

# Pacific Journal of Mathematics

**TORSION FREE ABELIAN GROUPS QUASI-PROJECTIVE  
OVER THEIR ENDOMORPHISM RINGS. II**

CHARLES IRVIN VINSONHALER

## TORSION FREE ABELIAN GROUPS QUASI-PROJECTIVE OVER THEIR ENDOMORPHISM RINGS II

C. VINSONHALER

**Let  $R$  be a commutative ring with 1, and  $X$  an  $R$ -module. Then  $M = X \oplus R$  is quasi-projective as an  $E$ -module, where  $E$  is either  $\text{Hom}_Z(M, M)$  or  $\text{Hom}_R(M, M)$ . In this note it is shown that any torsion free abelian group  $G$  of finite rank, quasi-projective over its endomorphism ring, is quasi-isomorphic to  $X \oplus R$ , where  $R$  is a direct sum of Dedekind domains and  $X$  is an  $R$ -module.**

**Introduction.** If  $R$  is a ring with identity, an  $R$ -module  $M$  is said to be quasi-projective if for any submodule  $N$  of  $M$ , and  $R$ -map  $f: M \rightarrow M/N$ , there is an  $R$ -map  $\bar{f}: M \rightarrow M$  such that  $\bar{f}$  followed by the factor map  $M \rightarrow M/N$  is equal to  $f$ . Results on quasi-projective modules appear in [3], [6], and [7]. In this note, we will be concerned with the case where  $M = G$  is a torsion free abelian group of finite rank and  $R = \text{Hom}_Z(G, G) = E(G)$ , and call  $G$  "*Eqp*" if  $G$  is quasi-projective as an  $E(G)$ -module. The strongly indecomposable *Eqp* groups have been characterized in [6], so we will focus on those groups  $G$  (always torsion free abelian of finite rank) such that  $nG \subseteq G_1 \oplus G_2 \subseteq G$  for some integer  $n \neq 0$  and subgroups  $G_1 \neq 0$ ,  $G_2 \neq 0$  of  $G$ . In fact, any group  $G$  can be quasi-decomposed into a direct sum of strongly indecomposable summands,  $nG \subseteq G_1 \oplus G_2 \oplus \cdots \oplus G_k \subseteq G$ . It is well-known that such a decomposition is unique up to order and the quasi-isomorphism class of the summands. It is therefore desirable to work with a slightly more general notion of quasi-projectivity which is invariant under quasi-isomorphism:

**DEFINITION.** An  $R$ -module  $M$  is almost quasi-projective (*aqp*) if there exists an integer  $n \neq 0$  such that given any submodule  $N$  of  $M$ , and  $R$ -map  $f: M \rightarrow M/N$ , there is an  $R$ -map  $\bar{f}: M \rightarrow M$  such that  $\bar{f}$  followed by the factor map  $M \rightarrow M/N$  is equal to  $nf$ .

In case  $M$  is a group  $G$  and  $R = E(G)$ ,  $G$  is called *almost E-quasi-projective* (*aEqp*).

**PROPOSITION 1.** *Let  $G$  and  $H$  be quasi-isomorphic groups (notation:  $G \sim H$ ). If  $G$  is aEqp, then  $H$  is aEqp.*

*Proof.* Assume that  $mG \subseteq H \subseteq G$  for some integer  $m \neq 0$ . Then if  $\alpha \in E(G)$ ,  $m\alpha|_H \in E(H)$ ; and if  $\beta \in E(H)$ ,  $\beta m \in E(G)$ , so we say

$mE(G) \subseteq E(H)$  and  $mE(H) \subseteq E(G)$ . Now let  $K$  be a fully invariant subgroup ( $E(H)$ -submodule) of  $H$ , and  $f: H \rightarrow H/K$ . Then  $K^* = E(G)(K)$  satisfies  $mK^* \subset K \subset K^*$  and  $f$  induces  $f^*: G \rightarrow G/K^*$  via  $f^*(x) = f(mx) + K^*$ . By assumption this lifts to a map  $g \in E(G)$  such that  $\pi g = n f^*$ , where  $\pi: G \rightarrow G/K^*$  is the natural factor map. Let  $y \in H$ . Then  $g(y) = n f^*(y) \bmod K^* = n f(my) \bmod K^* = n m f(y) \bmod K^*$ . This implies  $mg(y) = n m^2 f(y) \bmod K$ , so that  $mg$  is a lifting of  $n m^2 f$  and  $H$  is  $aEqp$ .

By the preceding proposition we may, without loss of generality, work with a group  $G = G_1 \oplus G_2 \oplus \dots \oplus G_n$  where each  $G_i$  is strongly indecomposable. The following notation is also used:

$$E = E(G) = \text{Hom}_Z(G, G).$$

$$E_i = E(G_i).$$

$$J_i = J(E_i) = \text{Jacobson radical of } E_i.$$

$$EG_i = E(G)G_i = E\text{-submodule of } G \text{ generated by } G_i.$$

Now, a sequence of lemmas leads to the main result.

LEMMA 2. *Suppose  $G$  is  $E$ -indecomposable. Then any  $E$ -map of  $G$  into  $G$  (any map in the center of  $E$ ) is either monic or nilpotent.*

*Proof.* Let  $f$  be an  $E$ -map of  $G$  into  $G$ . Then  $f = \bigoplus_{i=1}^n f_i$  where  $f_i: G_i \rightarrow G_i$  is monic or nilpotent (see [4]). Let  $H_1 = \bigoplus G_i f_i$  is nilpotent and  $H_2 = \bigoplus G_j f_j$  is monic. Since  $G$  is  $E$ -indecomposable, there is a nonzero map between  $H_1$  and  $H_2$  (or  $H_2$  and  $H_1$ ), say  $h: H_1 \rightarrow H_2$ ,  $h \neq 0$ . Letting  $g_1 = \bigoplus f_i$  nilpotent and  $g_2 = \bigoplus f_j$  monic, we have  $g_2 h = h g_1$  so that  $g_2^k h = h g_1^k$  for all  $k > 0$ . Since  $g_1$  is nilpotent,  $g_2^k h = 0$  for some  $k > 0$ . Since  $g_2$  is monic, this says  $h = 0$ , a contradiction.

LEMMA 3. *Let  $G$  be  $aEqp$  and  $E$ -indecomposable. Then for any nontrivial decomposition  $G = H \oplus K$ , either  $EH \sim G$  or  $EK \cap H \sim H$ .*

*Proof.* Suppose the conclusion is false. Then the map given by the identity on  $H/EK \cap H$  and zero on  $K/K \cap EH$  is an  $E$ -map and can be quasi-lifted to an  $E$ -endomorphism of  $G$ . But the lifting can be neither monic nor nilpotent, contradicting Lemma 2,

PROPOSITION 4. *Let  $G$  be  $aEqp$  and  $E$ -indecomposable. Then for each  $G_i$ , either  $G/EG_i$  is bounded or there is a  $j \neq i$  such that  $G_i/EG_j \cap G_i$  is bounded.*

*Proof.* Without loss of generality, assume  $i = 1$ . By Lemma

3, either  $G/EG_1$  or  $G_1/E(\bigoplus_{i=2}^n G_i) \cap G_1$  is bounded. In the latter case, let  $H_1 = EG_2 \cap G_1$  and  $H_2 = E(\bigoplus_{i=3}^n G_i) \cap G_1$ . Then  $(H_1 \cap H_2) \oplus [E(\bigoplus_{i=3}^n G_i) \cap G_2] \oplus (EG_2 \cap \bigoplus_{i=3}^n G_i) = K$  is an  $E$ -submodule of  $G$ , and if  $G_1/H_1 \cap H_2$  has a nontrivial quasi-decomposition, then the quasi-projections can be extended to  $E$ -maps of  $G/K$  into  $G/K$  which can be quasi-lifted to  $E$ -endomorphisms of  $G$ . Again, the liftings can be neither monic nor nilpotent, contradicting Lemma 2. Therefore,  $G/H_1 \cap H_2$  has no nontrivial quasi-decompositions, so that either  $H_1 \cap H_2 \sim H_1$  or  $H_1 \cap H_2 \sim H_2$ , since  $G_1/H_1 + H_2$  is bounded.

*Case I.* If  $H_1 \cap H_2 \sim H_2$ , then  $H_1 \sim G_1$  and we are done.

*Case II.* If  $H_1 \cap H_2 \sim H_1$ , then  $H_2 \sim G_1$ .

In this case, let  $H'_1 = EG_3 \cap G_1$  and  $H'_2 = E(\bigoplus_{i=4}^n G_i) \cap G_1$  and let  $K' = H'_1 \cap H'_2 \oplus E(\bigoplus_{i=4}^n G_i) \cap EG_3 \cap G_2 \oplus E(\bigoplus_{i=4}^n G_i) \cap G_3 \oplus EG_3 \cap \bigoplus_{i=4}^n G_i$ . Then it is straightforward to check that  $K'$  is fully invariant and that, as in the first paragraph, quasi-projections of  $G_1/H'_1 \cap H'_2$  can be extended to  $E$ -maps of  $G/K'$  into  $G/K'$ , which quasi-lift to maps in  $E$ . (It follows from Lemma 3 that  $E(\bigoplus_{i=3}^n G_i) \cap G_2 \sim G_2$ .) Thus as before, either  $H'_1 \cap H'_2 \sim H'_2$  or  $H'_1 \cap H'_2 \sim H'_1$ . In Case I we are done, and in Case II we may repeat the above argument with slight modifications to eventually get  $G_1/G_1 \cap EG_j$  bounded for some  $j$ .

**COROLLARY 5.** *There is a  $G_i$  such that  $G/EG_i$  is bounded.*

*Proof.* By the preceding proposition, either  $G/EG_1$  is bounded or  $EG_{i_1} \cap G_1 \sim G_1$  for some  $i_1$ . Then either  $G/EG_{i_1}$  is bounded or  $EG_{i_2} \cap G_{i_1} \sim G_{i_1}$  for some  $i_2$ . Inductively obtain a sequence  $1, i_1, i_2, \dots, i_{n-1}$  such that  $EG_{i_k} \cap G_{i_{k-1}} \sim G_{i_{k-1}}$ . (Unless the process stops, in which case  $G/EG_{i_k}$  is bounded for some  $k$ .) It follows that  $G/EG_{i_{n-1}}$  is bounded.

Henceforth the  $G_i$  of Corollary 5 will be denoted by  $G_0$ . That is,  $G/EG_0$  is bounded.

**LEMMA 6.** *If  $G_i/EG_0 \cap G_i$  is bounded, then either  $G_i \sim G_0$  or  $EG_i \cap G_0 \subseteq J_0G_0$ .*

*Proof.* Consider  $G_0 \xrightarrow{f} G_i \xrightarrow{g} G_0$ . If  $gf$  is monic, then  $kg^{i-1}f$  has an inverse in  $E(G_0)$  for some  $0 < k \in \mathbb{Z}$ . Then  $G_0 \xrightarrow{f} G_i \xrightarrow{g} G_0 \xrightarrow{(k^{-1}gf)^{-1}} G_0$  gives a quasi-splitting of  $G_i$ . Since  $G_i$  is strongly indecomposable,  $G_0 \sim G_i$ .

On the other hand, if  $gf$  is nilpotent for all possible  $g$  and  $f$ , then  $EG_i \cap G_0 \subseteq J_0G_0$  since  $EG_0 \cap G_i$  is of bounded index in  $G_i$ .

**LEMMA 7.**  $G_0/J_0G_0 \sim E_0/J_0$ .

*Proof.* Let  $\bar{E}_0 = E_0/J_0$ . Then  $Q \otimes_Z \bar{E}_0$  is a division ring. Let  $\bar{x}_1 = x_1 + J_0G_0, \dots, \bar{x}_r = x_r + J_0G_0$  be a maximal  $\bar{E}_0$ -independent set in  $G_0/J_0G_0 = \bar{G}_0$ . Then  $A = \bar{G}_0/\sum_{i=1}^r \bar{E}_0\bar{x}_i$  is torsion and furthermore must be bounded. If  $A$  were unbounded it would have uncountably many endomorphisms which would have to be induced by different endomorphisms of  $G$ . Now consider  $\bar{G}_0/\bar{E}_0\bar{x}_1 \cap \sum_{i=2}^r \bar{E}_0\bar{x}_i$ . If  $r \geq 2$ , this group has a nontrivial quasi-decomposition, and the quasi-projections can be lifted to maps in  $E$  which are neither monic nor nilpotent, a contradiction. Thus  $r = 1$  and  $\bar{G}_0 \sim \bar{E}_0\bar{x}_1 \cong \bar{E}_0$ .

LEMMA 8. *Let  $Z_0$  be the center of  $E_0$ . Then  $Z_0 + J_0 = E_0$ .*

*Proof.* For any  $x \in E_0$ , right multiplication by  $x$  is an  $E_0$ -map  $E_0/J_0 \xrightarrow{x_r} E_0/J_0$ . Using the previous lemma,  $x_r$  quasi-lifts to an  $E_0$  map of  $G_0$ ,  $\hat{x}_r$ , which is in  $Z_0$  since it is an  $E_0$ -map. Clearly  $\hat{x}_r - x \in J_0$ .

The next lemma is well-known but is included for completeness.

LEMMA 9. *Let  $E$  be a ring with identity and nilpotent ideal  $J$ . Let  $M$  be an  $E$ -module and  $L$  a submodule such that  $L + JM = M$ . Then  $L = M$ . (This says  $JM$  is small in  $M$ .)*

*Proof.*  $J(L + JM) = JM \Rightarrow JM \subseteq L + J^2M \Rightarrow M = L + J^2M \Rightarrow M = L + J^kM$  for all  $k > 0$  by induction. Since  $J$  is nilpotent,  $M = L$ .

PROPOSITION 10.  *$G_0$  is strongly irreducible, and hence  $G_0 \sim E_0 = Z_0$ .*

*Proof.* Choose a subring  $S$  of  $E_0$  maximal with respect to

$$(1) \quad S \cap J_0 = 0 \quad (2) \quad S \subset Z_0 \quad (3) \quad Q \cap E_0 \subset S.$$

Note that  $S$  is a pure subgroup of  $E_0$  and is an integral domain. Suppose  $z_0 \notin S + J_0$  for some  $z_0 \in Z_0$ . Then  $S[z_0]$  properly contains  $S$  and satisfies (1), (2), and (3), a contradiction. Thus  $Z_0 \subset S + J_0 \Rightarrow S \oplus J_0 = E_0$ .

Now from the proof of Lemma 7 it follows that  $G_0 \sim E_0x_1 + J_0G_0$  for some  $x_1 \in G_0$ . Hence, by Lemma 9,  $G_0 \sim E_0x_1$ , and  $K = \text{Ker}(E_0 \rightarrow E_0x_1) \subseteq J_0$ . Thus  $G_0 \sim E_0/K = S \oplus J_0/K$ . Since  $G_0$  is strongly indecomposable,  $G_0 \sim S$ . Therefore  $G_0$  is strongly irreducible, and hence  $G_0 \sim E_0 = Z_0$  by the results of [5].

It now follows from Lemma 6 that for any  $G_i$ , either  $G_i \sim G_0$

or  $\text{Hom}(G_i, G_0) = 0$ . Thus, up to quasi-isomorphism and relabeling,  $G = \bigoplus_{j=1}^k H_j \bigoplus_{i=1}^l G_i$  where  $H_j = G_0$ ,  $1 \leq j \leq k$  and  $\text{Hom}(G_i, G_0) = 0$ ,  $1 \leq i \leq l$ .

In the following, let  $G' = \bigoplus_{i=1}^l G_i$ , an  $E$ -submodule of  $G$ . For any map  $\phi \in E_0 = E(G_0)$ ,  $\phi \underbrace{\bigoplus \cdots \bigoplus}_{k \text{ times}} \phi$  is an  $E$ -map of  $G/G'$  into  $G/G'$  and hence can be quasi-lifted to an  $E$ -map,  $\psi$ , of  $G$ . The map  $\psi$  is unique, since if  $\psi'$  were another lifting  $(\psi - \psi')(\bigoplus_{j=1}^k H_j) = 0$ , so that  $\psi - \psi' = 0$  because  $EH_j \cap G_i \sim G_i$  for all  $i, j$ . Since  $\psi$  commutes with projections,  $\psi(G_i) \subseteq G_i$  for each  $i$ . Thus a ring isomorphism  $0 \rightarrow E_0 \rightarrow E(G_i)$  is obtained via  $\phi \mapsto \psi|_{G_i}$ . This yields a unitary  $E_0$ -module structure on  $G_i$ .

Now if  $R_0$  is the ring of integers in  $Q \otimes E_0$ , then  $G_0 \sim E_0 \sim R_0 \otimes E_0$ , and  $R_0 \otimes E_0$  is a Dedekind domain.

We are now ready for the main result.

**THEOREM 11.** *If  $G$  is a torsion free abelian group of finite rank, then  $G$  is a  $Eqp$  if and only if  $G \sim R \oplus X$ , where  $R$  is a direct sum of Dedekind domains, and  $X$  is a unitary  $R$ -module.*

*Proof.* The "if" direction has been demonstrated.

If  $G$  is  $E$ -indecomposable, let  $R = R_0 \otimes \bigoplus_{j=1}^k E_0$  and  $X = R_0 \otimes \bigoplus_{i=1}^l G_i$  in the notation of the preceding lemma and remarks. The general case follows by taking direct sums.

**REMARK.** If  $G = R \oplus X$  in the above, it is clear that  $G$  is actually  $Eqp$ . It would be nice to know exactly which quasi-isomorphic images of  $G$  were also  $Eqp$ .

#### REFERENCES

1. R. A. Beaumont and R. S. Pierce, *Torsion free rings*, Illinois J. Math., **5** (1961), 61-98.
2. L. Fuchs, *Infinite Abelian groups*, Vol. II, Academic Press, New York, 1973.
3. ———, *On torsion Abelian Groups quasi-projective over their endomorphism rings*, Proc. Amer. Math. Soc., **42** (1), Jan. (1974), 13-15.
4. J. D. Reid, *On the Ring of Quasi-Endomorphisms of a Torsion Free Group*, Topics in Abelian Groups, Chicago, (1963), 51-58.
5. ———, *On rings on groups*, Pacific J. Math., **53** (1974), 229-237.
6. C. Vinsonhaler and W. J. Wickless, *Torsion free Abelian groups quasi-projective over their endomorphism rings*, (to appear in the Pacific J. Math.).
7. L. E. T. Wu and J. P. Jans, *On quasi-projectives*, Illinois J. Math., **11** (1967), 439-448.

Received April 21, 1977.

THE UNIVERSITY OF CONNECTICUT  
STORRS, CT 06268



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)

University of California  
Los Angeles, California 90024

C. W. CURTIS

University of Oregon  
Eugene, OR 97403

C. C. MOORE

University of California  
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. FINN AND J. MILGRAM

Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA, RENO  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
OSAKA UNIVERSITY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan



Gerald Arthur Anderson, <i>Computation of the surgery obstruction groups</i> $L_{4k}(1; Z_p)$ .....	1
R. K. Beatson, <i>The degree of monotone approximation</i> .....	5
Sterling K. Berberian, <i>The character space of the algebra of regulated functions</i> .....	15
Douglas Michael Campbell and Jack Wayne Lamoreaux, <i>Continua in the plane with limit directions</i> .....	37
R. J. Duffin, <i>Algorithms for localizing roots of a polynomial and the Pisot Vijayaraghavan numbers</i> .....	47
Alessandro Figà-Talamanca and Massimo A. Picardello, <i>Functions that operate on the algebra <math>B_0(G)</math></i> .....	57
John Erik Fornæss, <i>Biholomorphic mappings between weakly pseudoconvex domains</i> .....	63
Andrzej Granas, Ronald Bernard Guenther and John Walter Lee, <i>On a theorem of S. Bernstein</i> .....	67
Jerry Grossman, <i>On groups with specified lower central series quotients</i> .....	83
William H. Julian, Ray Mines, III and Fred Richman, <i>Algebraic numbers, a constructive development</i> .....	91
Surjit Singh Khurana, <i>A note on Radon-Nikodým theorem for finitely additive measures</i> .....	103
Garo K. Kiremidjian, <i>A Nash-Moser-type implicit function theorem and nonlinear boundary value problems</i> .....	105
Ronald Jacob Leach, <i>Coefficient estimates for certain multivalent functions</i> .....	133
John Alan MacBain, <i>Local and global bifurcation from normal eigenvalues. II</i> .....	143
James A. MacDougall and Lowell G. Sweet, <i>Three dimensional homogeneous algebras</i> .....	153
John Rowlay Martin, <i>Fixed point sets of Peano continua</i> .....	163
R. Daniel Mauldin, <i>The boundedness of the Cantor-Bendixson order of some analytic sets</i> .....	167
Richard C. Metzler, <i>Uniqueness of extensions of positive linear functions</i> .....	179
Rodney V. Nillsen, <i>Moment sequences obtained from restricted powers</i> .....	183
Keiji Nishioka, <i>Transcendental constants over the coefficient fields in differential elliptic function fields</i> .....	191
Gabriel Michael Miller Obi, <i>An algebraic closed graph theorem</i> .....	199
Richard Cranston Randell, <i>Quotients of complete intersections by <math>C^*</math> actions</i> .....	209
Bruce Reznick, <i>Banach spaces which satisfy linear identities</i> .....	221
Bennett Setzer, <i>Elliptic curves over complex quadratic fields</i> .....	235
Arne Stray, <i>A scheme for approximating bounded analytic functions on certain subsets of the unit disc</i> .....	251
Nicholas Th. Varopoulos, <i>A remark on functions of bounded mean oscillation and bounded harmonic functions. Addendum to: "BMO functions and the <math>\bar{\partial}</math>-equation"</i> .....	257
Charles Irvin Vinsonhaler, <i>Torsion free abelian groups quasi-projective over their endomorphism rings. II</i> .....	261
Thomas R. Wolf, <i>Characters of <math>p'</math>-degree in solvable groups</i> .....	267
Toshihiko Yamada, <i>Schur indices over the 2-adic field</i> .....	273