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We consider complete intersections V in C^m which have an isolated singularity at $\underline{0}$. When V admits a C^* action, one has the orbit space $V^* = V - \{\underline{0}\}/C^*$. In this paper we determine when V^* is a topological manifold, or in some cases, the precise dimension of the set Σ along which V^* is not a manifold. For proper actions we consider a natural complex structure on the space V^* and determine some equivalences among V^* for different V. Our methods are topological; the results are expressed numerically in terms of weighted degrees of the polynomials defining V.

1. Introduction. Let $f^{(1)}, \cdots, f^{(k)}$ be complex polynomials in $\underline{z} = (z_1, \cdots, z_m)$. Let $V(f^{(j)}) = \{\underline{z} \in C^m \mid f^{(j)}(\underline{z}) = 0\}$, and suppose that V has an isolated singularity at $\underline{0}$ and is the complete intersection of the $V(f^{(j)})$; $\dim_C V = m - k$. We set $n = m - k$. Further suppose that there is an action of $C^* = C - \{0\}$ on C^m of the form

$$\sigma(t; z_1, \cdots, z_m) = (t^{q_1} z_1, \cdots, t^{q_m} z_m) \tag{1.1}$$

leaving V invariant, with (i) $q_i \in Z$, $i = 1, \cdots, m$ and (ii) g.c.d. $(q_1, \cdots, q_m) = 1$. Such an action will be called a *diagonal action* of type $(q_1, \cdots, q_m)$. We assume that V is not contained in any hyperplane so that (ii) implies the C^* action is effective. Also, $q_i \neq 0$ implies that $\underline{0}$ is the only fixed point, while $q_i > 0$ implies that the action is proper (i.e., the map $\psi: C^* \times C^m \to C^m \times C^m$ given by $\psi(t, \underline{z}) = (\underline{z}, \sigma(t; \underline{z}))$ is proper.) We shall call such actions *fixed-point free* and *proper*, respectively.

Results of Holmann [5] show that for proper actions there is a unique complex structure on $V^* = V - \{0\}/C^*$ such that the orbit map is holomorphic. Later we will describe this structure in more detail.

By [10, Proposition (1.1.3)], any algebraic variety V admitting a C^* action given by a morphism of algebraic varieties may be embedded in some C^m so that the given action is induced by a diagonal action on C^m. By the above, the action is proper and without fixed-points on $V - \{0\}$ precisely when $q_i > 0$. Actions with $q_i \leqq 0$ are also of interest, as they arise when considering C^* actions on versal deformations.

We next note that [10, Proposition (1.1.2)] allows us to assume that V is defined by weighted (or quasi-) homogeneous polynomials.

Recall that given an m-tuple $\underline{w} = (w_1, \cdots, w_m)$ of positive rationals we say that a polynomial is *weighted homogeneous* with weights $\underline{w}$ (or, f is of type $\underline{w}$) if $a_1/w_1 + \cdots + a_m/w_m = 1$ for every monomial $\alpha z_1^{a_1} \cdots z_m^{a_m}$ of f. Write $w_i = u_i/v_i$, $(u_i, v_i) = 1$ and let $d = \text{l.c.m.}(u_1, \cdots, u_m)$, $q_i = d/w_i$. Then

$$f(t^{q_1}z_1, \cdots, t^{q_m}z_m) = t^d f(z_1, \cdots, z_m) .$$

We call d the *polynomial degree* of f and q_i, $i = 1, \cdots, m$ the *coordinate degrees* of f. The coordinate degrees are related to the q_i of (1.1).

We may thus restate our situation: V is a complete intersection of varieties $V(f^{(j)})$, $j = 1, \cdots, k$, where $f^{(j)}$ is a weighted homogeneous polynomial with degree $d^{(j)}$ and coordinate degrees $q_i^{(j)}$, $i = 1, \cdots, m$. Furthermore, there are integers $\lambda^{(j)}$ with g.c.d. $(\lambda^{(1)}, \cdots, \lambda^{(k)}) = 1$ so that $(q_1, \cdots, q_m) = \lambda^{(j)}(q_1^{(j)}, \cdots, q_m^{(j)})$, $j = 1, \cdots, k$.

Since V is a complete intersection we may conclude from work of Hamm [3] that $K = V \cap S^{2m-1}$ is a $(2n-1)$-dimensional manifold with an effective action of $S^1 \subset C^*$. It is easily seen that $K^* = K/S^1$ is homeomorphic to V^*, and we will often work with K^*.

In § 2 we state some results on S^1 actions due to Neumann [8] which we use in § 4, where we determine necessary conditions for K^* to be a topological manifold. The most easily stated result is (with $q_i \neq 0$).

COROLLARY 4.4. *Suppose $n > 3$ and K^* is a manifold. If the weights $\underline{w}^{(j)}$ are the same for all j, then the weights are integers, and V is therefore equivariantly homeomorphic to a variety defined by Pham-Brieskorn polynomials.*

In § 5 we determine number-theoretic conditions sufficient to ensure that certain K^* are manifolds, and in fact we determine precisely the dimension of the singular set. The final section studies the complex structure of V^* if $q_i > 0$. We show that V^* is nonsingular as a complex space precisely when K^* is a topological manifold. We also give a general criterion to determine when different V yield biholomorphically equivalent V^*.

Many authors have studied these varieties. J. Milnor [7] was perhaps the first to notice that weighted homogeneous polynomials are topologically pleasant to work with. W. Neumann [8] considered many of the same problems for the Pham-Brieskorn polynomials $\Sigma z_i^{a_i}$; we often use his techniques. G. Edmunds [2] gave an explicit embedding of V^* into projective space. Finally, P. Orlik and P. Wagreich have contributed extensively to the study of varieties with C^* actions [10, 11, 12, 13]; it is a pleasure to thank them for many useful conversations and comments.

2. Slices and S^1 actions. It is convenient to work with the action of S^1 on K. In this section we briefly recall some language of slice diagrams (see Jänich [6] for more details) and state some results of Neumann [8] for quotients of linear actions of S^1 and finite cyclic groups.

Let G be a compact Lie group. At every point x of a G-manifold X there is a slice W_x transverse to the orbit $G(x)$ of G at x. W_x is a real vector space and the isotropy group $G_x = \{g \in G \mid gx = x\}$ acts effectively on W_x via a representation σ. The slice theorem [6, 1.3] yields the following easy result.

THEOREM. *Suppose G is a compact Lie group acting effectively on a smooth manifold X. Then X/G is a manifold if and only if W_x/G_x is a manifold for every $x \in X$.*

We will write $[G_x, \sigma]$ to indicate the action of G_x on W_x via σ, and we will call $[G_x, \sigma]$ the slice type at x. If W_x/G_x is a manifold we say $[G_x, \sigma]$ has QM.

In our situation we have an effective action of S^1 on K. Possible isotropy groups are $\{1\}$, cyclic groups $\boldsymbol{Z}_q$, and S^1 (possible only if some $q_i = 0$). For $W = \boldsymbol{R}^2$ or $\boldsymbol{C}$, we denote by σ_p the real or complex representation of S^1 or $\boldsymbol{Z}_q$ on W given by

$$\exp(i\theta) \longrightarrow \exp(i\theta p) .$$

Every representation of S^1 or $\boldsymbol{Z}_q$ as an isotropy group of the S^1 action on K on the vector space W_x is equivalent to one of the form $\sigma_{p_1} \oplus \cdots \oplus \sigma_{p_r} \oplus j$, where j denotes a j-dimensional trivial representation.

Thus the following result of Neumann [8, Theorem 2.2] is crucial. As usual, we write $[G_x, \sigma_{p_1} \oplus \cdots \oplus \sigma_{p_r} \oplus j]$ for a slice with $G_x = \boldsymbol{Z}_q$ or S^1 and indicated linear action of G_x on W_x.

THEOREM (Neumann's criterion).

(i) *Let* g.c.d. $(p_1, \cdots, p_r, q) = 1$ *and let* $\bar{p}_i =$ g.c.d. $(p_1, \cdots, \hat{p}_i, \cdots, p_r, q)$. *Then for $r \geqq 1$, $[\boldsymbol{Z}_q, \sigma_{p_1} \oplus \cdots \oplus \sigma_{p_r} \oplus j]$ has QM if and only if $\bar{p}_1 \cdots \bar{p}_r = q$.*

(ii) *Let* g.c.d. $(p_1, \cdots, p_r) = 1$. *Then $[S^1, \sigma_{p_1} \oplus \cdots \oplus \sigma_{p_r} \oplus j]$ has QM if and only if $r \leqq 2$.*

Thus, $[\boldsymbol{Z}_6, \sigma_2 \oplus \sigma_3]$ has QM, while $[\boldsymbol{Z}_6, \sigma_2 \oplus \sigma_5]$ does not.

3. The slice representation. From the preceding section it is clear that we must determine the various slice types of the action (1.1) on S^{2m-1} and K.

On S^{2m-1} the problem is trivial. At a point $\underset{\sim}{z}$ with precisely the first r coordinates nonzero the slice type is $[\boldsymbol{Z}_q, (r-1)\oplus\sigma_{q_{r+1}}\oplus\cdots\oplus\sigma_{q_m}]$, where $q = \text{g.c.d.}\,(q_1, \cdots, q_r)$.

On K the problem is slightly less trivial. Given an r-element subset I_r of $\{1, \cdots, m\}$, we will write $q(I_r) = \text{g.c.d.}\,\{q_i, i \in I_r\}$ and we will denote by $T(I_r)$ the slice type of K at a point $\underset{\sim}{z}$ whose nonzero coordinates are precisely those with subscripts in I_r. $O(I_r)$ will denote the orbit bundle of $T(I_r)$, that is, the set of the points of K with slice type $T(I_r)$. It is easily seen that $\dim_{\boldsymbol{R}} O(I_r) \geqq 2(r-k)-1$.

LEMMA 3.1. *If* $\dim_{\boldsymbol{R}} O(I_r) = 2(r-k)-1$, *then* $T(I_r) = [\boldsymbol{Z}_{q(I_r)}, (r-k-1)\oplus\sigma_{q_{r+1}}\oplus\cdots\oplus\sigma_{q_m}]$.

Proof. As in [8], this lemma is a consequence of the following general fact: Suppose Y is an invariant submanifold of X, and suppose that at some point $y \in Y$ the codimension of Y in X is the same as the codimension of the orbit bundle of y in Y in the orbit bundle of y in X. Then the slice type of y in Y is the same, up to trivial factors, as the slice type of x in X.

REMARK 3.2. In general, the slice representation at $\underset{\sim}{z}$ in K is a subrepresentation of the slice representation at $\underset{\sim}{z}$ in S^{2m-1}.

4. Bounds of the dimension of the singular set. Let Σ be the subset of K^* consisting of points where K^* is not locally homeomorphic to $\boldsymbol{R}^{2n-2}$. We will call Σ the *singular set* of K^*. Suppose $q_i \neq 0$ for all i. Recall that we have weights $w_i^{(j)} = u_i^{(j)}/v_i^{(j)}$, $(u_i^{(j)}, v_i^{(j)}) = 1$, for $i = 1, \cdots, m$; $j = 1, \cdots, k$. Let $t(I_r) = \dim_{\boldsymbol{C}}(V \cap \{z_i = 0, i \notin I_r\})$.

THEOREM 4.1. *Suppose V is a complete intersection with isolated singularity at the origin, and suppose V is defined by weighted homogeneous polynomials $f^{(j)}$ with weights $\underline{w}^{(j)}$. Further suppose that V is invariant under a fixed-point free diagonal action of type $(q_1, \cdots, q_m)$. If* (i) *there are sets $I_r \subset \{1, \cdots, m\}$ and $J_s \subset \{1, \cdots, k\}$ with r and s elements respectively, so that some prime p divides $v_i^{(j)}$ for $i \in I_r$, $j \in J_s$ and if* (ii) $n - 2(k-s) > 3$, *then*

$$(*) \qquad \dim_{\boldsymbol{R}} \Sigma \geqq 2(t(I_r)-1) \geqq 2(r-(k-s)-1)\,.$$

Before proving this we state several corollaries and give some examples. As shown in [12] (and certainly to be expected), one is particularly interested in the question of when K^* is a manifold.

COROLLARY 4.2. *Suppose* K^* *is a manifold and* $n > 2k + 1$. *Then for every* $j \in \{1, \cdots, k\}$ *and any* k*-element set* I_k, *one has* g.c.d. $\{v_i^{(j)}, i \in I_k\} = 1$.

COROLLARY 4.3. *Suppose* $n > 3$. *If there is a set* I_r *so that* p *divides* $v_i^{(j)}$ *for* $i \in I_r$, $j = 1, \cdots, k$ *then* $\dim_R \Sigma \geqq 2(r - 1)$.

COROLLARY 4.4. *Suppose* $n > 3$, K^* *is a manifold, and the weights* $\underline{w}^{(j)}$ *are the same for all* j. *Then the weights are integers and* V *is equivariantly homeomorphic to a variety defined by a complete intersection of Brieskorn varieties.*

The last statement of 4.4 follows from the straightforward generalization of [10, Theorem 3.1.4].

These results are essentially the best possible: If $n = 2$, K^* is always a manifold. If $n = 3$ we have the following

EXAMPLE 4.5. Let $n = 3$, $k = 1$, and define V by $f(z_1, \cdots, z_4) = z_1^5 + z_1 z_2^6 + z_3^3 + z_3 z_4^5$. Then the weights are (5, 15/2, 3, 15/2), but one may compute slice types and apply Neumann's criterion to see that K^* is a 4-manifold.

EXAMPLE 4.6. The variety V' defined by the equations

$$\begin{aligned} &z_1^4 + z_2^6 + z_3^{20} + z_4^{28} + z_5^{44} + z_6^{52} = 0 \\ &z_1^3 + z_1 z_2^3 + z_3^{15} + z_4^{21} + z_5^{33} + z_6^{39} = 0 \end{aligned} \tag{4.6.1}$$

has $n = 4$, $k = 2$, and $w_2^{(2)} = 9/2$. The reader may use Neumann's criterion and 3.1 to verify that K^* is a manifold. (This will also follow from 5.3.) This example should be compared with 4.4.

Proof of 4.1. Suppose we have I_r and J_s satisfying (i) so that the first inequality of (*) fails. Then we will show that (ii) also fails. In the course of doing this we will show that (i) implies $t(I_r) \geqq r - (k - s)$, giving the second inequality.

For convenience we will assume $I_r = \{1, \cdots, r\}$ and $J_s = \{1, \cdots, s\}$. Then for any monomial $az_1^{a_1} \cdots z_r^{a_r}$ of $f^{(j)}$, $j \in J_s$, one has

$$a_1/w_1^{(j)} + \cdots + a_r/w_r^{(j)} = 1 .$$

But since p divides $v_i^{(j)}$, $i \in I_r$, $j \in J_s$, the above equation implies that p divides $u_i^{(j)}$, $i \in I_r$, $j \in J_s$. Since $(u_i^{(j)}, v_i^{(j)}) = 1$ this is a contradiction, and no such monomial appears in $f^{(j)}$, $j \in J_s$.

Therefore the set $S = \{\underset{\sim}{z} \in C^m \mid z_i = 0, i > r\}$ is contained in $\{f^{(1)} = \cdots = f^{(s)} = 0\}$, so that $\dim_C V \cap S = t(I_r) \geqq r - (k - s)$.

Now let $S^* = S \cap K/S^1$. Then $\dim_R S^* \geqq 2(t(I_r) - 1)$, so that if we let $\underset{\sim}{z} \in S \cap K$ be a point with precisely the first r coordinates nonzero, and if we assume that (*) fails, then the slice type at $\underset{\sim}{z}$ must have QM.

Let this slice type be $[Z_q, \sigma]$. Then $q = \text{g.c.d.}(q_1, \cdots, q_r)$. Since p divides $v_i^{(j)}, i = 1, \cdots, r$, p divides $q_i^{(j)} = d^{(j)} v_i^{(j)} u_i^{(j)}$, and thus p divides q. Since σ has QM it follows easily from Neumann's criterion and 3.2 that p must divide at least $n - 1$ of the q_i, say p divides q_i, $i \in I_{n-1}$, where $I_r \subset I_{n-1}$. We may assume $I_n = \{1, \cdots, n - 1\}$.

We next claim that in fact, p divides $v_i^{(j)}$, $i \in I_{n-1}$, $j \in J_s$. By assumption p divides $v_i^{(j)}$, $i \in I_r \subset I_{n-1}$, $j \in J_s$. For $i \in I_{n-1}$, $j \in J_s$, p divides $q_i = \lambda^{(j)} d^{(j)} v_i^{(j)} / u_i^{(j)}$. If p does not divide $v_i^{(j)}$, then p divides $\lambda^{(j)} d^{(j)}$. This implies p^2 divides q which in turn implies that p^2 divides $\lambda^{(j)} d^{(j)}$, etc. Thus p divides $v_i^{(j)}$, $i \in I_{n-1}$, $j \in J_s$.

Now consider the $k \times m$ matrix $D = (d_{\alpha\beta})$, where $d_{\alpha\beta} = \partial f^{(\alpha)} / \partial z_\beta$. We have seen that every monomial in $f^{(j)}$, $j \in J_s$, which contains a variable z_i, $i = 1, \cdots, n - 1$, must also contain some z_i, $i > n - 1$. Let $P_0 = \{z_n = \cdots = z_m = 0\}$, and let $P = P_0 \cap V$. Then $f^{(j)}(z) = 0$, for $\underset{\sim}{z} \in P_0$, $j \in J_s$, so $\dim_C P \geqq (n - 1) - (k - s)$. On P we clearly have $d_{\alpha\beta} = 0$, $1 \leqq \alpha \leqq s$, $1 \leqq \beta \leqq n - 1$.

Of course, V as a complete intersection is singular wherever D has rank less than k. Let D_s be the $s \times m$ matrix consisting of the first s rows of D, and let D_s' be the $s \times (m - (n - 1)) = s \times (k + 1)$ matrix consisting of the last $k + 1$ columns of D_s. If the rank of D_s' is less than s at any point $\underset{\sim}{z}_0$ of P, then V is singular at $\underset{\sim}{z}_0$.

But D_s' will have rank less than s if $k - s + 2$ minors of size $s \times s$ vanish. Thus V will be singular on a set of complex dimension at least $\dim_C P - (k - s + 2) \geqq (n - 1) - (k - s) - (k - s + 2) = n - 2(k - s) - 3$. Since V has an isolated singularity, $n - 2(k - s) - 3 \leqq 0$, contradicting (ii) and thus completing the proof.

We conclude this section with two trivial consequences of Neumann's criterion.

PROPOSITION 4.7. *Suppose V is a complete intersection with isolated singularity and diagonal C^* action, and suppose $q_1 = \cdots = q_r = 0$, $q_i \neq 0$, $i > r$. If K^* is a manifold, $n - \dim_C V \cap \{z_i = 0, i > r\} \leqq 2$.*

Proof. The S^1 action on K fixes $K \cap \{z_i = 0, i > r\}$.

The next proposition is a topological analogue of a phenomenon noticed by G. Edmunds [2, § 5].

PROPOSITION 4.8. *The real codimension of Σ in K is at least* 4.

Proof. This is a trivial consequence of Neumann's criteria, as at any point the isotropy is S^1 or Z_q, and K^* can fail to be a manifold at the point only if the slice representation has at least three or two nontrivial summands, respectively.

5. Totally complete intersections.

In general one needs to know the form of the polynomials defining V in order to determine the exact dimension of Σ. There is, however, one class of complete intersections for which a knowledge of the polynomial and coordinate degrees will suffice.

DEFINITION 5.1. $V^n \subset C^m$ is called a *totally complete intersection* if the intersection of V with all coordinate subspaces of C^m has minimal dimension.

An example is an intersection of Brieskorn varieties with suitable coefficients (see Hamm [4]). The complete intersection V' of 4.6 is another such example.

DEFINITION 5.2. Given a complete intersection V^n with diagonal C^* action of type $(q_1, \cdots, q_m)$, $q_i \neq 0$, we define $t_i = \text{g.c.d.}\,(q_1, \cdots, \hat{q}_i, \cdots, q_m)$, and $s_i = q_i/t_1 \cdots \hat{t}_i \cdots t_m$.

Since $\text{g.c.d.}\,(q_1, \cdots, q_m) = 1$ we easily see that $(t_i, t_j) = 1$, $i \neq j$, $s_i \in Z$, and $(s_i, t_i) = 1$, $i = 1, \cdots, m$. Let γ be the largest integer such that there exist γ of the s_i with common divisor greater than one.

THEOREM 5.3. *Suppose $V^n \subset C^m$ is a totally complete intersection with isolated singularity at $\underline{0}$ admitting a diagonal C^* action of type $(q_1, \cdots, q_m)$, with $q_i \neq 0$, $i = 1, \cdots, m$. Then the real dimension of the singular set Σ of the orbit space $V - \{0\}/C^*$ is* $\max\{-1, 2(n - m - 1 + \gamma)\}$.

Proof. We consider the associated S^1 action on K. At a point $\underline{z}$ of K with precisely the first γ coordinates nonzero, we have cyclic isotropy of order $q = \text{g.c.d.}\,(q_1, \cdots, q_\gamma)$. By 3.1, the slice representation is $\sigma = \sigma_{q_{\gamma+1}} \oplus \cdots \oplus \sigma_{q_m} \oplus (n + \gamma - m - 1)$.

We now apply Neumann's criterion: K^* will be a manifold if

$$\prod_{s=1}^{m-\gamma} \text{g.c.d.}\,(q_1, \cdots, q_\gamma, q_{\gamma+1}, \cdots \hat{q}_{\gamma+s}, \cdots, q_m) = q \tag{5.3.1}$$

(5.3.1) holds, by definition, if $t_{\gamma+1} \cdots t_m = \text{g.c.d.}\,(q_1, \cdots, q_\gamma)$. The latter equation easily is seen to hold if and only if $\text{g.c.d.}\,(s_1, \cdots, s_\gamma) = 1$.

Since the set $V \cap \{\underline{z} \mid$ precisely $z_1, \cdots, z_\gamma$ are nonzero$\}$ has complex dimension $n-(m-\gamma)$, the result follows.

In particular, K^* is a manifold if and only if no collection of $(k+1)$ of the s_i has a common divisor.

For various applications (cf. [12, § 4]) one wishes to construct V with C^* action so that K^* is a manifold.

PROPOSITION 5.4. *Suppose integers* $t_i, s_i, i = 1, \cdots, m$ *and* c_j, $j = 1, \cdots, k$ *are given such that* $(t_i, t_j) = 1$, $i \neq j$, *and* $(s_i, t_i) = 1$, *for all* i. *Define* $a_{ij} = (c_j t_i)[\text{l.c.m.}(s_1, \cdots, s_m)]/s_i$. *Then a totally complete intersection* V *defined by the equations*

$$\sum_{i=1}^{m} \alpha_{ij} z_i^{a_{ij}} = 0\ , \quad j = 1, \cdots, k \tag{5.4.1}$$

has a C^* *action. The associated* K^* *is a manifold if and only if no* $k+1$ *of the numbers* $s_1, \cdots, s_m$ *have a common divisor.*

Proof. This follows from 5.3 and easy computations which yield

$$d^{(j)} = c_j[\text{l.c.m.}(s_1, \cdots, s_m)]t_1 \cdots t_m$$
$$q_i = t_1 \cdots t_{i-1} s_i t_{i+1} \cdots t_m\ .$$

Neumann proved 5.4 for $k = 1$. We should emphasize that 5.3 does not depend on the polynomials themselves, but only on the polynomial and coordinate degrees.

6. The complex spaces V^*. We now change our viewpoint somewhat and require $q_i > 0$, $i = 1, \cdots, m$, so that the action (1.1) is proper.

We give $V^* = V - \{0\}/C^*$ a complex structure as in Brieskorn and Van de Ven [1]: Define a holomorphic operation of C on $V - \{0\}$ by

$$t(z_1, \cdots, z_m) = (\exp(tq_1)z_1, \cdots, \exp(tq_m)z_m)\ . \tag{6.0.1}$$

Notice that an orbit of the C action on V intersects K in an orbit of the S^1 action on K. In fact the imaginary axis from 0 to $2\pi i$ moves any point of K through its S^1 orbit. Thus $V - \{0\}/C \cong V^* \cong K^*$.

Consider $Z \subset C$ as an additive subgroup and let $H = V - \{0\}/Z$. It is easily seen that $H \cong K \times S^1$. Let Γ be the discrete subgroup of C generated by 1 and $2\pi i$. The torus $T = C/\Gamma$ acts on H by (6.0.1), and by results of Holmann [5], H/T is a complex space homeomorphic to V^* or K^*.

THEOREM 6.1 (Neumann [8] for Brieskorn varieties). *Suppose* V *is a complete intersection with proper diagonal* C^* *action, and* *suppose V has an isolated singularity at $\underline{0}$. Then K^* is a manifold if and only if the complex structure on V^* is nonsingular.*

Proof. A theorem of Prill [14] asserts that the complex structure is nonsingular if and only if the isotropy group at every point p is generated by elements of T with complex codimension one fixed-point sets passing through p.

Let K denote the intersection of V with the *unit* sphere and let $\underline{z} \in K$, $\theta \in S^1$. Then the T action on H is given by $(a, b)(\underline{z}, \theta) = (b(\underline{z}), a(\theta))$, where (a, b) are coordinates of T in the direction of 1 and $2\pi i$. Clearly the isotropy at $(\underline{z}, \theta) \in H$ is the same as the isotropy of the $S^1 \subset T$, $S^1 = \{(0, 2\pi i\theta) | 0 \leqq \theta < 1\}$ and this in turn is the same as the isotropy of the S^1 action on K at $\underline{z}$.

The result then follows by direct comparison of the criterion of Neumann for K^* to be a manifold with the above criterion of Prill.

We now generalize the concept of the cone over V, [10]. We no longer assume that V is a complete intersection or has an isolated singularity at $\underline{0}$. We do of course continue to assume that V is invariant under a proper diagonal C^* action.

DEFINITION 6.2. Suppose V is defined by polynomials $f^{(j)} = \Sigma \alpha z_1^{a_1} \cdots z_m^{a_m}$. The variety V_0 defined by $g^{(j)} = \Sigma \alpha z_1^{a_1 r_1} \cdots z_m^{a_m r_m}$ is called the *weighted cone over V with weights* $(r_1, \cdots, r_m) \in (\boldsymbol{Z}^+)^m$.

Note that $\phi(z_1, \cdots, z_m) = (z_1^{r_1}, \cdots, z_m^{r_m})$ defines a map $\phi: V_0 \to V$, and that ϕ has degree $r_1 \cdots r_m$ so long as V is contained in no coordinate hyperplane. In [10], the weighted cone with weights $(q_1, \cdots, q_m)$ was called simply the cone over V. We will call this special case the *minimal homogeneous cone* over V. V_0 admits a proper diagonal C^* action which commutes with ϕ, so that one obtains a map $\psi: V_0^* \to V^*$ of complex spaces. Thus if V_0 is the minimal homogeneous cone, V^* is branch covered by a projective variety.

We next ask for the degree of ψ, and in particular, when is ψ biholomorphic?

THEOREM 6.3. *Let V_0 be a variety with proper diagonal C^* action of type $(q_1, \cdots, q_m)$. Define $t_i = \text{g.c.d.}\,(q_1, \cdots, \hat{q}_i, \cdots, q_m)$. Suppose V_0 is the weighted cone over V of type $(r_1, \cdots, r_m)$, and define $e_i = \text{g.c.d.}\,(r_i, t_i)$. Then the degree of $\psi: V_0^* \to V^*$ is $r_1 \cdots r_m / e_1 \cdots e_m$.*

Proof. The finite group $G = \boldsymbol{Z}_{r_1} \oplus \cdots \oplus \boldsymbol{Z}_{r_m}$ acts on V_0 and $V_0/G \cong V$. Similarly, G acts on V_0^*, and $V_0^*/G = V^*$. However, the latter action is not effective in general. Setting $G' = \{g \in G | gz^* = z^*$, for all $z^* \in V_0^*\}$, we must show that the order of G' is $e_1 \cdots e_m$.

Let β_i generate $\boldsymbol{Z}_{r_i}$, so that $\beta^{(i)} = (1, \cdots, \beta_i, \cdots, 1) \in G$ acts on

V_0 by fixing all coordinates except the ith, which is multiplied by $\exp(2\pi i/r_i)$. Let $\gamma_i = \beta_i^{r_i/e_i}$ and $\gamma^{(i)} = (1, \cdots, \gamma_i, \cdots, 1)$. We claim that G' is generated by the $\gamma^{(i)}$.

We show first that $\gamma^{(i)}z^* = z^*$. That is, we show that $\underline{z}$ and $\gamma^{(i)}(\underline{z})$ are in the same orbit of the C^* action on V_0. Let $\zeta = \exp(2\pi i/e_i)$. Then $\zeta(z_1, \cdots, z_m) = (z_1, \cdots, z_{i-1}, \zeta^{q_i}z_i, z_{i+1}, \cdots, z_m)$, since $\zeta^{q_j} = \exp(2\pi i q_j/e_i) = 1$ because e_i divides t_i and t_i divides q_j, $i \neq j$. Further, since g.c.d. $(q_i, e_i) = 1$, some power of ζ maps $\underline{z}$ to $\gamma^{(i)}(\underline{z})$. Thus $\gamma^{(i)}z^* = z^*$.

A similar argument shows that any element of G' must be a product of $\gamma^{(i)}$, and the result follows.

COROLLARY 6.4. *Let V_0 be the minimal homogeneous cone over V. Then* $\deg\phi = \deg\psi = q_1 \cdots q_m$.

Proof. g.c.d. $(q_i, t_i) = 1$, $i = 1, \cdots, m$.

COROLLARY 6.5. *ψ is biholomorphic if and only if r_i divides t_i, $i = 1, \cdots, m$.*

This was proved by Neumann for Brieskorn varieties.

REMARK 6.6. The restriction of ψ to coordinate hyperplanes may not have the expected degree. For instance, if V_0 is defined by $z_1^6 + z_2^6 + z_3^6$ and V is defined by $z_1^2 + z_2^3 + z_3^6$, $\deg\psi = 6$ but $\deg\psi|_{z_1=0} = 2$, since the restricted C^* action is not effective.

Corollary 6.5 shows that one cannot obtain biholomorphic complex spaces by considering weighted cones between V and the minimal homogeneous cone. One *can* obtain biholomorphic complex spaces by dividing the exponents of the defining polynomial by t_i, assuming that such division yields a polynomial. Our final result shows that one does get a polynomial.

PROPOSITION 6.7. *Suppose V is a hypersurface with an isolated singularity at $\underline{0}$ and suppose V admits a proper diagonal C^* action of type $(q_1, \cdots, q_m)$. If V is defined by f, with*

$$f(z_1, \cdots, z_m) = \Sigma \alpha z_1^{a_1} \cdots z_m^{a_m} .$$

Then t_i divides a_i for every monomial of f.

Proof. Let $z_1^{a_1} \cdots z_m^{a_m}$ be a monomial of f, with polynomial degree d. Then, since $w_i = d/q_i$, we have

$$a_1 q_1 + \cdots + a_m q_m = d .$$

Since t_i divides q_j for $i \neq j$, and $(t_i, q_i) = 1$ we see that t_i divides a_i if and only if t_i divides d. Since t_i divides q_j, t_i divides dv_j/u_j, so if t_i does not divide d, t_i must divide v_j, $i \neq j$. Then, as in the proof of 4.1, we see that f has at most 2 variables, if the singularity is isolated. So we are done for $m > 2$. For $m = 1$ the result is trivial, and for $m = 2$ it may be checked by direct computation.

EXAMPLE. $z_1^{a_1} + z_1 z_2^{a_2} + z_2 z_3^{a_2}$, with $(a_1 - 1, a_2) = 1$, $(a_2 a_3, a_1 a_2 - a_1 + 1) = 1$. The weights, in reduced form, are

$$w_1 = a_1 , \quad w_2 = a_1 a_2/(a_1 - 1) , \quad w_3 = a_1 a_2 a_3/(a_1 a_2 - a_1 + 1) .$$

Thus, $q_1 = a_2 a_3$, $q_2 = a_3(a_1 - 1)$, $q_3 = a_1 a_2 - a_1 + 1$. Then

$$t_1 = \text{g.c.d.}\,((a_1 - 1), (a_1 a_2 - a_1 + 1)) = 1$$
$$t_2 = \text{g.c.d.}\,(a_2 a_3, a_1 a_2 - a_1 + 1) = 1$$
$$t_3 = \text{g.c.d.}\,(a_2 a_3, a_3(a_1 - 1)) = a_3 .$$

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