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## **UNIQUENESS OF EXTENSIONS OF POSITIVE LINEAR FUNCTIONS**

RICHARD C. METZLER

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**In this paper necessary and sufficient conditions that every approximated function has a unique maximal approximated extension are given. When applied to the Choquet situation this gives a new approach to known uniqueness results for representing measures.**

Extensions of positive linear functions preserving a certain approximation property were studied in a previous paper [4]. This led to a unified approach to integration theory and the Choquet-Bishop-de Leeuw theorem.

Notation will be that of [4]. In particular  $V$  and  $Y$  will designate ordered vector spaces;  $G$  will be a subspace of  $V$ ;  $W$  will be a wedge in  $V$  such that  $G \subset W \subset G + V^+$  and  $\alpha$  will be a positive linear function from  $G$  to  $Y$ .

If  $f \in V$  and  $A \subset V$  we say that  $f$   $\alpha$ -dominates  $A$  if, for every  $g$  in  $A$  such that  $g \leq f$ , the following holds; for every pair  $y, z$  in  $Y$  such that  $y \geq \alpha(f - V^+)$  and  $z \leq \alpha(g + V^+)$  we have  $y \geq z$ . This is the condition of Theorem 2.1 in [4]. If  $\underline{\alpha}(f) = \sup \{\alpha(h) : f \geq h \in G\}$  and  $\bar{\alpha}(g) = \inf \{\alpha(h) : g \leq h \in G\}$  both exist in  $Y$ , this condition is equivalent to the requirement  $\underline{\alpha}(f) \geq \bar{\alpha}(g)$ . Now the proof of Theorem 2.1 can be modified easily to yield the following: If  $f \in W$  and  $f$   $\alpha$ -dominates  $(-W)$  then  $f$  is in  $\text{dmn } \underline{\alpha}$  if and only if all maximal  $W$ -extensions are defined and give the same value on  $f$ . If  $Y$  is assumed Dedekind complete the converse holds; the equivalence implies that  $f$   $\alpha$ -dominates  $(-W)$ .

We define a "closure",  $G_1$ , of a subspace  $G \subset V$  as  $G_1 = \{f \in V : \exists g \in G^+ \text{ with } (f - \lambda g + V^+) \cap (f + \lambda g - V^+) \cap G \neq \emptyset \text{ for all } \lambda > 0\}$ . It is easy to see that  $G_1$  is a subspace containing  $G$ . If  $G$  contains an order unit  $u$ , then  $G_1$  is just the closure of  $G$  in the order-unit normed space  $(Ru + V^+) \cap (Ru - V^+)$ .

If  $f \in V$  and  $A, B \subset V$  we say that  $A$  separates  $f$  and  $B$  if, for each  $g \in B$ , there exists  $h \in A$  such that  $f \geq h \geq g$ .

**THEOREM 1.** *Let  $Y$  be an Archimedean space. If  $f \in W$  is such that  $G_1$  separates  $f$  and  $(-W) \cap (f - V^+)$  then  $f$   $\alpha$ -dominates  $(-W)$  for every  $\alpha$ . Consequently if the separation holds for all  $f \in W$  then every  $\alpha$  has a unique maximal  $W$ -extension.*

*Proof.* Suppose  $f \geq g \in (-W)$ . By hypothesis there exists  $h \in G_1$

such that  $f \geq h \geq g$ . By definition of  $G_1$  there is  $p \in G^+$  such that, for any  $\delta > 0$ , there is  $q_\delta \in G$  with  $h - \delta p \leq q_\delta \leq h + \delta p$ . Then  $q_\delta + \delta p \in (g + V^+) \cap G$  so if  $z \leq \alpha(g + V^+)$  we have  $z \leq \alpha(q_\delta + \delta p)$ . Similarly if  $y \geq \alpha(f - V^+)$  we find  $y \geq \alpha(q_\delta - \delta p)$ . Then  $z - y \leq 2\delta\alpha(p)$  and the Archimedean property of  $Y$  gives  $y \geq z$ .

Now we wish to investigate under what conditions the unique extension property implies the  $G_1$  separation of Theorem 1.

**LEMMA.** *Assume  $G$  has an order unit  $u$  and suppose  $Y^+ \neq \{0\}$ . If every  $\alpha$  has a unique maximal  $W$ -extension then, for every  $f \in W \cap (G - V^+)$  and  $\varepsilon > 0$ ,  $G$  separates  $f + \varepsilon u$  and  $(-W) \cap (f - V^+) \cap (G + V^+)$ .*

*Proof.* We will suppose that there is  $f \in W \cap (G - V^+)$  and  $\varepsilon > 0$  such that  $G$  does not separate  $f + \varepsilon u$  and  $(-W) \cap (f - V^+) \cap (G + V^+)$ . Then there exists  $g \in (-W) \cap (f - V^+) \cap (G + V^+)$  such that if  $A = \{h \in G: h \geq g + (\varepsilon/2)u\}$ ,  $B = \{h \in G: h \leq f + \varepsilon u\}$  and  $U = \{h \in G: -(\varepsilon/2)u \leq h \leq (\varepsilon/2)u\}$  then  $(A + U) \cap B = \emptyset$ . Since  $U$  is radial at the origin as a subset of  $G$  a standard separation result [3; p. 23] shows that there exists a linear functional  $\varphi$  on  $G$  which strongly separates  $A$  and  $B$ . By taking  $-\varphi$  if necessary we can assume that  $r_0 = \sup\{\varphi(p): p \in B\} < \inf\{\varphi(q): q \in A\} = s_0$ . Now let  $p \in G^+$ . Then if  $f_1 \in B$  and  $f_2 \in A$  we have  $\varphi(f_2 + rp) = \varphi(f_2) + r\varphi(p) \geq \varphi(f_1)$  for all  $r \geq 0$ . This shows that  $\varphi(p) \geq 0$  and we see that  $\varphi$  is a positive linear functional on  $G$ . Then we have  $\varphi(f + \varepsilon u) = r_0 < s_0 = \varphi(g + (\varepsilon/2)u)$ . Since  $g + (\varepsilon/2)u \in (-W) + G \subset (-W)$  we see that  $f + \varepsilon u$  does not  $\varphi$ -dominate  $(-W)$ . Since  $R$  is Dedekind complete Theorem 2.1 of [4] shows that  $\varphi$  does not have a unique maximal  $W$ -extension. Now choose  $y > 0$  in  $Y$  and define  $\alpha: G \rightarrow Y$  by  $\alpha(f) = \varphi(f)y$ . Then it is easy to see that  $\alpha$  does not have a unique maximal  $W$ -extension.

**THEOREM 2.** *In addition to the assumptions of the lemma we assume that  $V$  is Dedekind  $\sigma$ -complete and  $W$  is closed under finite infs. Then, if every  $\alpha$  has a unique maximal  $W$ -extension,  $G_1$  separates  $f$  and  $(f - V^+) \cap (-W)$  for all  $f \in W$ .*

*Proof.* Given any  $f \in W$  and  $g \leq f$  such that  $g \in -W$  we wish to show that  $G_1$  separates  $f$  and  $g$ . Now we can assume, without loss of generality, that  $f \in G - V^+$  and  $g \in G + V^+$ . If this were not so we could choose  $g' \in G \cap (f - V^+)$  and (using the assumptions that  $W$  is closed under finite infs and  $W \subset G + V^+$ )  $f' \in G \cap (g \vee g' + V^+)$ . Then we could replace  $f$  by  $f \wedge f'$  and  $g$  by  $g \vee g'$ . Clearly any element which separates  $f \wedge f'$  and  $g \vee g'$  will separate  $f$  and  $g$ .

We adapt a technique of Edwards [2] to find  $h \in G_1$  such that

$f \geq h \geq g$ . Let  $g_0 = g - u$  and  $f_0 = f + u$  and use the lemma to choose  $h_0 \in G$  such that  $g_0 \leq h_0 \leq f_0$ . Now assume that for  $m = 1, 2, \dots, n$  we have  $f_m \in W$ ,  $g_m \in -W$  and  $h_m \in G$  such that  $g - 2^{-m}u \leq g_m \leq h_m \leq f_m \leq f + 2^{-m}u$  and  $-3 \cdot 2^{-m-1}u \leq h_m - h_{m-1} \leq 3 \cdot 2^{-m-1}u$ .

Let

$$g_{n+1} = (g - 2^{-n-1}u) \vee (h_n - 3 \cdot 2^{-n-2}u) \in -W$$

while

$$\begin{aligned} f_{n+1} &= (f + 2^{-n-1}u) \wedge (h_n + 3 \cdot 2^{-n-2}u) \\ &= (f + 2^{-n-2}u) \wedge (h_n + 2^{-n-1}u) + 2^{-n-2}u \in W. \end{aligned}$$

Now  $g_{n+1} + 2^{-n-2}u \leq f_{n+1}$  results from the following inequalities:

$$\begin{aligned} g - 2^{-n-1}u &\leq f + 2^{-n-2}u; & g - 2^{-n-1}u &\leq h_n + 2^{-n-1}u; \\ h_n - 3 \cdot 2^{-n-2}u &\leq f + 2^{-n-2}u; & \text{and } h_n - 3 \cdot 2^{-n-2}u &\leq h_n + 2^{-n-1}u. \end{aligned}$$

Hence we can use the lemma to choose  $h_{n+1} \in G$  such that  $g - 2^{-n-1}u \leq g_{n+1} \leq h_{n+1} \leq f_{n+1} \leq f + 2^{-n-1}u$  and  $-3 \cdot 2^{-n-2}u \leq g_{n+1} - h_n \leq h_{n+1} - h_n \leq f_{n+1} - h_n \leq 3 \cdot 2^{-n-2}u$ . This completes the inductive definition.

Now  $-3 \cdot 2^{-n-2}u \leq h_{n+1} - h_n \leq 3 \cdot 2^{-n-2}u$  implies  $-3 \cdot 2^{-m-1}u \leq h_p - h_m \leq 3 \cdot 2^{-m-1}u$  for all  $p \geq m$ .

Now let  $h = \inf_n (\sup_{k \geq n} h_k)$  which exists by the inequality for  $h_p - h_m$  and the fact that  $V$  is Dedekind  $\sigma$ -complete. From the inequalities  $g - 2^{-n-1}u \leq h_{n+1} \leq f + 2^{-n-1}u$  we conclude, since a Dedekind  $\sigma$ -complete space is Archimedean, that  $g \leq h \leq f$ . Since we can replace  $h_p$  by  $h$  in the inequality for  $h_p - h_m$  we see that  $h \in G_1$  as desired.

Now in the approach to Choquet boundary theory given in [4] we assume that  $V$  is the space of continuous functions on a compact Hausdorff space  $X$ ,  $G$  is a closed subspace and  $W$  is a wedge of bounded continuous functions on  $X$  closed under finite infs. Then  $G = G_1$  and, since  $W$ -approximated linear functionals are maximal measures, we see that uniqueness of representing "Choquet" measures implies the separation of Theorem 2. This gives the "geometric simplex" result of Boboc and Cornea [1, Th. 4]. If we let  $X$  be a convex compact subset of a locally convex space,  $G$  the continuous affine functions and  $W$  the wedge of finite infs from  $G$  then we find that the separation property reduces in this case to the interpolation version of the Riesz decomposition property. This gives the "Choquet simplex" result of Edwards [2].

We now investigate an alternate characterization of the space  $G_1$ . We define  $G_Y$  to be the largest subspace of  $V$  such that every positive linear  $\alpha: G \rightarrow Y$  has a unique positive linear extension to  $G_Y$ . In the notation of [4] we can write  $G_Y$  as  $\bigcap \{ \text{dmn } \alpha_G : \alpha \text{ positive} \}$

and linear from  $G$  to  $Y$ ).

**THEOREM 3.** *If  $Y$  is Dedekind  $\sigma$ -complete  $G_1 \subset G_Y$ .*

*Proof.* If  $f \in G_1$  then there exists  $g \in G^+$  and a sequence  $\{h_n\} \subset G$  such that  $h_n - (1/n)g \leq f \leq h_n + (1/n)g$  for all  $n$ . This gives us

$$-\left(\frac{1}{n} + \frac{1}{m}\right)g \leq h_n - h_m \leq \left(\frac{1}{n} + \frac{1}{m}\right)g$$

for all  $n$  and  $m$ . Now let  $\alpha$  be any positive linear function from  $G$  to  $Y$ . Then

$$-\left(\frac{1}{n} + \frac{1}{m}\right)\alpha(g) \leq \alpha(h_n) - \alpha(h_m) \leq \left(\frac{1}{n} + \frac{1}{m}\right)\alpha(g)$$

gives  $-(1/m)\alpha(g) \leq y - \alpha(h_m) \leq (1/m)\alpha(g)$  for  $y = \inf_n (\sup_{k \geq n} \alpha(h_k))$ . Then  $h_n - (1/n)g \leq f \leq h_n + (1/n)g$  for all  $n$  implies  $\bar{\alpha}(f) \leq y \leq \underline{\alpha}(f)$ . From this it is not hard to see that every maximal positive extension of  $\alpha$  assumes the value  $y$ . Since  $\alpha$  was arbitrary we conclude that  $f \in G_Y$ .

**THEOREM 4.** *If  $G$  has an order-unit and  $Y^+ \neq \{0\}$  then  $G_Y \subset G_1$ .*

*Proof.* Note first that if  $f \in G_Y$  we must have  $f \in (G - V^+) \cap (G + V^+)$ . For if  $f$  is not in  $G - V^+$  let  $\hat{\alpha}$  be a maximal positive extension of a positive linear  $\alpha$  from  $G$  to  $Y$ . Choose  $y > 0$  in  $Y$  and define  $\alpha_1$  by  $\alpha_1(g + rf) = \alpha(g) + r(\hat{\alpha}(f) + y)$ . Then since  $\hat{\alpha}$  is positive and  $f \notin G - V^+$  it is easy to see that  $\alpha_1$  is positive. Then any maximal extension of  $\alpha_1$  contradicts  $f \in G_Y$ . A symmetric argument gives  $f \in G + V^+$ .

Now if  $f \in G_Y$  let  $W = G + Rf$ . Then the lemma applies and we can assume that  $G$  separates  $f + \varepsilon u$  and  $(-W) \cap (f - V^+) \cap (G + V^+)$  for all  $\varepsilon > 0$ . Since  $f \in (W) \cap (-W) \cap (f - V^+) \cap (G + V^+)$  we see there exists  $h_\varepsilon \in G$  such that  $f \leq h_\varepsilon \leq f + \varepsilon u$  for all  $\varepsilon > 0$ . Hence  $f \in G_1$  as desired.

#### REFERENCES

1. N. Boboc and A. Cornea, *Cônes des fonctions continues sur un espace compact*, C. R. Acad. Sc. Paris, t., **261** (October 4, 1965), 2564-2567.
2. D. A. Edwards, *Séparation des fonctions réelles définies sur un simplexe de Choquet*, C. R. Acad. Sc. Paris, t., **261** (October 1965), 2798-2800.
3. J. Kelley and I. Namioka, *Linear Topological Spaces*, D. Van Nostrand Co., Inc., Princeton, New Jersey, (1963).
4. R. C. Metzler, *Positive linear functions, integration, and Choquet's theorem*, Pacific J. Math., **60** (1975), 277-296.

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