A NOTE ON RADON-NIKODÝM THEOREM FOR FINITELY ADDITIVE MEASURES

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The Radon-Nikodym theorem for finitely additive measures is deduced from the corresponding result for countably additive measures.

In ([4], Theorem 1, p. 35) a Radon-Nikodym type result is proved for finitely additive measures. In this note we prove that this result is a simple consequence of the corresponding result for the countably additive case.

Let $\mathcal{A}_0$ be an algebra of subsets of a set $X$; without loss of generality we assume that $\mathcal{A}_0$ is reduced, i.e., separates points of $X$ ([5], p. 68). We denote by $\rho$ the isomorphism between $\mathcal{A}_0$ and $\mathcal{A}$ the algebra of all clopen subsets of $\hat{X}$, the compact Hausdorff, totally disconnected space which is the Boolean space for $\mathcal{A}_0$ ([5], p. 70).

**THEOREM** ([4], Theorem 1, p. 35). Let $\lambda$ and $\mu$ be two complex-valued finite-additive measures on $\mathcal{A}_0$ such that $\mu$ is bounded and $\lambda$ is absolutely continuous relative to $\mu$ ($\varepsilon - \delta$ meaning of absolute continuity). Then there exists a sequence $\{f_n\}$ of $\mathcal{A}_0$-simple functions on $X$ such that

1. $\lim_{n \to \infty} \int_A f_n \, d\mu = \lambda(A)$, unif. for $A \in \mathcal{A}_0$

and

2. $\lim_{m,n \to \infty} \int |f_n - f_m| \, d|\mu| = 0$, $|\mu|$ being the total variation of $\mu$ ([2]).

**Proof.** For any disjoint sequence $\{A_n\} \subset \mathcal{A}_0$, $|\mu|(A_n) \to 0$ (note $\mu$ is bounded) and so $\lambda(A_n) \to 0$. This means $\lambda$ is exhaustive ($\equiv$ strongly bounded) and so $\lambda$ is bounded ([1]). $\lambda$ and $\mu$ naturally give rise to countably additive measures $\lambda'$ and $\mu'$ on $\mathcal{A}$ and as such can be uniquely extended to the $\sigma$-algebra $\mathcal{B}_\infty$ generated by $\mathcal{A}$; $\mathcal{B}_\infty$ is also the class of all Baire subsets of $\hat{X}$ ([5], p. 70). We claim $|\lambda'|$ is absolutely continuous with respect to $|\mu'|$: suppose $|\mu'|(B) = 0$ but $|\lambda'(B)| > 0$ for some $B \in \mathcal{B}_\infty$. This means there exists a $C \subset B$, $C \in \mathcal{B}_\infty$ such that $|\lambda'(C)| > \varepsilon$ for some $\varepsilon > 0$. Fix $\delta > 0$ such that $P \in \mathcal{A}_0$, $|\mu|(P) < \delta$ implies $|\lambda(P)| < \varepsilon$. Since Baire measures are regular, there exists an open subset $V$ of $\hat{X}$ such that $V \supset C$, $|\mu'|(V) < \delta$, and $|\lambda'(V)| > \varepsilon$. Again by regularity and total disconnectedness of $\hat{X}$ there is a clopen subset $U \subset V$ such that $|\mu'|(U) < \delta$ and $|\lambda'(U)| > \varepsilon$. Taking $P = \rho^{-1}(U)$ we get $|\mu|(P) < \delta$ and $|\lambda(P)| > \varepsilon$, a contradiction.
By ([2], Theorem 7, p. 181) there exists an \( f \in L^1(X, \mathcal{B}_\infty, |\mu'|) \) such that \( \lambda' = f \mu' \). Since \( \mathcal{U} \)-simple functions are dense in \( L^1(X, \mathcal{B}_\infty, |\mu'|) \) there exists a sequence \( \{f_n\} \) of \( \mathcal{U} \)-simple functions such that \( \lim \int_E |f_n - f| d|\mu'| = 0 \). From this it follows that \( \int_E f_n d|\mu'| \to \int_E f d|\mu'| \) uniformly for \( E \in \mathcal{U} \). Note on \( \mathcal{U} \) the variation \( |\mu'| \) of \( \mu' \) is the same whether this variation is calculated relative to \( \mathcal{U} \) or \( \mathcal{B}_\infty \) ([2]), Theorem 3, p. 76). The results (1) and (2) of the theorem are obvious now.

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