

Pacific Journal of Mathematics

**LOCAL AND GLOBAL BIFURCATION FROM NORMAL
EIGENVALUES. II**

JOHN ALAN MACBAIN

LOCAL AND GLOBAL BIFURCATION FROM NORMAL EIGENVALUES II

JOHN A. MACBAIN

This paper studies the bifurcation of solutions of non-linear eigenvalue problems of the form $Lu = \lambda u + H(\lambda, u)$, where L is linear and H is $o(\|u\|)$ uniformly on bounded λ intervals. This paper shows that isolated eigenvalues of L having odd multiplicity are bifurcation points if H merely has a "degree" of compactness, but is not necessarily compact (treated in [3], [5]). Moreover, a global alternative theorem follows.

Introduction. In this paper we study the bifurcation of solutions of nonlinear eigenvalue problems. The equations to be studied are of the form

$$(0.1) \quad Lu = \lambda u + H(u)$$

where all operators are defined in a real Banach space \mathcal{B} . L is assumed to be linear, bounded or unbounded; I , the identity map, and H , $o(\|u\|)$ near $u = 0$. Clearly, $(\lambda, 0)$ is a solution for each real λ , and these are called the trivial solutions of (0.1). Of more interest are the nontrivial solutions, pairs (λ, u) satisfying (0.1) with $u \neq 0$. In particular, one is interested in how solutions of (0.1) are related to solutions of the linear equation

$$(0.2) \quad Lu = \lambda u.$$

The study of this led to the following definition.

DEFINITION. A point $(\lambda_0, 0)$ is a bifurcation point for (0.1) if every neighborhood of $(\lambda_0, 0)$ in $\mathbf{R} \times \mathcal{B}$ contains a nontrivial solution of (0.1).

Under quite general conditions, it is easy to show that in order for $(\lambda_0, 0)$ to be a bifurcation point of (0.1), it is necessary that λ_0 be in the spectrum of L . [8].

The first general existence theorem for bifurcation points was obtained by Krasnoseljskii [2]. He considered equations of the type

$$(0.3) \quad u + \lambda Lu + H(u)$$

where L is linear and compact, and H compact. He proved that if λ_0 is a characteristic value of L having odd algebraic multiplicity,

then $(\lambda_0, 0)$ is a bifurcation point.

More recently, Rabinowitz [6] studied the same problem as Krasnosel'kii and proved a much stronger result. The bifurcation from such points is a global property, with a continuous branch of solutions joining $(\lambda_0, 0)$ to infinity in $\mathbf{R} \times \mathcal{B}$, or if the branch is bounded, containing $(\lambda_1, 0)$ with $\lambda_1 \neq \lambda_0$.

The author ([3] and [5]) eliminated the compactness assumption on L while maintaining the strength of the result. The main result of this paper is that the compactness assumption on H can be relaxed. The proofs of the theorems mentioned involve the use of degree theory.

1. Preliminaries. Let \mathcal{B} be a real Banach space and let \mathcal{E} denote $\mathbf{R} \times \mathcal{B}$ with the product topology. By a nonlinear eigenvalue problem we mean an equation of the form

$$(1.1) \quad Lu = \lambda u + H(u)$$

where $L: \mathcal{B} \rightarrow \mathcal{B}$ is linear and $H: \mathcal{B} \rightarrow \mathcal{B}$ is a nonlinear operator satisfying hypothesis $H-1$:

- ($H-1$) (i) H is continuous, and bounded on each ball centered at 0.
(ii) H is $o(\|u\|)$ for u near 0.

A nontrivial solution of (1.1) is a pair (λ, u) with $u \neq 0$ which satisfies (1.1), and the trivial solutions are the pairs $(\lambda, 0)$.

In what follows, $L: \mathcal{B} \rightarrow \mathcal{B}$ will be a densely defined linear operator (bounded or unbounded) with domain $\text{dom}(L)$. The resolvent set of L , $\rho(L)$, will be all real values of λ for which there exists a bounded linear operator $C: \mathcal{B} \rightarrow \mathcal{B}$ such that

$$(1.2) \quad \begin{aligned} C(L - \lambda)x &= x, & x \in \text{dom}(L) \\ (L - \lambda)Cx &= x, & x \in \text{range}(L - \lambda). \end{aligned}$$

C will be denoted by $(L - \lambda)^{-1}$.

DEFINITION 1.1. The (algebraic) multiplicity of an eigenvalue λ of L is defined to be the dimension of the subspace $\bigcup_{j=1}^{\infty} \ker(L - \lambda)^j$ where $\ker(L - \lambda)^j$ denotes the nullspace of $(L - \lambda)^j$. $\bigcup_{j=1}^{\infty} \ker(L - \lambda)^j$ will be referred to as the principal manifold of L associated with λ .

DEFINITION 1.2. An eigenvalue λ of L is defined to be normal if

- (i) the multiplicity of L is finite.
(ii) \mathcal{B} is the direct sum of subspaces $\mathcal{L}_\lambda \oplus \mathcal{N}_\lambda$ where $\mathcal{L}_\lambda = \bigcup_{j=1}^{\infty} \ker(L - \lambda)^j$, \mathcal{N}_λ is invariant under L , and $(L - \lambda)$ is invertible on \mathcal{N}_λ .

The projection of \mathcal{B} onto \mathcal{L}_λ along \mathcal{N}_λ is denoted by P_λ .

Hence $P_\lambda \mathcal{B} = \mathcal{L}_\lambda$ and $(I - P_\lambda) \mathcal{B} = \mathcal{N}_\lambda$. Let $Q_\lambda = I - P_\lambda$.

An eigenvalue λ of L is isolated if there exists $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon)$ contains no other members of $\text{sp } L$. The set of isolated normal eigenvalues of L is called the discrete spectrum of L which we denote by $\text{sp}_d(L)$. The remaining part of the spectrum will be called nondiscrete and is denoted by $\text{sp}_{nd}(L)$.

REMARK. If L is self-adjoint, the nondiscrete spectrum is the essential spectrum of L .

DEFINITION 1.3. $(\lambda, 0)$ is a bifurcation point for (1.1) if every neighborhood in \mathcal{E} of $(\lambda, 0)$ contains a nontrivial solution of (1.1).

DEFINITION 1.4. If \mathcal{V} is a subset of \mathcal{E} , \mathcal{V} , and \mathcal{V}_R are defined to be $\mathcal{V}^\lambda = \{u | (\lambda, u) \in \mathcal{V}\}$ and $\mathcal{V}_R = \{\lambda | (\lambda, u) \in \mathcal{V} \text{ for some } u\}$. For $W \subset R$, \mathcal{B} , or \mathcal{E} , \bar{W} denotes the closure of W in the respective space.

Some of the material that follows in this section was presented in [8], and is repeated here without proof.

DEFINITION. The set measure of compactness of a bounded set Ω , expressed by $\alpha(\Omega)$, is defined to be the infimum of all $\delta > 0$ such that Ω can be covered by a finite number of balls having radius δ .

Some useful results in this area include:

- (i) $\alpha(\Omega) = \alpha(\bar{\Omega})$ for all bounded sets Ω .
- (ii) If Ω is bounded, Ω is relatively compact if and only if $\alpha(\Omega) = 0$.
- (iii) $\alpha(\Omega_1 + \Omega_2) \leq \alpha(\Omega_1) + \alpha(\Omega_2)$.
- (iv) If $\lim_{n \rightarrow \infty} x_n = 0$, then $\alpha(\{x_n\}_{n=1,2,\dots}) = 0$.

DEFINITION. An operator $T: B \rightarrow B$ is called a k -set contraction if it is continuous and $\alpha(T(\Omega)) \leq k\alpha(\Omega)$ for all bounded sets Ω . Let $\gamma(T) = \inf \{k | T \text{ is a } k\text{-set contraction}\}$. The following results concerning k -set contractions hold.

- (i) T is compact if and only if T is a 0-set contraction.
- (ii) If L is a bounded linear operator with operator norm $\|L\|$, then L is a $\|L\|$ -set contraction. (This need not be true if L is nonlinear. (See § 4.))
- (iii) If L is a bounded, linear, and self-adjoint operator, $\gamma(L) = \rho_e(L)$ where $\rho_e(L)$ is the radius of the essential spectrum of L . [8].
- (iv) If $F = GH$ with G linear, $\gamma(F) \leq \|G\|\gamma(H)$. In general, for all G and H , $\gamma(F) \leq \gamma(G)\gamma(H)$.

A degree theory for nonlinear operators of the form $I - T$, where $T: B \rightarrow B$ is a k -set contraction with $k < 1$, was developed by

Nussbaum in his thesis. The results of Nussbaum's to be used are given below, together with a theorem of Stuart.

Let $T: B \rightarrow B$ be a k -set contraction ($k < 1$). Then an integer-valued function, denoted by \deg , can be defined so as to have the following properties.

(1) $\deg(\Omega, I - T, 0)$ is well defined for each open, bounded subset $\Omega \subset B$ such that T has no fixed points on the boundary $\partial\Omega$ of Ω .

(2) If $\deg(\Omega, I - T, 0) \neq 0$, then there is a point $x \in \Omega$ such that $x = Tx$.

(3) If Ω_1 and Ω_2 are open subsets of Ω , itself a bounded, open subset of such that T has no fixed points in $[\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2)] \cup \overline{(\Omega_1 \cap \Omega_2)}$, then $\deg(\Omega, I - T, 0) = \deg(\Omega_1, I - T, 0) + \deg(\Omega_2, I - T, 0)$.

(4) If T is compact, then $\deg(\Omega, I - T, 0) = d(\Omega, I - T, 0)$, where d denotes the Leray-Schauder degree, whenever the left-hand side is defined. [8].

The arguments of this paper will closely follow those of [5]. Thus, a notation of index is helpful. Define

$$\text{index}(T, x_0) = \deg(B, I - T, 0)$$

where B is an open ball in B centered at x_0 with a radius small enough so that x_0 is the only fixed point of T in \bar{B} .

In [5], critical use was made of a theorem in Leray-Schauder degree theory which has been extended to the Nussbaum degree theory by Toland and Stuart [8].

THEOREM. *Let $T: X \rightarrow X$ be a k -set contraction ($k < 1$) and let x_0 be a fixed point of T . Suppose that T has Frechet derivative $T'(x_0)$ at x_0 and that unity is not an eigenvalue of $T'(x_0)$.*

Then x_0 is an isolated fixed point of T , and

$$\text{ind}(T, x_0) = (-1)^\nu,$$

where ν is the sum of the multiplicities of the eigenvalues greater than unity of $T'(x_0)$.

Proof. See [8].

2. Local bifurcation theorem. The first theorem shows that bifurcation from an isolated eigenvalue λ_0 of L having odd multiplicity is not dependent upon H being compact, but rather on how "close" H is to being compact.

THEOREM 2.1. *Let L be as above and let H satisfy $H - 1$. λ_0*

is an isolated normal eigenvalue of L having odd multiplicity. Assume that for $|\lambda - \lambda_0| < \varepsilon'$, $\|(L - \lambda)^{-1}Q_{\lambda_0}\|\gamma(H) \leq K < 1$. Then, $(\lambda_0, 0)$ is a bifurcation point for (1.1).

Proof. In order to prove this theorem, (1.1) will be rewritten in the form $u - C(\lambda, u) = 0$. Split (1.1) by

$$(2.1) \quad \begin{aligned} LP_{\lambda_0}u &= \lambda P_{\lambda_0}u + P_{\lambda_0}H(\lambda, u) \\ LQ_{\lambda_0}u &= \lambda Q_{\lambda_0}u + Q_{\lambda_0}H(\lambda, u). \end{aligned}$$

A solution of (1.1) is equivalent to a simulation solution of the two equations in (2.1). Select $\mu_0 \in \rho(L)$. Instead of (2.1) we may write

$$(2.2) \quad \begin{aligned} P_{\lambda_0}u &= \frac{(L - \mu_0)P_{\lambda_0}u}{\lambda - \mu_0} - \frac{P_{\lambda_0}H(\lambda, u)}{\lambda - \mu_0} \\ Q_{\lambda_0}u &= (L - \lambda)^{-1}Q_{\lambda_0}H(\lambda, u) \end{aligned}$$

where $(L - \lambda)^{-1}$ is to be interpreted as $(L - \lambda)^{-1}|_{\mathcal{N}_{\lambda_0}}$. Thus, (2.2) is valid for $\lambda \in \{\lambda_0\} \cup \{\rho(L) \setminus \{\mu_0\}\}$. Adding these equations we get

$$(2.3) \quad \begin{aligned} u &= C_1(\lambda, u) + C_2(\lambda, u) \\ C_1(\lambda, u) &= \frac{(L - \mu_0)P_{\lambda_0}u}{\lambda - \mu_0} \\ C_2(\lambda, u) &= \left((L - \lambda)^{-1}Q_{\lambda_0} - \frac{P_{\lambda_0}}{\lambda - \mu_0} \right) H. \end{aligned}$$

Note that $C_1: \mathcal{E} \rightarrow \mathcal{B}$ is compact and linear in u for each fixed λ . $C_2: \mathcal{E} \rightarrow \mathcal{B}$ satisfies $H - 1$. Define

$$(2.4) \quad \Phi(\lambda, \cdot) = I - C_1(\lambda, \cdot) - C_2(\lambda, \cdot).$$

Clearly, (2.3) or $\Phi(\lambda, u) = 0$ is equivalent to (1.1) for the specified values of λ when L is bounded. If L is unbounded, the question arises as to whether u is in $\text{dom}(L)$ if (λ, u) is a zero of Φ . Noting (2.2), which is obtained from (2.3) by projecting onto \mathcal{L}_{λ_0} , \mathcal{N}_{λ_0} respectively, we see that $Q_{\lambda_0}u$ is in $\text{dom}(L)$. Since $P_{\lambda_0}u$ is in an eigenspace of L , $u = P_{\lambda_0}u + Q_{\lambda_0}u$ is in $\text{dom}(L)$.

If the assertion of the theorem is not true we can find a neighborhood \mathcal{O} of $(\lambda_0, 0)$ such that the only solutions of (1.1) in \mathcal{O} are trivial solutions, $\rho(L) \setminus \mathcal{O}_R \neq \emptyset$, and $\mathcal{O}_R \cap \text{sp } L = \{\lambda_0\}$. Select $\mu_0 \in \rho(L) \setminus \mathcal{O}_R$ such that (1.1) is equivalent to (2.3) for all \mathcal{O}_R . Select $\varepsilon > 0$, $0 < \varepsilon < \varepsilon'$, that $[-\varepsilon + \lambda_0, \lambda_0 + \varepsilon] \times \{0\} \subset \mathcal{O}$. Applying the homotopy property of degree theory we obtain

$$(2.5) \quad \deg(\Phi(\lambda, \cdot), \mathcal{O}^\lambda, 0) = \text{constant}, \quad |\lambda - \lambda_0| < \varepsilon.$$

Select $\underline{\lambda}$ and $\bar{\lambda}$ such that $\lambda_0 - \varepsilon < \underline{\lambda} < \lambda_0 < \bar{\lambda} < \lambda_0 + \varepsilon$. Then

$$(2.6) \quad \begin{aligned} \deg(\Phi(\lambda, \cdot), \mathcal{O}^i, 0) &= \text{index}(I - C_1(\lambda, \cdot), (\lambda, 0)) \\ \deg(\Phi(\bar{\lambda}, \cdot), \mathcal{O}^{\bar{i}}, 0) &= \text{index}(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)). \end{aligned}$$

Thus, using (2.5) and (2.6),

$$(2.7) \quad \begin{aligned} &\text{index}(I - C_1(\lambda, \cdot), (\lambda, 0)) \\ &= \text{index}(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)). \end{aligned}$$

However, since the multiplicity of λ_0 is odd,

$$(2.8) \quad \begin{aligned} &\text{index}(I - C_1(\lambda, \cdot), (\lambda, 0)) \\ &= -\text{index}(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)). \end{aligned}$$

Since the indices in (2.7) and (2.8) are either $+1$ or -1 , we have a contradiction. Thus, such a neighborhood can never be found. This proves that $(\lambda_0, 0)$ is a bifurcation point.

REMARK 1. If $\lambda_0 \neq 0$ is an eigenvalue of L having odd multiplicity, then the hypotheses of Theorem 1 are satisfied if L is compact or if L is self-adjoint with λ_0 isolated in σL .

REMARK 2. The condition on $\gamma((L - \lambda)^{-1}Q_{\lambda_0}H)$ can be relaxed. If one restricted the operators to a ball B_0 centered at $(\lambda_0, 0)$ and then extended them to all of $R \times B_0$ in a linear manner, one could apply Theorem 1.1 if $\gamma((L - \lambda)^{-1}Q_{\lambda_0}H|B_0) = K < 1$, for $|\lambda - \lambda_0| < \varepsilon'$. This would handle the case that H is well behaved near $u = 0$ but grows too large for u far from 0.

(H - 2) $H: \mathcal{E} \rightarrow \mathcal{B}$ satisfies:

- (i) H is continuous, and bounded on each ball centered at 0.
- (ii) H is $o(\|u\|)$ uniformly on bounded λ intervals.

REMARK 3. The theorem remains true if H satisfies hypothesis H - 2 rather than the more restricted H - 1. The proof is very similar.

3. A global alternative theorem. In this section we will show that the local bifurcation exhibited in Theorem 2.1 is a global property with an alternative-type result.

For $\mathcal{V} \subset \mathcal{E}$, a subcontinuum of \mathcal{V} is a subset of \mathcal{V} which is closed and connected in \mathcal{E} . \mathcal{S} will denote the closure of the set of nontrivial solutions of (1.1) in \mathcal{E} . Let \mathcal{E}_{λ_0} denote the maximal subcontinuum of $\mathcal{S} \cup (\lambda_0, 0)$ containing $(\lambda_0, 0)$. B_ρ will denote the open ball in \mathcal{B} centered at 0 and having radius ρ . L and H will be as in § 2.

LEMMA 3.1. *Suppose λ_0 and λ_1 are distinct normal eigenvalues of L . Then $\mathcal{B} = \mathcal{L}_{\lambda_0} \oplus \mathcal{L}_{\lambda_1} \oplus \mathcal{N}$, a direct sum of subspaces, where $\mathcal{N} = \mathcal{N}_{\lambda_0} \cap \mathcal{N}_{\lambda_1}$, and $P = P_{\lambda_0} + P_{\lambda_1}$ projects onto $\mathcal{L}_{\lambda_0} \oplus \mathcal{L}_{\lambda_1}$ along \mathcal{N} . [5].*

LEMMA 3.2. *Let K be a compact metric space and A and B disjoint closed subsets of K . Then either there exists a subcontinuum of K meeting both A and B , or $K = K_A \cup K_B$ where K_A and K_B are disjoint compact subsets of K containing A and B respectively.*

Proof. See [9].

For λ_0 as before, define

$$\begin{aligned} \alpha_1(\lambda_0) &= \sup \{ \lambda \mid \lambda < \lambda_0, \lambda \in \text{sp}_{nd}(L) \} \\ \beta_1(\lambda_0) &= \inf \{ \lambda \mid \lambda > \lambda_0, \lambda \in \text{sp}_{nd}(L) \}. \end{aligned}$$

These values will be $\pm \infty$ respectively if the vacuous case results. For $\varepsilon_j > 0$, consider $I(\varepsilon_1, \varepsilon_2) = [\alpha_1(\lambda_0) + \varepsilon_1, \beta_1(\lambda_0) - \varepsilon_2]$. (Here assume both are finite.) Let $P_{\varepsilon_1, \varepsilon_2} = \sum P_\lambda$ where the summation is over all eigenvalues of L in $I(\varepsilon_1, \varepsilon_2)$, and let $Q_{\varepsilon_1, \varepsilon_2} = I - P_{\varepsilon_1, \varepsilon_2}$. Select $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\| (L - \lambda)^{-1} Q_{\varepsilon_1, \varepsilon_2} \| \gamma(H) < 1$ on $I(\varepsilon_1, \varepsilon_2)$. Let $[\alpha(\lambda_0), \beta(\lambda_0)] = I(\varepsilon_1, \varepsilon_2)$. If $\alpha_1(\lambda_0)$ or $\beta_1(\lambda_0)$ are infinite, select $\alpha(\lambda_0)$ or $\beta(\lambda_0)$ to be any appropriate finite number.

LEMMA 3.3 *Suppose λ_0 is an isolated normal eigenvalue of L having finite multiplicity. Assume \mathcal{E}_{λ_0} is bounded, $(\overline{\mathcal{E}_{\lambda_0}})_R \cap \{\alpha(\lambda_0), \beta(\lambda_0)\} = \emptyset$, and $\mathcal{E}_{\lambda_0} \cap \{\mathbf{R} \times \{\mathbf{0}\}\} = (\lambda_0, \mathbf{0})$. Then \mathcal{E}_{λ_0} is compact and there exists a bounded open set \mathcal{O} such that $\mathcal{E}_{\lambda_0} \subset \mathcal{O}$, $\partial \mathcal{O} \cap \mathcal{S} = \emptyset$, $(\overline{\mathcal{O}}_R) \cap \text{sp}_{nd}(L) = \emptyset$, the trivial solutions contained in \mathcal{O} are the points $(\lambda, \mathbf{0})$ where $|\lambda - \lambda_0| < \varepsilon$ for some $\varepsilon < \varepsilon_0 = \text{dist}(\lambda_0, \text{sp } L \setminus \{\lambda_0\})$, and $\|(\lambda, u) - (\mu, \mathbf{0})\| \geq 2\varepsilon_1$ for some positive ε_1 whenever $(\lambda, u) \in \partial \mathcal{O}$ and $\mu \in \text{sp } L$.*

Proof. \mathcal{E}_{λ_0} is compact. Indeed, let $\{(\lambda_n, u_n)\}_{n \in \mathcal{N}}$ be elements of \mathcal{E}_{λ_0} . Since \mathcal{E}_{λ_0} is bounded, we may find a λ and a subsequence \mathcal{N}_1 such that $\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}_1}} \lambda_n = \lambda$. Let P be the projection for $(\overline{\mathcal{E}_{\lambda_0}})_R$ and $Q = I - P$. Consider $\{u_n\}_{n \in \mathcal{N}_1}$.

$$\begin{aligned} \alpha(\{u_n\}_{n \in \mathcal{N}_1}) &= \alpha(\{C_1(\lambda_n, u_n) + C_2(\lambda_n, u_n)\}_{n \in \mathcal{N}_1}) \\ &\leq \alpha(\{C_1(\lambda_n, u_n)\}_{n \in \mathcal{N}_1} + \{C_2(\lambda_n, u_n)\}_{n \in \mathcal{N}_1}) \\ &\leq \alpha(\{C_2(\lambda_n, u_n)\}_{n \in \mathcal{N}_1}) \end{aligned}$$

$$\begin{aligned}
(\text{similarly}) &\leq \alpha(\{(L - \lambda_n)^{-1}QH(u_n)\}_{n \in \mathcal{S}_1}) \\
&\leq \alpha(\{(L - \lambda)^{-1}QH(u_n)\}_{n \in \mathcal{S}_1}) \\
&\quad + \alpha(\{(L - \lambda_n)^{-1} - (L - \lambda)^{-1}\}QH(u_n)\}_{n \in \mathcal{S}_1}) \\
&= \alpha(\{(L - \lambda)^{-1}QH(u_n)\}_{n \in \mathcal{S}_1}) \\
&\leq \|(L - \lambda)^{-1}Q\| \alpha(\{H(u_n)\}_{n \in \mathcal{S}_1}) \\
&\leq \|(L - \lambda)^{-1}Q\| \gamma(H) \alpha(\{u_n\}_{n \in \mathcal{S}_1}) \\
&< \alpha(\{u_n\}_{n \in \mathcal{S}_1}).
\end{aligned}$$

Thus $\alpha(\{u_n\}_{n \in \mathcal{S}_1}) = 0$ meaning the set is compact, meaning it has a convergent subsequence. Thus, C_{λ_0} is compact.

The remainder of the proof follows from [5] and [8] using Lemma 3.1.

The following theorem is modeled after one in [5] given for the case when H is compact.

THEOREM 3.1. *Suppose λ_0 is an isolated eigenvalue of L of odd multiplicity. L is as before and H satisfies $H - 1$. Furthermore, let $\|(L - \lambda_0)^{-1}Q_{\lambda_0}\| \gamma(H) < 1$. Then $(\lambda_0, 0)$ is a bifurcation point of (1.1) possessing a maximal continuous branch \mathcal{E}_{λ_0} such that exactly one of the following alternatives occurs.*

- (i) \mathcal{E}_{λ_0} is unbounded.
- (ii) \mathcal{E}_{λ_0} is bounded and $\overline{(\mathcal{E}_{\lambda_0})_R} \cap \{\alpha(\lambda_0), \beta(\lambda_0)\} \neq \emptyset$.
- (iii) \mathcal{E}_{λ_0} is compact, $(\mathcal{E}_{\lambda_0})_R \cap \{\alpha(\lambda_0), \beta(\lambda_0)\} = \emptyset$ and $\mathcal{E}_{\lambda_0} \cap \{\mathbf{R} \times \{0\}\} = \{\lambda_0, \lambda_1, \dots, \lambda_n\} \times \{0\}$ where $\lambda_1, \dots, \lambda_n$ are normal eigenvalues of L distinct from λ_0 , and the sum of the multiplicities of $\lambda_0, \lambda_1, \dots, \lambda_n$ is even.

Proof. With the use of Lemma 3.3, the proof is similar to Theorem 2.1 and Theorem 2.2 [5].

REMARK 1. The hypotheses of this theorem are unnecessarily stringent. The same results hold with $H(\lambda, u)$ if $\|(L - \lambda_0)^{-1}Q_{\lambda_0}\| \gamma(H(\lambda_0, \cdot)) < 1$, where H satisfies $H - 2$. The preceding proofs, however, become a little more complicated mainly due to notation.

REMARK 2. Suppose that $H(\lambda, \cdot)$, when restricted to a ball centered at $u = 0$, has $\|(L - \lambda_0)^{-1}Q_{\lambda_0}\| \gamma(H(\lambda_0, \cdot)) < 1$ (but this hypothesis fails on the entire space). One can do the degree work on these balls (by reworking all previous proofs) and obtain a theorem similar to that in [4]. (It was necessary to make a change in that theorem due to an error committed in [4] (see the next section).)

Assign $F(\varepsilon) = [\alpha(\lambda_0) + \varepsilon, \beta(\lambda_0) - \varepsilon]$. Let P_ε correspond to $F(\varepsilon)$, and $Q_\varepsilon = I - P_\varepsilon$. When restricted to a ball of radius r centered at

0, let $H_{(\lambda, \cdot)}$ be a $\gamma_r(H(\lambda, \cdot))$ -set contraction, and define $\gamma_r(H)$ to be strictly monotone increasing.

THEOREM 3.2. *Let λ_0 be an isolated eigenvalue of L having odd algebraic multiplicity. L is as before and H satisfies $H - 2$. Then $(\lambda_0, 0)$ is a bifurcation point of (1.1) and emanating from it is a maximal continuous branch \mathcal{E}_{λ_0} which obeys exactly one of the following alternatives for each suitably small $\varepsilon > 0$.*

(i) \mathcal{E}_{λ_0} is unbounded.

(ii) \mathcal{E}_{λ_0} is bounded and $\overline{\mathcal{E}_{\lambda_0}}$ meets $S_\varepsilon = \{(\lambda, u) \mid \lambda \in F(\varepsilon) \text{ and } \|u\| = r \text{ where } \gamma_r(H(\lambda, \cdot)) = \|(L - \lambda)^{-1}Q_\varepsilon\|^{-1}\} \cup \{(\alpha(\lambda_0) + \varepsilon) \times \mathcal{B}\} \cup \{(\beta(\lambda_0) - \varepsilon) \times \mathcal{B}\}$.

(iii) \mathcal{E}_{λ_0} is compact, $\overline{\mathcal{E}_{\lambda_0}}$ does not meet S_ε , and $\mathcal{E}_{\lambda_0} \cap \{0 \times B\} = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$, each a distinct normal eigenvalue of L , and the sum of their algebraic multiplicities is even.

REMARK 1. In the case where L is self-adjoint, $\|(L - \lambda)^{-1}Q_\varepsilon\| = 1/\text{dist}(\lambda, \text{sp}(L)/F(\varepsilon))$ where $\text{dist}(\cdot)$ is the standard distance function in \mathbf{R} . This simplifies the statement of (ii).

REMARK 2. If $(\alpha(\lambda_0), \beta(\lambda_0)) \cap \text{sp}(L)$ consists of a finite list of eigenvalues, there is an $\varepsilon_0 > 0$ such that whenever $0 < \varepsilon_1 < \varepsilon_2 \leq \varepsilon_0$, S_{ε_1} and S_{ε_2} are identical in $F(\varepsilon_2) \times B$. This is because $\|(L - \lambda)^{-1}Q_\varepsilon\|$ is constant in ε for $0 < \varepsilon < \varepsilon_0$. This leads to an improvement in (ii) and (iii).

(ii)' \mathcal{E}_{λ_0} is bounded and $\overline{\mathcal{E}_{\lambda_0}}$ meets $S = \{(\lambda, u) \mid \lambda \in (\alpha(\lambda_0), \beta(\lambda_0)) \text{ and } \|u\| = r \text{ where } \gamma_r(H(\lambda, \cdot)) = \|(L - \lambda)^{-1}Q_{\varepsilon_0}\|^{-1}\} \cup \{\alpha(\lambda_0) \times B\} \cup \{\beta(\lambda_0) \times B\}$.

(iii)' \mathcal{E}_{λ_0} is compact, $\overline{\mathcal{E}_{\lambda_0}}$ does not meet S , and $\mathcal{E}_{\lambda_0} \cap \{0 \times B\} = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$, each a distinct normal eigenvalue of L , and the sum of their algebraic multiplicities is even.

4. **Other results.** The theorems I proposed in [4] are unfortunately incorrect as stated and require modification as in §3 of this paper. The hypothesis of continuity on H had to be strengthened. My error was in a proof that if one restricted H to a ball centered at 0 in B and on that ball $\|H\| = k$, then H was a k -set contraction on the ball. This is true for linear operators.

This error was found by Professor Norman Dancer, The University of New England, Armidale N.S.W., Australia. He constructed a counterexample to Theorem I of [4], which I present here. There is an operator $V: c_0 \rightarrow c_0$ such that if $x = \lambda V(x)$, then $x = 0$ and $\lambda = 0$. Set $B = c_0 \times \mathbf{R}$, $L: B \rightarrow B$ is defined by $L(w, t) = (2w, t)$ and $H: \mathbf{R} \times B \rightarrow B$ is defined by $H(\lambda, (w, t) = (0, \lambda t^2 V(w))$.

$\lambda = 1$ is an eigenvalue of L of multiplicity 1. If λ is near 1 and $Lu = \lambda u + H(\lambda, u)$ where $u = (w, t)$, then $t = \lambda t$ and $2w = \lambda w + \lambda t^2 V(w)$. $w = \lambda t^2 V(w)/(2 - \lambda)$ which implies $w = 0$, and together with λ being near 1 imply $t = 0$. Thus, for λ near 1, the only solution is $u = 0$. Many thanks to Professor Dancer. The operator V is due to Ana and Vasile Istratescu and appeared in the Proceeding of the Amer. Math. Soc., Vol. 48, No. 1, page 197.

REFERENCES

1. N. Dunford and J. T. Schwarz, *Linear Operators*, Part II, Interscience, New York, 1963).
2. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, New York, 1964.
3. J. A. MacBain, *Global bifurcation theorems for noncompact operators*, Bulletin Amer. Math. Soc., **80** (1974),
4. ———, *Global bifurcation theorems for nonlinearly perturbed operator equation*, Bulletin Amer. Math. Soc., **82** (1976).
5. ———, *Local and global bifurcation from normal eigenvalues*, Pacific J. Math., **63** (1976), 445-466.
6. P. H. Rabinowitz, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math., **3** (1973).
7. F. Riesz, B. Sz.-Nagy, *Functional Analysis*, tr. L. Boron, Ungar, New York, 1971.
8. C. A. Stuart, *Some bifurcation theory for k -set contractions*, Proc. London Math. Soc., (1973).
9. G. T. Whyburn, *Topological Analysis*, Princeton University Press, Princeton, 1958.

Received February 16, 1977.

AIR FORCE INSTITUTE OF TECHNOLOGY
WRIGHT-PATTERSON AIR FORCE BASE, OH 45433

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

C. W. CURTIS

University of Oregon
Eugene, OR 97403

C. C. MOORE

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. FINN AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
OSAKA UNIVERSITY

Gerald Arthur Anderson, <i>Computation of the surgery obstruction groups</i> $L_{4k}(1; Z_p)$	1
R. K. Beatson, <i>The degree of monotone approximation</i>	5
Sterling K. Berberian, <i>The character space of the algebra of regulated functions</i>	15
Douglas Michael Campbell and Jack Wayne Lamoreaux, <i>Continua in the plane with limit directions</i>	37
R. J. Duffin, <i>Algorithms for localizing roots of a polynomial and the Pisot Vijayaraghavan numbers</i>	47
Alessandro Figà-Talamanca and Massimo A. Picardello, <i>Functions that operate on the algebra $B_0(G)$</i>	57
John Erik Fornæss, <i>Biholomorphic mappings between weakly pseudoconvex domains</i>	63
Andrzej Granas, Ronald Bernard Guenther and John Walter Lee, <i>On a theorem of S. Bernstein</i>	67
Jerry Grossman, <i>On groups with specified lower central series quotients</i>	83
William H. Julian, Ray Mines, III and Fred Richman, <i>Algebraic numbers, a constructive development</i>	91
Surjit Singh Khurana, <i>A note on Radon-Nikodým theorem for finitely additive measures</i>	103
Garo K. Kiremidjian, <i>A Nash-Moser-type implicit function theorem and nonlinear boundary value problems</i>	105
Ronald Jacob Leach, <i>Coefficient estimates for certain multivalent functions</i>	133
John Alan MacBain, <i>Local and global bifurcation from normal eigenvalues. II</i>	143
James A. MacDougall and Lowell G. Sweet, <i>Three dimensional homogeneous algebras</i>	153
John Rowlay Martin, <i>Fixed point sets of Peano continua</i>	163
R. Daniel Mauldin, <i>The boundedness of the Cantor-Bendixson order of some analytic sets</i>	167
Richard C. Metzler, <i>Uniqueness of extensions of positive linear functions</i>	179
Rodney V. Nillsen, <i>Moment sequences obtained from restricted powers</i>	183
Keiji Nishioka, <i>Transcendental constants over the coefficient fields in differential elliptic function fields</i>	191
Gabriel Michael Miller Obi, <i>An algebraic closed graph theorem</i>	199
Richard Cranston Randell, <i>Quotients of complete intersections by C^* actions</i>	209
Bruce Reznick, <i>Banach spaces which satisfy linear identities</i>	221
Bennett Setzer, <i>Elliptic curves over complex quadratic fields</i>	235
Arne Stray, <i>A scheme for approximating bounded analytic functions on certain subsets of the unit disc</i>	251
Nicholas Th. Varopoulos, <i>A remark on functions of bounded mean oscillation and bounded harmonic functions. Addendum to: "BMO functions and the $\bar{\partial}$-equation"</i>	257
Charles Irvin Vinsonhaler, <i>Torsion free abelian groups quasi-projective over their endomorphism rings. II</i>	261
Thomas R. Wolf, <i>Characters of p'-degree in solvable groups</i>	267
Toshihiko Yamada, <i>Schur indices over the 2-adic field</i>	273