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MOMENT SEQUENCES OBTAINED FROM RESTRICTED POWERS

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Let $(m_n)_{n=1}^{\infty}$ be an increasing divergent sequence of positive numbers. Then we are interested in characterising those sequences $(\alpha_n)_{n=1}^{\infty}$ for which $\alpha_n = \int_0^1 x^{m_n} f(x) dx$ for $n = 1, 2, \dots$ and some $f \in L^2([0, 1])$. It is shown that if $(m_n)_{n=1}^{\infty}$ diverges sufficiently rapidly, then $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ if and only if $\alpha_n = \sqrt{m_n} \int_0^1 x^{m_n} f(x) dx$ for $n = 1, 2, \dots$ and some $f \in L^2([0, 1])$. It is also shown that if $(m_n)_{n=1}^{\infty}$ is a lacunary sequence of integers then the Hilbert subspace of $L^2([0, 1])$ generated by the functions x^{m_n} ($n = 1, 2, \dots$) has a reproducing kernel.

1. Introduction. Let $C([0, 1])$ denote all the complex valued continuous functions on $[0, 1]$. If $\alpha \geq 0$ let e_α be the function in $C([0, 1])$ given by $x \rightarrow x^\alpha$. Throughout the paper, S will be a given sequence $(m_n)_{n=1}^{\infty}$ of positive real numbers so that $0 \leq m_1 < m_2 < m_3 < \dots$ and $\lim_{n \rightarrow \infty} m_n = \infty$. The subspace of $C([0, 1])$ obtained by taking the uniform closure of the vector space generated by $\{e_{m_i} : i = 1, 2, \dots\}$ will be denoted by $M(S)$.

A classical result due to Müntz and Szasz (see, for example, [2] p. 272) says that if $m_1 = 0$, $M(S) = C([0, 1])$ if and only if $\sum_{i=2}^{\infty} 1/m_i = \infty$. In the case where $\sum_{i=2}^{\infty} 1/m_i < \infty$ it can be shown that e_α is not in $M(S)$ unless $\alpha = m_i$ for some i ([7], p. 305). It follows from this, using the Hahn-Banach theorem, that if $\sum_{i=2}^{\infty} 1/m_i < \infty$ and if j is given, then there is a measure μ so that $\int_0^1 x^{m_i} d\mu(x) = 0$ if $i \neq j$ and $\int_0^1 x^{m_j} d\mu(x) \neq 0$. Among the results of this paper, it is shown that if S satisfies a certain stronger condition than $\sum_{i=2}^{\infty} 1/m_i < \infty$, the measure μ can be chosen to be absolutely continuous with respect to Lebesgue measure and at the same time be supported by $[0, \delta]$, where $\delta > 0$ is preassigned.

We also let L^2 be the Hilbert space of all square integrable functions on $[0, 1)$ and we denote by $A(S)$ the subspace of L^2 obtained by taking the closure, in the L^2 norm, of the vector space generated by $\{e_{m_n} : n = 1, 2, \dots\}$. Any function in $M(S)$, if restricted to $[0, 1)$, belongs to $A(S)$. $S = (m_n)_{n=1}^{\infty}$ is said to be *lacunary* (or a *Hadamard set*) if there is $\alpha > 1$ so that $m_{n+1} > \alpha m_n$, for $n = 1, 2, 3, \dots$. Lacunary sets are well known in complex analysis ([8], pp. 314-316) and in harmonic analysis ([7], pp. 100-118). We show that if S is lacunary then $A(S)$ has a reproducing kernel.

Finally we shall be concerned with characterizing those sequences

$(\alpha_n)_{n=1}^\infty$ which are of the form $\alpha_n = \int_0^1 f(x)x^{m_n}dx$ for $n = 1, 2, \dots$, for some $f \in L^2$. As noted in [3], vol. II, pp. 139-140, any such sequence $(\alpha_n)_{n=1}^\infty$ belongs to ℓ^2 , the Hilbert space of all square summable sequences on the positive integers. It is shown that, regardless of what S is, it is never possible to obtain all of ℓ^2 simply by taking different functions f in L^2 . However it is possible to obtain all of ℓ^2 in this way if we consider instead sequences $(\alpha_n)_{n=1}^\infty$ of the form $\alpha_n = \sqrt{m_n} \int_0^1 x^{m_n} f(x) dx$, provided the sequence S diverges rapidly enough.

The basic idea underlying a number of our results is that provided there are "sufficiently large" gaps between m_i and m_{i+1} for $i = 1, 2, \dots$, then the functions $\{e_{m_i}; i = 1, 2, \dots\}$ are "sufficiently orthogonal" for them to be treated (in a certain sense) as orthogonal functions.

2. Properties of $A(S)$. In L^2 the Gram-Schmidt process can be applied to the functions e_{m_1}, e_{m_2}, \dots to obtain an orthonormal sequence p_1, p_2, \dots . Of course we can write $p_n = \sum_{j=1}^n a_{nj} e_{m_j}$ or

$$(2.1) \quad p_n(x) = \sum_{j=1}^n a_{nj} x^{m_j}, \quad \text{for } 0 \leq x < 1 \text{ and } n = 1, 2, 3, \dots$$

If the inner product in $L^2(0, 1)$ is denoted by (\cdot, \cdot) then also we have

$$(2.2) \quad (p_n, p_m) = 0 \quad \text{if } m \neq n, \text{ and } 1, \text{ if } m = n.$$

Then functions in $A(S)$ are precisely those functions of the form $\sum_{n=1}^\infty \alpha_n p_n$ for some sequence $(\alpha_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$, where the series is to be interpreted in terms of the L^2 norm. We shall assume that the Gram-Schmidt process has been carried out so that $a_{nn} > 0$ for all n , in which case the constants a_{nj} in (2.1) are uniquely determined. If S is a sequence of integers, it should be noted that the p_n are polynomials and can be regarded as being defined on $D = \{\lambda: |\lambda| < 1\}$.

LEMMA 2.1.

$$(2.3) \quad a_{11} = \sqrt{2m_1 + 1} \quad \text{and} \quad a_{21} = -\sqrt{2m_2 + 1} \left[\frac{2m_1 + 1}{m_2 - m_1} \right].$$

If $n > 2$ and $j < n$ we have

$$(2.4) \quad a_{nj} = (-1)^{n+j} \sqrt{2m_n + 1} \left[\frac{2m_j + 1}{m_n - m_j} \right] \prod_{\substack{i=1 \\ i \neq j}}^{n-1} \left[\frac{m_i + m_j + 1}{|m_i - m_j|} \right].$$

If $n > 1$ we have

$$(2.5) \quad a_{nn} = \sqrt{2m_n + 1} \prod_{i=1}^{n-1} \left[\frac{m_n + m_i + 1}{m_n - m_i} \right].$$

Proof. If $f_1, f_2, \dots, f_n \in L^2([0, 1])$, let $g(f_1, f_2, \dots, f_n)$ be the determinant of $((f_i, f_j))_{1 \leq i, j \leq n}$ (the Gramian determinant, see [2], p. 177). Then by [2], p. 183 we have

$$(2.6) \quad p_n = \frac{1}{\sqrt{g(e_{m_1}, e_{m_2}, \dots, e_{m_n})g(e_{m_1}, e_{m_2}, \dots, e_{m_{n-1}})}} \begin{vmatrix} (e_{m_1}, e_{m_1}) & \dots & (e_{m_n}, e_{m_1}) \\ \vdots & & \vdots \\ (e_{m_1}, e_{m_{n-1}}) & \dots & (e_{m_n}, e_{m_{n-1}}) \\ e_{m_1} & \dots & e_{m_n} \end{vmatrix}.$$

Since $(e_{m_i}, e_{m_j}) = 1/(m_i + m_j + 1)$, it is possible to use (2.6) to find the a_{nj} and we find that (2.3), (2.4), and (2.5) are true. The calculation is tedious but straight forward and is similar to one used in one proof of the Müntz-Szasz theorem (see [2], pp. 270-271).

LEMMA 2.2. *S is lacunary if and only if there is a number C so that for all $n \geq 2$,*

$$(2.7) \quad \left| \prod_{\substack{i=1 \\ i \neq j}}^n \left(\frac{m_i + m_j + 1}{m_i - m_j} \right) \right| \leq C, \quad \text{when } j \leq n.$$

Proof. Each term of the product in (2.7) is greater than 1, so that if (2.7) holds, $C > 1$. In this case, for $n > 1$ we will have $(m_n + m_{n-1} + 1) \leq C(m_n - m_{n-1})$ so that $m_n \geq (C + 1)/(C - 1)m_{n-1}$, and S must be lacunary.

Conversely, if S is lacunary and $m_1 > 0$, choose $\alpha > 1$ so that $m_{i+1} > \alpha m_i$ for $i = 1, 2, \dots$. Then if $i > j$ we have

$$(2.8) \quad m_i > \alpha^{i-j} m_j.$$

If $x \geq 0$ then $1 + x \leq e^x$ so that if $j \leq n - 1$,

$$(2.9) \quad \prod_{i=j+1}^n \left(\frac{m_i + m_j + 1}{m_i - m_j} \right) = \prod_{i=j+1}^n \left(1 + \frac{2m_j + 1}{m_i - m_j} \right) \leq e^{s_{j,n}},$$

where

$$s_{j,n} = \sum_{i=j+1}^n \left(\frac{2m_j + 1}{m_i - m_j} \right).$$

If $j > 1$, we also have

$$(2.10) \quad \prod_{i=1}^{j-1} \left(\frac{m_i + m_j + 1}{m_j - m_i} \right) = \prod_{i=1}^{j-1} \left(1 + \frac{2m_i + 1}{m_j - m_i} \right) \leq e^{t_j},$$

where

$$t_j = \sum_{i=1}^{j-1} \left(\frac{2m_i + 1}{m_j - m_i} \right).$$

Now

$$s_{j,n} = 2 \sum_{i=j+1}^n \left(\frac{1}{m_i/m_j - 1} \right) + \sum_{i=j+1}^n \left(\frac{1}{m_i - m_j} \right), \quad \text{by (2.9),}$$

$$< 2 \sum_{i=j+1}^{\infty} \frac{1}{\alpha^{i-j} - 1} + \sum_{i=j+1}^{\infty} \frac{1}{m_j(\alpha^{i-j} - 1)}, \quad \text{by (2.8),}$$

$$< 2 \sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1} + \frac{1}{m_1} \left(\sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1} \right), \quad \text{and}$$

$$t_j = 2 \sum_{i=1}^{j-1} \frac{1}{m_j/m_i - 1} + \sum_{i=1}^{j-1} \frac{1}{m_i(\alpha^{j-i} - 1)}, \quad \text{by (2.8) and (2.10),}$$

$$< 2 \sum_{i=1}^{j-1} \frac{1}{\alpha^{j-i} - 1} + \sum_{i=1}^{j-1} \frac{1}{m_i(\alpha^{j-i} - 1)},$$

$$< 2 \sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1} + \frac{1}{m_1} \sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1}.$$

These inequalities for $s_{j,n}$ and t_j are sufficient to deduce that (2.7) holds if for C we take

$$C = e^{2(2+1/m_1)\sum_{i=1}^{\infty} 1/(\alpha^i-1)}.$$

If $m_1 = 0$ a similar argument suffices to deduce the conclusion.

THEOREM 2.3. *Let S be lacunary, consist of integers and for $z_1, z_2 \in D$ let*

$$(2.11) \quad K(z_1, z_2) = \sum_{n=1}^{\infty} p_n(z_1)p_n(z_2).$$

Then the series in (2.11) converges absolutely and uniformly on compact subsets of $D \times D$. Also $A(S)$ is a Hilbert space of analytic functions on $[0, 1)$ which has a reproducing kernel given by the restriction of K to $[0, 1) \times [0, 1)$.

Proof. Let $0 \leq \delta < 1$ and consider $p_n(z)$ where $z \in D$ and $|z| < \delta$. Let C be chosen so that (2.7) holds and use (2.1), (2.3), (2.4), and (2.5) to obtain

$$\begin{aligned} |p_n(z)| &\leq C\sqrt{2m_n + 1} \left(\sum_{j=1}^{n-1} \frac{2m_j + 1}{m_n - m_j} \delta^{m_j} + \delta^{m_n} \right), \\ &\leq C\sqrt{2m_n + 1} \left(\sum_{j=1}^{n-1} \frac{2m_j + 1}{1 - m_j/m_n} \delta^{m_j} + m_n \delta^{m_n} \right), \end{aligned}$$

$$\begin{aligned}
 &< C \frac{\sqrt{2 + \frac{1}{m_n}}}{\sqrt{m_n}} \left(\sum_{j=1}^{n-1} \frac{2m_j + 1}{1 - \alpha^{j-n}} \delta^{m_j} + m_n \delta^{m_n} \right), \\
 &< \frac{C_0}{\sqrt{m_n}} \left(\sum_{j=1}^{\infty} (2m_j + 1) \delta^{m_j} \right),
 \end{aligned}$$

for some C_0 , since $\lim_{n \rightarrow \infty} m_n = \infty$.

Hence we have for all k ,

$$(2.12) \quad \sum_{n=k}^{\infty} |p_n(z)|^2 \leq C_0^2 \left(\sum_{j=1}^{\infty} (2m_j + 1) \delta^{m_j} \right)^2 \left(\sum_{n=k}^{\infty} \frac{1}{m_n} \right) < \infty.$$

If we now use (2.12), which is true for all $|z| < \delta$, an application of Schwartz's inequality proves that the series in (2.11) has the stated convergence properties. We also see that if $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and if we let $F(z) = \sum_{n=1}^{\infty} \alpha_n p_n(z)$ for $|z| < 1$, then this series converges absolutely and uniformly on compact subsets of D so that F is analytic in D . From this we see that $A(S)$ consists of analytic functions on $[0, 1)$ and that if $0 \leq x < 1$ and $f \in A(S)$, $f(x) = \sum_{n=1}^{\infty} \alpha_n p_n(x)$, where α_n is the n th Fourier coefficient of f with respect to the orthonormal system (p_n) . We now see that K is a reproducing kernel for $A(S)$, for (2.12) shows that if $z \in D$ then $x \rightarrow K(z, x)$ belongs to $A(S)$, and if $f = \sum_{n=1}^{\infty} \alpha_n p_n \in A(S)$, where $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$, then

$$\begin{aligned}
 \int_0^1 K(z, x) f(x) dx &= \lim_{n \rightarrow \infty} \int_0^1 \left(\sum_{k=1}^n p_k(z) p_k(x) \right) \left(\sum_{k=1}^n \alpha_k p_k(x) \right) dx \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \alpha_k p_k(z), \quad \text{by (2.2),} \\
 &= f(z), \quad \text{if } z \in [0, 1).
 \end{aligned}$$

REMARK. It should be noted that if $m_1 = 0$ and $\sum_{n=2}^{\infty} 1/m_n = \infty$, then $A(S)$ does not have a reproducing kernel. This is because of the Müntz-Szasz theorem, which shows that in this case $A(S)$ will contain the restrictions to $[0, 1)$ of all functions in $C([0, 1])$. Hence, if $x \in [0, 1)$, the linear functional on $A(S)$ given by $f \rightarrow f(x)$ is not bounded, so that $A(S)$ does not have a reproducing kernel ([2], p. 317).

3. Moment sequences. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of complex numbers and let $f \in L^2([0, 1])$. Consider the condition

$$(3.1) \quad \alpha_n = \int_0^1 x^{m_n} f(x) dx, \quad \text{for } n = 1, 2, 3, \dots$$

THEOREM 3.1. Let $(\alpha_n)_{n=1}^{\infty}$ be a given sequence of complex numbers. Then if S is lacunary, a sufficient condition for (3.1) to hold for some f in L^2 is that

$$(3.2) \quad \sum_{n=1}^{\infty} m_n |\alpha_n| < \infty .$$

This condition is not necessary.

If S satisfies the condition

$$(3.3) \quad \sum_{n=2}^{\infty} \frac{\left(\sum_{j=1}^{n-1} m_j \right)}{m_n} < \infty ,$$

then a sufficient condition for (3.1) to hold for some $f \in L^2$ is that

$$(3.4) \quad \sum_{n=1}^{\infty} m_n |\alpha_n|^2 < \infty .$$

If S satisfies the condition

$$(3.5) \quad \sum_{n=2}^{\infty} \frac{\left(\sum_{j=1}^{n-1} \sqrt{m_j} \right)^2}{m_n} < \infty ,$$

then (3.4) is a necessary and sufficient condition for (3.1) to hold for some $f \in L^2$.

Proof. By virtue of [2], pp. 226-227, (3.1) holds for some $f \in L^2$ if and only if

$$(3.6) \quad \sum_{n=1}^{\infty} \left| \sum_{j=1}^n a_{n,j} \alpha_j \right|^2 < \infty ,$$

in which case $\sum_{n=1}^{\infty} (\sum_{j=1}^n a_{n,j} \alpha_j) p_n$ will do for f .

If S satisfies (3.3) or (3.5) then S is lacunary so in any case we may choose $\alpha > 1$ so that (2.8) holds when $i > j$. Also, choose C so that (2.7) holds and use (2.3), (2.4), and (2.5) to obtain for $n \geq 2$,

$$\left| \sum_{j=1}^n a_{n,j} \alpha_j \right| \leq C \sqrt{2m_n + 1} \left(|\alpha_n| + \sum_{j=1}^{n-1} \frac{2m_j + 1}{m_n - m_j} |\alpha_j| \right) ,$$

so that

$$(3.7) \quad \left| \sum_{j=1}^n a_{n,j} \alpha_j \right| \leq \frac{C \sqrt{2m_n + 1}}{m_n} \left(m_n |\alpha_n| + \frac{\alpha}{\alpha - 1} \sum_{j=1}^{n-1} (2m_j + 1) |\alpha_j| \right) .$$

If (3.2) holds this shows that there is F so that

$$\left| \sum_{j=1}^n a_{n,j} \alpha_j \right| \leq \frac{F}{\sqrt{m_n}} ,$$

for all $n \geq 2$, so that (3.6) and hence (3.1) hold. (3.2) is not necessary since

$$1/(m_n + 1) = \int_0^1 x^{m_n} dx, \quad \text{but} \quad \sum_{n=1}^{\infty} m_n/(m_n + 1) = \infty.$$

Now assume that (3.3) holds. Because of (2.5) and (2.7), (3.4) is equivalent to the condition

$$(3.8) \quad \sum_{n=1}^{\infty} |a_n \alpha_n|^2 < \infty.$$

Also the approach used to derive (3.7) shows that there is a constant G , depending only on S , so that

$$(3.9) \quad \left| \sum_{j=1}^{n-1} a_{n_j} \alpha_j \right| \leq \frac{G}{\sqrt{m_n}} \left(\sum_{j=1}^{n-1} (2m_j + 1) |\alpha_j| \right),$$

for $n \geq 2$.

Now let (3.4) hold. An application of Schwartz's inequality shows that

$$\left| \sum_{j=1}^{n-1} m_j \alpha_j \right|^2 \leq \left(\sum_{j=1}^{n-1} m_j \right) \left(\sum_{j=1}^{n-1} m_j |\alpha_j|^2 \right) \leq \left(\sum_{j=1}^{n-1} m_j \right) \left(\sum_{j=1}^{\infty} m_j |\alpha_j|^2 \right).$$

Since S is lacunary and (3.3) holds we have, for some J , $\sum_{n=2}^{\infty} |\alpha_n| \leq J \sum_{n=2}^{\infty} 1/\sqrt{m_j} < \infty$. These facts, together with (3.3), (3.8), and (3.9) imply that (3.6), and hence (3.1), hold (the latter for some $f \in L^2$).

Condition (3.5) is stronger than (3.3), so that if (3.5) holds then (3.4) implies (3.1). Conversely let (3.5) and (3.1) hold, the latter for some $f \in L^2$. An application of the Schwartz inequality to (3.1) shows that there is a constant H so that $|\alpha_n| \leq H/\sqrt{m_n}$, for $n = 2, 3, \dots$. Hence $\sum_{n=1}^{\infty} |\alpha_n| < \infty$ and also

$$\sum_{j=1}^{n-1} m_j |\alpha_j| \leq H \left(\sum_{j=1}^{n-1} \sqrt{m_j} \right).$$

Because of (3.5) we conclude from (3.9) that $\sum_{n=1}^{\infty} |\sum_{j=1}^{n-1} a_{n_j} \alpha_j|^2 < \infty$. Since (3.1) implies (3.6) we deduce that (3.8) holds and this is equivalent to (3.4).

This result suggests introducing a sequence space L_S as follows. A sequence $(\alpha_n)_{n=1}$ belongs to L_S if and only if there is $f \in L^2$ such that $\alpha_n = \int_0^1 x^{m_n} f(x) dx$, for $n = 1, 2, \dots$. It is shown in [5], p. 237 that if S consists of integers, then $L_S \subseteq \ell^2$.

COROLLARY 3.2. *If S satisfies (3.5), there is a subsequence S_1 of S so that $L_{S_1} \subseteq L_S$. We also have $L_S \neq \ell^2$, regardless of whether or not S satisfies (3.5).*

COROLLARY 3.3. *Let S be a lacunary sequence of integers and let $\alpha = (\alpha_n)_{n=1}^{\infty}$ be a given sequence of complex numbers. Then if*

there are real γ and δ , with $\delta < 1$, so that $|\alpha_n| \leq \gamma\delta^{m_n}$ for $n = 1, 2, \dots$, then $\alpha \in L_S$.

These results suggest that rather than using the functions x^{m_n} ($n = 1, 2, \dots$) in (3.1) it may be more appropriate to use the functions $m_n x^{m_n}$ or $\sqrt{m_n} x^{m_n}$. The following is essentially a rewording of Theorem 3.1.

THEOREM 3.4. *Let $(\alpha_n)_{n=1}^\infty$ be a sequence of complex numbers. If S is lacunary and $\sum_{n=1}^\infty |\alpha_n| < \infty$ then there is $f \in L^2$ so that $\alpha_n = m_n \int_0^1 x^{m_n} f(x) dx$ for $n = 1, 2, \dots$.*

If S satisfies (3.5) then $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$ if and only if there is $f \in L^2$ so that $\alpha_n = \sqrt{m_n} \int_0^1 x^{m_n} f(x) dx$, for $n = 1, 2, \dots$.

REMARKS. If μ is a measure supported by $[0, \delta]$, where $\delta < 1$, and we let $\alpha_n = \int_0^1 x^{m_n} d\mu(x)$ for $n = 1, 2, \dots$ then, assuming that S is a lacunary sequence of integers, Corollary 3.3 applies to give a function $f \in L^2$ so that (3.1) holds. That is, the measure μ can be absolutely continuous with respect to Lebesgue measure.

As remarked in the introduction, if $\sum_{i=2}^\infty 1/m_i < \infty$ and j is given, there is a measure μ on $[0, 1]$ so that $\int_0^1 x^{m_n} d\mu(x) = 0$ if $n \neq j$ and $\int_0^1 x^{m_j} d\mu(x) \neq 0$. If δ is given ($1 \geq \delta > 0$) and S is a lacunary sequence of integers, the measure μ can be chosen to be supported by $[0, \delta]$ and be absolutely continuous with respect to Lebesgue measure. To see this, apply Corollary 3.3 to the sequence $(\varepsilon_n \delta^{m_n})_{n=1}^\infty$, where $\varepsilon_n = 0$ if $n \neq j$ and $\varepsilon_j = 1$. We obtain $f \in L^2$ so that $\varepsilon_n \delta^{2m_n} = \int_0^1 (\delta x)^{m_n} f(x) dx$, from which the result follows by a change of variable.

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