\gamma) d\gamma dt_2 \cdots dt_k$$

implying

$$L_n(f, x)' = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} A_{2\lambda} f(x + t_2 + \cdots + t_k - \lambda) dt_2 \cdots dt_k ;$$

repeating the argument, j times, $j = 1, \dots, k$,

$$(10) \quad \begin{aligned} L_n(f, x)^{(j)} &= (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} A_{2\lambda}^j f(x + t_{j+1} + \cdots + t_k - j\lambda) dt_{j+1} \cdots dt_k . \end{aligned}$$

(5) follows immediately. (10) and the definition of λ imply

$$(11) \quad \|L_n(f)^{(j)}\|_{\alpha} \leq C_1 n^j \omega(f, n^{-1}) \quad (j = 1, \dots, k) .$$

(6), (7) follow from (11) on estimating the derivatives of the Taylor polynomials extending $L_n(f)$ to the larger interval.

To prove (8). The definition of $L_n(f, x)$ clearly implies

$$(12) \quad \|f - L_n(f)\|_{\alpha} \leq C_2 \omega(f, n^{-1}) .$$

Also

$$\begin{aligned} \|f - L_n(f)\|_{[\alpha, 1/4]} &\leq \|f - f(\alpha)\|_{[\alpha, 1/4]} + |f(\alpha) - L_n(f, \alpha)| \\ &\quad + \|L_n(f) - L_n(f, \alpha)\|_{[\alpha, 1/4]} ; \end{aligned}$$

so by (4); (12); (6), (7); and the manner in which $L_n(f)$ was extended

$$\|f - L_n(f)\|_{[\alpha, 1/4]} \leq C_3 \omega(f, n^{-1}) .$$

A similar result holds on $[-1/4, -\alpha]$; (8) follows.

To prove (9). Note that (8) implies both

$$\omega(L_n(f), n^{-1}) \leq C_4 \omega(f, n^{-1})$$

and

$$L_n(f, -1/4) \leq F_k \omega(f, n^{-1}) ;$$

the second since $f(-1/4) = 0$; (9) follows.

We proceed to prove the results for $f' \in C[-1/4, 1/4]$. Arguments analogous to those leading from (10) to (6), (7); lead from

$$\begin{aligned} L_n(f, x)^{(j)} &= (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} A_{2\lambda}^{j-1} f'(x + t_j + \cdots + t_k - (j-1)\lambda) dt_j \cdots dt_k , \end{aligned}$$

($j = 1, \dots, k$) to (6'), (7').

To show (8') we use the quantitative Korovkin type estimate (see e.g., DeVore [2, p. 28-32])

$$(13) \quad |L_n(f, x) - f(x)| \leq |f(x)| |1 - L_n(1, x)| + |f'(x)| |L_n((t-x), x)| \\ + (1 + \sqrt{L_n(1, x)}) \alpha_n(x) \omega(f', \alpha_n(x))$$

where

$$(14) \quad \alpha_n^2(x) = L_n((t-x)^2, x).$$

Now $\|1 - L_n(1)\| = \|L_n((t-x), x)\| = 0$, while

$$L_n((t-x)^2, x) = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} (t_1 + t_2 + \cdots + t_k)^2 dt_1 \cdots dt_k \\ = k(2\lambda)^{-1} \int_{-\lambda}^{\lambda} t^2 dt \leq C_5 n^{-2}.$$

Substituting into (13), (14) we find

$$(12') \quad \|L_n(f) - f\|_{\alpha} \leq C_6 n^{-1} \omega(f', n^{-1}).$$

Since for this particular operator

$$L_n(f, x)' = L_n(f', x), \quad x \in [-\alpha, \alpha]$$

and $L_n(f, x)'$ is continued outside $[-\alpha, \alpha]$ by adjoining the Taylor polynomials of degree $k-1$, corresponding to f' , at either end point; reasoning, similar to that yielding (8), implies

$$(15) \quad \|f' - L_n(f)'\|_{1/4} \leq C_7 \omega(f', n^{-1}).$$

Writing

$$\|f - L_n(f)\|_{[\alpha, 1/4]} \leq |f(\alpha) - L_n(f, \alpha)| + \int_{\alpha}^{1/4} |f'(t) - L_n(f, t)'| dt;$$

(12'); (4) and (15) imply

$$\|f - L_n(f)\|_{[\alpha, 1/4]} \leq C_8 n^{-1} \omega(f', n^{-1}).$$

Combining the above, the similar result on $[-1/4, -\alpha]$, and (12') proves (8').

To show (9'). Note (15) implies

$$\omega(L_n(f)', n^{-1}) \leq C_9 \omega(f', n^{-1})$$

and also

$$|L_n(f, \xi)'| \leq C_7 \omega(f', n^{-1}) \quad \text{where} \quad f'(\xi) = 0, \quad -\frac{1}{4} < \xi < \frac{1}{4};$$

the existence of such an ξ following from $f(-1/4) = f(1/4) = 0$. Hence

$$\|L_n(f)'\|_{1/4} \leq C_{10} n \omega(f', n^{-1}).$$

(9') follows since (8') implies

$$\left|L_n\left(f, -\frac{1}{4}\right)\right| \leq \bar{F}_k n^{-1} \omega(f', n^{-1}).$$

We now know how well $L_n(f)$ approximates f , and concern ourselves with how well $L_n(f)$ may be approximated by convolutions with positive polynomials.

LEMMA 2. *Suppose $k \geq 2$. Then there exist constants H_k, I_k and a sequence of even positive algebraic polynomials $\{\lambda_n\}_{n=k}^\infty$ satisfying*

$$(16) \quad \int_{-1}^1 \lambda_n(t) dt = 1,$$

and

$$(17) \quad \|\lambda_n^{(j)}\|_{[-1,1] \setminus [-1/4, 1/4]} \leq H_k n^{2-4k+2j} (\leq H_k n^{-2k}), \quad (j = 0, \dots, k-1).$$

Further if f satisfies the conditions of Lemma 1, $g = L_n(f)$ and

$$(18) \quad L_n^*(g) = \int_{-1/2}^{1/2} g(t) \lambda_n(t-x) dt;$$

then if $f \in C[-1/4, 1/4]$

$$(19) \quad \|g - L_n^*(g)\|_{1/4} \leq I_k \omega(f, n^{-1});$$

and if $f' \in C[-1/4, 1/4]$

$$(20) \quad \|g - L_n^*(g)\|_{1/4} \leq I_k n^{-1} \omega(f', n^{-1}).$$

Proof. Let $\lambda_k = \lambda_{k+1} = \dots = \lambda_{4k-1} \equiv 1/2$. For $n \geq 2k$, let

$$(21) \quad \lambda_{4n-4k}(t) = c_n [P_{2n}(t) / ((t^2 - x_{1,2n}^2) \dots (t^2 - x_{n,2n}^2))]^2,$$

where P_{2n} is the Legendre polynomial of degree $2n$ and $x_{1,2n}, \dots, x_{n,2n}$ are its positive zeros in increasing order. c_n is a normalizing constant for (16). Define the remaining λ_n 's with the relation

$$\lambda_{4n+1} = \lambda_{4n+2} = \lambda_{4n+3} = \lambda_{4n}, \quad n \geq k.$$

Observe firstly that a theorem of Bruns (see e.g., DeVore [2, p. 20]) implies

$$(22) \quad C_{11} n^{-1} \leq x_{1,2n} < \dots < x_{k,2n} \leq C_{12} n^{-1}, \quad n > k.$$

Using the normalization $\|P_n\|_{[-1,1]} = 1$ and the corresponding Taylor

expansion of P_n (see e.g., Davis [1, p. 365]),

$$(23) \quad |P_{2n}(0)| = 2^{-2n} \begin{bmatrix} 2n \\ n \end{bmatrix} = (1 + o(1))/\sqrt{\pi n},$$

the last equality being a consequence of Stirlings formula. (21), (22), and (23) together imply

$$(24) \quad \lambda_{4n-4k}(0) \geq C_{13} c_n n^{4k-1}, \quad n \geq 2k.$$

Let $n \geq 2k$. Write

$$1 = \int_{-1}^1 \lambda_{4n-4k}(t) dt = \sum_{k=-n}^n A_k(2n+1) \lambda_{4n-4k}(x_{k,2n+1});$$

where the $A_k(2n+1)$ are the weights of the Gaussian quadrature formula, exact for polynomials of degree $4n+1$, with nodes at the zeros of the Legendre polynomial of degree $2n+1$. Therefore

$$1 \geq A_0(2n+1) \lambda_{4n-4k}(0)$$

and since (Szegő [6, p. 350]), $A_0(2n+1) = \pi(1 + o(1))/(2n+1)$

$$(25) \quad \lambda_{4n-4k}(0) \leq C_{14} n.$$

(24) and (25) imply

$$c_n \leq C_{15} n^{2-4k};$$

which together with the normalization of the P_n , the definition of the λ_n , and (22) implies

$$\|\lambda_n\|_{[-1,1]/[-1/4,1/4]} \leq C_{16} n^{2-4k}.$$

(17) follows by means of Markov's inequality.

It remains to show the order of approximation results. We cannot use the standard quantitative Korovkin theorem as

$$\omega_{[-1/2,1/2]}(g, n^{-1}) \neq 0(\omega_{[-1/4,1/4]}(f, n^{-1}));$$

at least not in general. However a related method is applicable.

Again let $n \geq 2k$. $t^{2k} \lambda_{4n-4k}(t)$ is a polynomial of degree $4n-2k$. Therefore for $j = 1, \dots, k$

$$M_j = \int_{-1}^1 t^{2j} \lambda_{4n-4k}(t) dt = 2 \sum_{i=1}^n x_{i,2n}^{2j} A_i(2n) \lambda_{4n-4k}(x_{i,2n});$$

where the $A_i(2n)$ are the weights of the Gaussian quadrature formula, exact for polynomials of degree $4n-1$, with nodes at the zeros of the Legendre polynomial of degree $2n$. Since λ_{4n-4k} has zeros at $x_{k+1,2n}, \dots, x_{n,2n}$,

$$M_j = 2 \sum_{i=1}^k x_{i,2n}^{2j} A_i(2n) \lambda_{4n-4k}(x_{i,2n}) .$$

Since also λ_{4n-4k} has a local maximum on $[-x_{k+1,2n}, x_{k+1,2n}]$ at zero, and Szego [6, p. 350]

$$A_i(2n) \leq \frac{\pi}{2n} (1 + o(1)) \quad (i = 1, \dots, k) ,$$

(22), (25) and the definition of the λ_n imply

$$(26) \quad \int_{-1}^1 t^{2j} \lambda_n(t) dt \leq C_{17} n^{-2j} , \quad j = 1, \dots, k; n \geq k .$$

(26) and (17) may be used to estimate certain quantities involving L_n^* . All the estimates are uniform in $|x| \leq 1/4$.

$$(27) \quad \begin{aligned} 1 - L_n^*(1, x) &= \int_{-1}^1 \lambda_n(t) dt - \int_{-1/2-x}^{1/2-x} \lambda_n(t) dt \\ &\leq 2 \int_{1/4}^1 \lambda_n(t) dt \leq C_{18} n^{2-4k} . \\ L_n^*((t-x)^{2j}, x) &= \int_{-1/2}^{1/2} (t-x)^{2j} \lambda_n(t-x) dt \\ &= \int_{-1/2-x}^{1/2-x} t^{2j} \lambda_n(t) dt \\ &\leq \int_{-1}^1 t^{2j} \lambda_n(t) dt \end{aligned}$$

and applying (26)

$$(28) \quad L_n^*((t-x)^{2j}, x) \leq C_{19} n^{-2j} , \quad j = 1, \dots, k .$$

$$(29) \quad \begin{aligned} L_n^*(|t-x|^k, x) &\leq \int_{-1}^1 |t|^k \lambda_n(t) dt \\ &\leq \left[\int_{-1}^1 t^{2k} \lambda_n(t) dt \right]^{1/2} \\ &\leq C_{20} n^{-k} , \end{aligned}$$

where we have used the Schwartz inequality, (16) and (28).

For j odd,

$$\begin{aligned} |L_n^*((t-x)^j, x)| &= \left| \int_{-1/2-x}^{1/2-x} t^j \lambda_n(t) dt \right| \\ &\leq 2 \int_{1/4}^1 t^j \lambda_n(t) dt \end{aligned}$$

since λ_n is even. Applying (17)

$$(30) \quad |L_n^*((t-x)^j, x)| \leq C_{21} n^{2-4k} , \quad j = 1, 3, 5, \dots .$$

If $t \in [-1/2, 1/2]$ and $x \in [-1/4, 1/4]$, Taylor's theorem gives

$$(31) \quad g(t) = \left[\sum_{j=0}^{k-1} \frac{g^{(j)}(x)(t-x)^j}{j!} \right] + \frac{1}{(k-1)!} \int_x^t g^{(k)}(u)(t-u)^{k-1} du.$$

Since the last term on the right hand side is bounded in modulus by $(1/k!)|t-x|^k \|g^{(k)}\|_{[-1/2, 1/2]}$,

$$\begin{aligned} |L_n^*(g, x) - g(x)| &\leq |g(x)| |1 - L_n^*(1)| + \sum_{j=1}^{k-1} \frac{|g^{(j)}(x)|}{j!} |L_n^*((t-x)^j, x)| \\ &\quad + \frac{1}{k!} \|g^{(k)}\|_{[-1/2, 1/2]} L_n^*(|t-x|^k, x). \end{aligned}$$

Thus

$$\begin{aligned} \|L_n^*(g, x) - g(x)\|_{[-1/4, 1/4]} &\leq \|g\|_{[-1/4, 1/4]} |1 - L_n^*(1)|_{[-1/4, 1/4]} \\ &\quad + \sum_{j=1}^{k-1} \frac{\|g^{(j)}\|_{[-1/4, 1/4]}}{j!} |L_n^*((t-x)^j, x)|_{[-1/4, 1/4]} \\ &\quad + \frac{1}{k!} \|g^{(k)}\|_{[-1/2, 1/2]} |L_n^*(|t-x|^k, x)|_{[-1/4, 1/4]}. \end{aligned}$$

Combining the above, the estimates of all the terms involving g from Lemma 1 ($g = L_n(f)$), and the estimates (27), (28), (29), (30) of all the $L_n^*(\cdot, \cdot)$ yields (19), (20).

Given Lemmas 1 and 2 it remains to discuss how close $L_n^*(g)$ is to a polynomial with nonnegative k th derivative on $[-1/4, 1/4]$.

THEOREM. *For each $k \geq 2$ there exists a constant D_k , such that for all $h \in C[-1/4, 1/4]$ with k th difference nonnegative on $[-1/4, 1/4]$*

$$E_{n,k}(h) \leq D_k \omega_{[-1/4, 1/4]}(h, n^{-1}), \quad n = k, k+1, \dots$$

If in addition $h' \in C[-1/4, 1/4]$ then

$$E_{n,k}(h) \leq D_k n^{-1} \omega_{[-1/4, 1/4]}(h', n^{-1}), \quad n = k, k+1, \dots$$

Proof. Fix $k \geq 2$. Let $f = h - \rho$ where

$$\rho(x) = h\left(-\frac{1}{4}\right) + 2\left(h\left(\frac{1}{4}\right) - h\left(-\frac{1}{4}\right)\right)\left(x + \frac{1}{4}\right).$$

Clearly $\omega(f, n^{-1}) \leq 2\omega(h, n^{-1})$ and when h' exists $\omega(f', n^{-1}) = \omega(h', n^{-1})$. Lemmas 1 and 2 apply to f . Writing

$$\bar{L}_n(h) = \rho(x) + L_n^*(L_n(f))$$

Lemmas 1 and 2 imply

$$\begin{aligned}
 \|h - \bar{L}_n(h)\|_{1/4} &= \|f - L_n^*(L_n(f))\| \\
 &\leq \|f - L_n(f)\|_{1/4} + \|L_n(f) - L_n^*(L_n(f))\|_{1/4} \\
 (32) \qquad &\leq \begin{cases} C_{22}\omega(h, n^{-1}) & h \in C\left[-\frac{1}{4}, \frac{1}{4}\right], \\ C_{22}n^{-1}\omega(h', n^{-1}), & h' \in C\left[-\frac{1}{4}, \frac{1}{4}\right]. \end{cases}
 \end{aligned}$$

Let $g = L_n(f)$. Then

$$\begin{aligned}
 \bar{L}_n(h) &= \rho(x) + L_n^*(g) = \rho(x) + \int_{-1/2}^{1/2} g(t)\lambda_n(t-x)dt, \\
 \bar{L}_n(h, x)' &= \rho'(x) + \int_{-1/2}^{1/2} g(t) \cdot -\lambda_n'(t-x)dt \\
 &= \rho'(x) + [-g(t)\lambda_n(t-x)]_{-1/2}^{1/2} + \int_{-1/2}^{1/2} g'(t)\lambda_n(t-x)dt.
 \end{aligned}$$

$k \geq 2$ alternate differentiations and integrations by parts yield;

$$\begin{aligned}
 \bar{L}_n(h, x)^{(k)} &= (-1)^k \left[\sum_{j=0}^{k-1} (-1)^j \left[g^{(j)}(t)\lambda_n^{(k-1-j)}(t-x) \right]_{t=-1/2}^{t=1/2} \right] \\
 &\quad + \int_{-1/2}^{1/2} g^{(k)}(t)\lambda_n(t-x)dt \\
 &= r(x) + \int_{-1/2}^{1/2} g^{(k)}(t)\lambda_n(t-x)dt.
 \end{aligned}$$

(5) and the positivity of the kernels imply the second term on the right hand side is nonnegative. Lemma 1 implies

$$\|g^{(j)}\|_{1/2} \leq C_{23}n^k\omega(h, n^{-1}), \quad j = 0, \dots, k-1, h \in C, \left[-\frac{1}{4}, \frac{1}{4}\right].$$

Hence using (17)

$$\|r\|_{1/4} \leq C_{24}n^{-k}\omega(h, n^{-1}).$$

Let

$$p_n(x) = \bar{L}_n(h, x) + \frac{x^k}{k!}C_{25}n^{-k}\omega(h, n^{-1});$$

$p_n^{(k)}(x)$ is nonnegative on $[-1/4, 1/4]$, and by (32) p_n provides the first estimate of the theorem. Similarly, when $h' \in C[-1/4, 1/4]$

$$p_n(x) = \bar{L}_n(h, x) + \frac{x^k}{k!}C_{25}n^{-k-1}\omega(h', n^{-1})$$

provides the second estimate of the theorem.

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