

Pacific Journal of Mathematics

FUNCTIONS THAT OPERATE ON THE ALGEBRA $B_0(G)$

ALESSANDRO FIGÀ-TALAMANCA AND MASSIMO A. PICARDELLO

FUNCTIONS THAT OPERATE ON THE ALGEBRA $B_0(G)$

ALESSANDRO FIGÀ-TALAMANCA AND
MASSIMO A. PICARDELLO

Let G be a locally compact group and let $B(G)$ be the algebra of linear combinations of positive definite continuous functions. We let $B_0(G) = B(G) \cap C_0(G)$ be the subalgebra consisting of the elements of $B(G)$ which vanish at infinity. A complex valued function F , defined on an open interval of the real line containing zero, is said to *operate* on $B_0(G)$ if for every $u \in B_0(G)$ whose range is contained in the domain of F , the composition $F \circ u$ is an element of $B_0(G)$. In this paper we prove that, if G is a separable group with noncompact center or a separable nilpotent group, then every function which operates on $B_0(G)$ can be extended to an entire function. This result follows directly from the corresponding theorem for noncompact commutative groups, which is well known, via a lemma which states that every function on the center Z of G which belongs to $B_0(Z)$ can be extended to an element of $B_0(G)$.

We refer of the paper of N. Th. Varopoulos [11] for the treatment of the commutative case and to the paper of P. Eymard [4] for the definition and properties of the algebra $B(G)$.

We remark that $B(G)$ may also be defined as the algebra of all coefficients of unitary representations of G . A natural subalgebra and ideal of $B(G)$ consists of the coefficients of the regular representation. This subalgebra is called $A(G)$ and is also studied in [4], where it is proved that $A(G) \subseteq B_0(G)$, and that the maximal ideal space of $A(G)$ is G itself. This implies that all functions which are defined and analytic on a neighborhood of zero and vanish at zero operate on $A(G)$. Thus the existence of suitably defined analytic functions which do not operate on $B_0(G)$ is an indication that $B_0(G)$ is quite different from $A(G)$.

To be precise, if some analytic function vanishing at zero and defined on an open interval fails to operate on $B_0(G)$, then the quotient algebra $B_0(G)/A(G)$ is not a radical algebra.

It is important to observe that for several noncompact noncommutative groups $B_0(G)$ coincides with $A(G)$. (The simplest example of such groups is the affine group of the real line, discussed in [6]; other examples are studied in [9], [10], and [12]; a sufficient, and probably necessary, condition to ensure that $B_0(G)$ be different from $A(G)$, for a unimodular group, is given in [5].) But even when $A(G) \neq B_0(G)$, the two algebras may have the same maximal ideal space and therefore all suitably defined analytic functions may

operate on $B_0(G)$. In particular, J. R. Liukkonen and M. W. Mislove [7] have proved (under mild additional hypothesis) that if G is the semidirect product obtained from the action of a compact group on a finite dimensional vector space, and if the center of G is compact, then every element $u \in B_0(G)$ has an integral power u^n which belongs to $A(G)$. An analogous result was proved by M. Cowling [2] for the group $SL(2, R)$. Furthermore, M. Cowling has communicated to us that, if G is any semisimple Lie group, then $B_0(G)/A(G)$ is a radical algebra, and G is the maximal ideal space of $B_0(G)$.

These results seem to support the following conjecture: if G is a connected Lie group with compact center and the maximal ideal space of $B_0(G)$ is larger than G , then G contains a nontrivial nilpotent direct factor. This conjecture does not extend to the case of non-connected Lie groups: if G is the semidirect product defined by the action of the integers on R/Z given by multiplication, it is not difficult to prove, using the corresponding result for the integers, that only entire functions operate on $B_0(G)$, and yet G has trivial center and is not nilpotent.

2. The basic lemma. Before stating the lemma we recall a few facts concerning induced representations. Let G be a separable group and N a closed normal subgroup. It is known that there exists a "smooth cross-section" for G/N in G , that is a Borel measurable function s mapping G/N into G in such a way that s carries the identity of G/N into the identity of G , and maps compact subsets of G/N into relatively compact subsets of G [8]. As a consequence, if $g \in G$, then g is uniquely decomposable, in a measurable fashion, as $g = n \cdot s(\dot{g})$, with $n \in N$ and $\dot{g} = gN \in G/N$. Let now π be a unitary representation of N on the separable Hilbert space \mathcal{H}_π ; then it is easy to see that the induced representation $\text{Ind}_N^G \pi$ is unitarily equivalent to the representation π^* on $L^2(G/N)$ defined by

$$(1) \quad \pi^*(g)f(\dot{g}_0) = \pi(s(\dot{g}_0)ns(\dot{g})s(\dot{g}_0\dot{g})^{-1})f(\dot{g}_0\dot{g}),$$

where $g = n \cdot s(\dot{g})$, and $\dot{g} = gN \in G/N$. (See [1] for a proof.)

LEMMA. *Let G be a locally compact separable group and let Z be its center. Then every element of $B_0(Z)$ is the restriction to Z of an element of $B_0(G)$. Furthermore, if $u \in B_0(Z)$ is real valued and $-1 < u < 1$, there is an extension to G which is also real valued and satisfies the same inequalities.*

Proof. Since Z is commutative, every $u \in B_0(Z)$ can be written as

$$u(z) = \int_{\hat{Z}} \psi(z) d\mu(\psi),$$

where \hat{Z} is the character group of Z , μ is a bounded regular measure on \hat{Z} , and $z \in Z$.

For $\psi \in \hat{Z}$, denote by π_ψ the representation induced by ψ from Z to G . Let

$$\pi = \int_{\hat{Z}}^\oplus \pi_\psi d\mu(\psi);$$

let f be a continuous real valued function with compact support in G/Z , such that $\|f\|_{L^2(G/Z)} = 1$, and define $\tilde{u}(g) = \langle \pi(g)f, f \rangle$. We shall prove the following: (i) $\tilde{u}(z) = u(z)$, for $z \in Z$; (ii) if u is real valued and $-1 < u < 1$, then \tilde{u} is also real valued and satisfies the same inequalities; (iii) $\tilde{u} \in B_0(G)$. The first and second assertions follow readily from formula (1), which, in our case, yields (applying Fubini's theorem):

$$\begin{aligned} \tilde{u}(g) &= \langle \pi(g)f, f \rangle = \langle \pi(z_g s(\dot{g}))f, f \rangle \\ (2) \quad &= \int_{\hat{Z}} \int_{G/Z} \psi(z_g s(\dot{g}_0) s(\dot{g}) s(\dot{g}_0 \dot{g})^{-1}) f(\dot{g}_0 \dot{g}) f(\dot{g}_0) d\dot{g}_0 d\mu(\psi). \end{aligned}$$

The third assertion will be proved if we show that the last expression in (2) vanishes at infinity. We observe that, since f has compact support in G/Z , $f(\dot{g}_0 \dot{g}) f(\dot{g}_0) = 0$ identically in \dot{g}_0 , when \dot{g} lies outside a fixed compact set K , and therefore $\tilde{u}(g) = 0$, unless \dot{g} belongs to $K \subseteq G/Z$. Since s maps compact subsets of G/Z into relatively compact subsets of G , when $\tilde{u}(g) \neq 0$, g can be uniquely decomposed as $g = z_g s(\dot{g})$, where $z_g \in Z$ and $s(\dot{g})$ belongs to a compact subset K' of G . If $\tilde{u}(g)$ does not vanish at infinity, there exists a sequence $\{g_k\} \subseteq G$ such that g_k is eventually outside every fixed compact set and $|\tilde{u}(g_k)| \geq \varepsilon > 0$. If $g_k = z_k s(\dot{g}_k)$, then the condition $|\tilde{u}(g_k)| \geq \varepsilon$ implies that $\dot{g}_k \in K$ and $s(\dot{g}_k) \in K'$; therefore $g_k \in z_k K'$, and the sequence z_k belongs eventually to the complement of every fixed compact set. As a consequence, for every fixed element g_0 of G , the sequence $\zeta_k = z_k s(\dot{g}_0) s(\dot{g}_k) s(\dot{g}_0 \dot{g}_k)^{-1}$ tends to infinity: indeed $\dot{g}_0 \dot{g}_k \in \dot{g}_0 K$, as $\dot{g}_k \in K$, and both $s(\dot{g}_k)$ and $s(\dot{g}_0 \dot{g}_k)^{-1}$ must keep inside a fixed compact set because s is smooth and g_0 is fixed. Noticing that $\zeta_k \in Z$, we can define

$$I_k(\dot{g}_0) = \int_{\hat{Z}} \psi(\zeta_k) d\mu(\psi).$$

Since $\zeta_k \rightarrow \infty$, and $I_k(\dot{g}_0) = u(\zeta_k)$, one has $\lim_k I_k(\dot{g}_0) = 0$ for each $\dot{g}_0 \in G/Z$. On the other hand, by (2),

$$\tilde{u}(g_k) = \int_{G/Z} f(\dot{g}_0 \dot{g}_k) f(\dot{g}_0) I_k(\dot{g}_0) d\dot{g}_0.$$

The sequence of functions under the integral sign is uniformly bounded by $\sup |f(\dot{g})| \cdot \|\mu\|$; furthermore this sequence converges to zero for each $\dot{g}_0 \in G/Z$, and its elements all vanish outside the support of f , which is compact. We conclude, by the bounded convergence theorem, that $\lim_k \tilde{u}(g_k) = 0$, contradicting the assumption that $|\tilde{u}(g_k)| \geq \varepsilon$.

REMARK. The fact that functions in $B(Z)$ may be extended to $B(G)$ is well known [7]. For an interesting generalization see [3].

3. **The main theorem.** We recall that, if G is a locally compact abelian group, then the only functions which operate on $B_0(G)$ are the entire functions [11]. We shall use this result and the lemma to prove our main theorem.

THEOREM. *Let G be a separable locally compact group with noncompact center. Suppose that F is a complex valued function defined on an open interval of the real line containing 0, and that $F(0) = 0$. If, for each $u \in B_0(G)$ with range contained in the domain of F , the composition $F \circ u$ belongs to $B_0(G)$, then F may be extended to an entire function.*

Proof. Without loss of generality we may suppose that F is defined on the interval $(-1, 1)$. (If not, replace $F(t)$ by $F(rt)$ for a sufficiently small positive real number r .) We shall prove that F operates on $B_0(Z)$. If $u \in B_0(Z)$, and $-1 < u < 1$, let \tilde{u} be the extension of u to G , whose existence is asserted in the lemma. Then $F \circ \tilde{u} \in B_0(G)$ and its restriction to Z is $F \circ u$. Since the restriction map carries $B_0(G)$ into $B_0(Z)$, we have proved that F operates on $B_0(Z)$. This implies that F may be extended to an entire function, by the result of Varopoulos [11].

COROLLARY 1. *The conclusion of the theorem holds if G is a noncompact nilpotent group.*

Proof. Let $\{Z_j\}_{j=1}^k$ be an ascending central series for G . Since G is noncompact, there exists a positive integer $n < k$ such that Z_{n+1}/Z_n is noncompact and Z_n is compact; then Z_{n+1}/Z_n is the center of G/Z_n , and the theorem implies that every function that operates on $B_0(G/Z_n)$ is an entire function. As Z_n is compact, we may extend canonically an element $u \in B_0(G/Z_n)$ to an element $\tilde{u} \in B_0(G)$, letting $\tilde{u}(g) = u(gZ_n)$. Then it is immediate to see that, if F operates on $B_0(G)$, it operates also on $B_0(G/Z_n)$, and therefore it is extendable to an entire function.

COROLLARY 2. *If G satisfies the hypothesis of the theorem or of Corollary 1, then $B_0(G)$ is not symmetric on its maximal ideal space.*

Proof. This follows from the theorem and Corollary 1 by a standard argument. We sketch the argument here for completeness. Let $F(t) = t(t-i)^{-1}$ for $-1 < t < 1$; then F cannot be extended to an entire function, therefore there exists a real valued u in $B_0(G)$ such that $-1 < u(x) < 1$ and $F \circ u \in B_0(G)$. In other words $F(Z) = z(z-i)^{-1}$ cannot be analytic on the spectrum of u , and therefore the spectrum of u contains i ; this implies that $B_0(G)$ is not symmetric.

REFERENCES

1. L. Baggett, *A weak containment theorem for groups with a quotient R -group*, Trans. Amer. Math. Soc., **128** (1967), 277-290.
2. M. Cowling, *$B(\text{SL}(2, R))$ is symmetric*, (to appear).
3. M. Cowling and P. Rodway, *Restriction of certain function spaces to normal subgroups*, (to appear).
4. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France, **92** (1964), 181-236.
5. A. Figà-Talamanca, *Positive definite functions vanishing at infinity*, Pacific J. Math., **69** (1977), 355-363.
6. I. Khalil, *L'analyse harmonique de la droite et du groupe affine de la droite*, Thèse de doctorat d'état, Université de Nancy, (1973).
7. J. R. Liukkonen and M. W. Mislove, *Symmetry in Fourier-Stieltjes Algebras*, Math. Ann., **217** (1975), 97-112.
8. G. Mackey, *Induced representations of locally compact groups I*, Ann. of Math., **55** (1952), 101-139.
9. G. Mauzeri, *Square integrable representations and the Fourier algebra of a unimodular group*, (to appear Pacific J. Math.)
10. G. Mauzeri and M. A. Picardello, *Noncompact unimodular groups with purely atomic Plancherel measure*, (to appear).
11. N. Th. Varopoulos, *The functions which operate on $B_0(\Gamma)$ of a discrete group Γ* , Bull. Soc. Math. France, **93** (1965), 301-321.
12. M. Walter, *On a theorem of Figà-Talamanca*, Proc. Amer. Math. Soc., **60** (1976), 72-74.

Received February 15, 1977.

UNIVERSITÀ DI PERUGIA
06100 PERUGIA, ITALY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

C. W. CURTIS

University of Oregon
Eugene, OR 97403

C. C. MOORE

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. FINN AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
OSAKA UNIVERSITY

Gerald Arthur Anderson, <i>Computation of the surgery obstruction groups</i> $L_{4k}(1; Z_p)$	1
R. K. Beatson, <i>The degree of monotone approximation</i>	5
Sterling K. Berberian, <i>The character space of the algebra of regulated functions</i>	15
Douglas Michael Campbell and Jack Wayne Lamoreaux, <i>Continua in the plane with limit directions</i>	37
R. J. Duffin, <i>Algorithms for localizing roots of a polynomial and the Pisot Vijayaraghavan numbers</i>	47
Alessandro Figà-Talamanca and Massimo A. Picardello, <i>Functions that operate on the algebra $B_0(G)$</i>	57
John Erik Fornæss, <i>Biholomorphic mappings between weakly pseudoconvex domains</i>	63
Andrzej Granas, Ronald Bernard Guenther and John Walter Lee, <i>On a theorem of S. Bernstein</i>	67
Jerry Grossman, <i>On groups with specified lower central series quotients</i>	83
William H. Julian, Ray Mines, III and Fred Richman, <i>Algebraic numbers, a constructive development</i>	91
Surjit Singh Khurana, <i>A note on Radon-Nikodým theorem for finitely additive measures</i>	103
Garo K. Kiremidjian, <i>A Nash-Moser-type implicit function theorem and nonlinear boundary value problems</i>	105
Ronald Jacob Leach, <i>Coefficient estimates for certain multivalent functions</i>	133
John Alan MacBain, <i>Local and global bifurcation from normal eigenvalues. II</i>	143
James A. MacDougall and Lowell G. Sweet, <i>Three dimensional homogeneous algebras</i>	153
John Rowlay Martin, <i>Fixed point sets of Peano continua</i>	163
R. Daniel Mauldin, <i>The boundedness of the Cantor-Bendixson order of some analytic sets</i>	167
Richard C. Metzler, <i>Uniqueness of extensions of positive linear functions</i>	179
Rodney V. Nillsen, <i>Moment sequences obtained from restricted powers</i>	183
Keiji Nishioka, <i>Transcendental constants over the coefficient fields in differential elliptic function fields</i>	191
Gabriel Michael Miller Obi, <i>An algebraic closed graph theorem</i>	199
Richard Cranston Randell, <i>Quotients of complete intersections by C^* actions</i>	209
Bruce Reznick, <i>Banach spaces which satisfy linear identities</i>	221
Bennett Setzer, <i>Elliptic curves over complex quadratic fields</i>	235
Arne Stray, <i>A scheme for approximating bounded analytic functions on certain subsets of the unit disc</i>	251
Nicholas Th. Varopoulos, <i>A remark on functions of bounded mean oscillation and bounded harmonic functions. Addendum to: "BMO functions and the $\bar{\partial}$-equation"</i>	257
Charles Irvin Vinsonhaler, <i>Torsion free abelian groups quasi-projective over their endomorphism rings. II</i>	261
Thomas R. Wolf, <i>Characters of p'-degree in solvable groups</i>	267
Toshihiko Yamada, <i>Schur indices over the 2-adic field</i>	273