BIHOLOMORPHIC MAPPINGS BETWEEN WEAKLY PSEUDOCONVEX DOMAINS

JOHN ERIK FORNAESS
BIHOLOMORPHIC MAPPINGS BETWEEN WEAKLY PSEUDOCONVEX DOMAINS

JOHN ERIK FORNAESS

Assume we have a biholomorphic mapping between weakly pseudoconvex domains. It is an old question whether this extends to a diffeomorphism between their closures. The well known theorem of Fefferman states that this is true for strongly pseudoconvex domains. We will show that if the map has a smooth extension to the boundary, then it cannot map an analytic disc in the boundary to a single point.

This is an immediate consequence of the following theorem.

**Theorem.** Assume $\Omega, W$ are bounded pseudoconvex sets with $C^2$ boundary in $\mathbb{C}^n$, and assume $\Phi: \Omega \to W$ is a biholomorphic map with a $C^2$-extension $\Phi: \bar{\Omega} \to \bar{W}$. Then $\Phi$ is a $C^2$-diffeomorphism between $\bar{\Omega}$ and $\bar{W}$.

This theorem generalizes a similar result for strongly pseudoconvex domains by the author [2], see also Pinchuk [3].

The theorem is false in general for $C^1$-domains and maps $\Phi$ with $C^1$-extensions. To see this, let $\Omega = \{z \in \mathbb{C}; |z + 1| < 1\}$ and let $\Phi(z) = z/\log z$ and $W = \Phi(\Omega)$.

Also the theorem is false in general for proper holomorphic maps. For example, let $\Omega = \{(z, w) \in \mathbb{C}^2; |z|^2 + |w|^2 < 1\}$, $W = \{(z, w); |z|^2 + |w|^2 < 1\}$ and $\Phi(z, w) = (z, w^2)$.

**Proof of the Theorem.** Let us fix a point $p \in \partial \Omega$. We want to show that $\Phi'(p)$ is a nonsingular linear transformation. The proof is complete if this is true for all $p$ in the boundary of $\Omega$.

To simplify the computations, we will make affine complex changes of coordinates such that $p$ becomes the origin in $\mathbb{C}^n$ with variables $(z_1, \cdots, z_n)$ and $\Phi(p)$ becomes the origin in $\mathbb{C}^n$ with variables $(w_1, \cdots, w_n)$. We may arrange this such that

$$\Omega = \{z = (z_1, \cdots, z_n); \rho(z) = \text{Re } z_1 + R(z_1, \cdots, z_n) < 0\} \text{ and }$$

$$W = \{w = (w_1, \cdots, w_n); \sigma(w) = \text{Re } w_1 + S(w_1, \cdots, w_n) < 0\}$$

where $R, S$ are $C^2$-functions which vanish to at least second order at the origin.

From Diederich, Fornaess [1] it follows that there exists a $C^2$-defining function $\delta$ of $W$ such that $-(\delta)^{2/3}$ is strictly plurisubharmonic in $W$ near the origin. It follows that $-(\delta)^{2/3} \circ \Phi$ is strictly
plurisubharmonic in $\Omega$ near the origin.

We apply the Hopf lemma to points in $\Omega$ of the form $(t, 0, \ldots, 0)$ with $-1 < t < 0$. There must exist a $K > 0$ such that

$$-(\xi)^{3/4}((t, 0, \ldots, 0)) \leq K \xi$$

where we have written $\xi = (\xi_1, \ldots, \xi_n)$. Hence $(\xi_1, \xi_2, \ldots, 0) > 0$.

Consider the $C^2$-map $\Lambda(z_1, \ldots, z_n) = (\varphi_1(z_1, \ldots, z_n), z_2, \ldots, z_n)$. Then $\Lambda'(0)$ is invertible. Therefore $\Lambda$ maps the germ of $\Omega$ at the origin to the germ of a pseudoconvex set $U$ with $C^2$ boundary at the origin. We can describe $U$ by

$$U = \{\eta = (\eta_1, \ldots, \eta_n); \tau(\eta) = \Re \eta_1 + T(\eta_1, \ldots, \eta_n) < 0\}$$

for some $C^2$-function $T$ vanishing to at least second order at the origin.

We will study the map $\Psi = \Phi \circ \Lambda^{-1}, \Psi = (\psi_1, \ldots, \psi_n)$. It suffices to show that $\Psi'(0)$ is a nonsingular linear map. We can describe $\Psi$ by

$$\Psi(\eta_1, \ldots, \eta_n) = (\eta_1, \psi_2(\eta_1, \ldots, \eta_n), \ldots, \psi_n(\eta_1, \ldots, \eta_n))$$

If $\Psi'(0)$ is singular, then we may assume, after a linear change in the $(w_2, \ldots, w_n)$ — and $(\eta_2, \ldots, \eta_n)$ — direction, that

$$\Psi'(0) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \frac{\partial \psi_2}{\partial \eta_1}(0) & \frac{\partial \psi_2}{\partial \eta_2}(0) & \cdots & \frac{\partial \psi_2}{\partial \eta_n}(0) \\
\vdots & \vdots & \vdots & & \vdots \\
0 & \frac{\partial \psi_n}{\partial \eta_1}(0) & \frac{\partial \psi_n}{\partial \eta_2}(0) & \cdots & \frac{\partial \psi_n}{\partial \eta_n}(0)
\end{bmatrix}.$$ 

Next we consider points $t_i = (t, 0, \ldots, 0)$ and $t_2 = (t, t, 0, \ldots, 0)$ in $U$ with $-1 < t < 0$. We then have the estimates, for $\tau_i = \Psi(t_i)$,

$$\tau_i = \begin{bmatrix} t, \frac{\partial \psi_2}{\partial \eta_1}(0) \cdot t + 0(t^2), \cdots, \frac{\partial \psi_n}{\partial \eta_1}(0) \cdot t + 0(t^2) \end{bmatrix},$$

$i = 1, 2$.

Define $W_t$ to be the set

$$W_t = \{(w_1, \ldots, w_n) \in W; w_1 = t\}.$$ 

There exists some $\delta > 0$ independent of $t$ such that
Let us write $\Phi^{-1}: W \rightarrow \Omega$ as $\Phi^{-1} = (\mu_1, \cdots, \mu_n)$. We then have that for some constant $K > 0$, independent of $t$, that
\[ |\mu_n(t, w_2, \cdots, w_n)| \leq K \]
for all points in $\tilde{W}$, since $\Omega$ is bounded. The points $\tau_1, \tau_2$ are in $\tilde{W}$ and satisfy the estimates $||\tau_1|| \leq K_1 |t|$, $||\tau_2|| \leq K_1 |t|$ and $||\tau_1 - \tau_2|| \leq K_1 |t|^2$ for some constant $K_1$ independent of $t$. It follows from Schwarz's lemma that for some constant $K_2$, independent of $t$, we have
\[ |\mu_n(\tau_1) - \mu_n(\tau_2)| \leq K_2 |t|^{3/2}. \]
However, by construction we know that $|\mu_n(\tau_1) - \mu_n(\tau_2)| = |t|$ which is a contradiction for all small enough $|t|$.

REFERENCES


Received May 12, 1977.

PRINCETON UNIVERSITY
PRINCETON, NJ 08540
Pacific Journal of Mathematics
Vol. 74, No. 1 May, 1978

Gerald Arthur Anderson, Computation of the surgery obstruction groups $L_{4k}(1; \mathbb{Z}_p)$ ................................................................. 1
R. K. Beatson, The degree of monotone approximation ........................................... 5
Sterling K. Berberian, The character space of the algebra of regulated functions ....... 15
Douglas Michael Campbell and Jack Wayne Lamoreaux, Continua in the plane with limit directions ......................................................... 37
R. J. Duffin, Algorithms for localizing roots of a polynomial and the Pisot Vijayaraghavan numbers ......................................................... 47
Alessandro Figà-Talamanca and Massimo A. Picardello, Functions that operate on the algebra $B_0(G)$ ....................................................... 57
John Erik Fornaess, Biholomorphic mappings between weakly pseudoconvex domains ................................................................. 63
Andrzej Granas, Ronald Bernard Guenther and John Walter Lee, On a theorem of S. Bernstein ................................................................. 67
Jerry Grossman, On groups with specified lower central series quotients ................. 83
William H. Julian, Ray Mines, III and Fred Richman, Algebraic numbers, a constructive development ....................................................... 91
Surjit Singh Khurana, A note on Radon-Nikodým theorem for finitely additive measures .................................................................................... 103
Garo K. Kiremidjian, A Nash-Moser-type implicit function theorem and nonlinear boundary value problems ........................................... 105
Ronald Jacob Leach, Coefficient estimates for certain multivalent functions .......... 133
John Alan MacBain, Local and global bifurcation from normal eigenvalues. II ....... 143
James A. MacDougall and Lowell G. Sweet, Three dimensional homogeneous algebras ................................................................. 153
John Rowlay Martin, Fixed point sets of Peano continua ......................................... 163
R. Daniel Mauldin, The boundedness of the Cantor-Bendixon order of some analytic sets ............................................................................... 167
Richard C. Metzler, Uniqueness of extensions of positive linear functions .......... 179
Rodney V. Nillsen, Moment sequences obtained from restricted powers .......... 183
Keiji Nishioka, Transcendental constants over the coefficient fields in differential elliptic function fields ....................................................... 191
Gabriel Michael Miller Obi, An algebraic closed graph theorem ............................ 199
Richard Cranston Randell, Quotients of complete intersections by $C^*$ actions .... 209
Bruce Reznick, Banach spaces which satisfy linear identities ............................... 221
Bennett Setzer, Elliptic curves over complex quadratic fields ............................... 235
Arne Stray, A scheme for approximating bounded analytic functions on certain subsets of the unit disc ....................................................... 251
Nicholas Th. Varopoulos, A remark on functions of bounded mean oscillation and bounded harmonic functions. Addendum to: “BMO functions and the $\overline{\partial}$-equation” ........................................................................ 257
Charles Irvin Vinsonhaler, Torsion free abelian groups quasi-projective over their endomorphism rings. II ........................................... 261
Thomas R. Wolf, Characters of $p'$-degree in solvable groups ................................ 267
Toshihiko Yamada, Schur indices over the 2-adic field .......................................... 273