CHARACTERS OF $p'$-DEGREE IN SOLVABLE GROUPS

THOMAS R. WOLF
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We prove that $|I_p(G)| = |I_p(N(P))|$ for $P \in \text{Syl}(G)$, for solvable $G$. Here $p$ is a prime and $I_p(G)$ is the set of irreducible characters $\psi$ such that $(\psi(1), p) = 1$.

1. Introduction. The groups considered are finite and the group characters are defined over the complex numbers. McKay conjectured $|I_p(G)| = |I_p(N(P))|$ where $P \in \text{Syl}(G)$ for simple $G$ and $p = 2$ [6]. I. M. Isaacs has proven the result when $|G|$ is odd and $p$ is any prime (Theorem 10.9 of [4]). We prove the result for solvable $G$. In fact we generalize this slightly to sets of primes and normalizers of Hall subgroups.

For characters $\chi$ and $\psi$ of $G$, we let $[\chi, \psi]$ denote the inner product of $\chi$ and $\psi$. Let $N \leq G$ and $\theta \in \text{IRR}(N)$. We write $I_\theta(\theta)$ to denote the inertia group $\{g \in G | \theta^g = \theta\}$. We also write $\text{IRR}(G | \theta) = \{\chi \in \text{IRR}(G) | [\chi_N, \theta] \neq 0\}$. Of course, character induction yields a one-to-one map from $\text{IRR}(I_\theta(\theta) | \theta)$ onto $\text{IRR}(G | \theta)$. If $\chi \in \text{IRR}(G | \theta)$; we say $\chi$ (or $\theta$) is fully ramified with respect to $G/N$ if $\chi_N = e\theta$ and $e^2 = |G:N|$. This will occur if $I_\theta(\theta) = G$ and $\chi$ vanishes off $N$.

Suppose that $K/L$ is an abelian chief factor of $G$; $\gamma \in \text{IRR}(K)$; $\phi \in \text{IRR}(L)$; and $[\gamma_L, \phi] \neq 0$. If $K \cdot I_\theta(\phi) = G$, then one of the following occur:

(a) $\gamma_L = \phi$;
(b) $\gamma$ and $\phi$ are fully ramified with respect to $K/L$, or
(c) $\phi^K = \gamma$.

We note that $K \cdot I_\theta(\phi) = G$ whenever $I_\theta(\gamma) = G$. The results of these last two paragraphs are well known (e.g. see Chapter 6 of [5]); and we will use them without reference. In Theorem 3.3, we use known results about character triple isomorphisms (see §8 of [4] or Chapter 11 of [5]); otherwise, everything should be self-explanatory.

I would like to thank E. C. Dade for his preprint [1].

2. Extendability. A straightforward proof of Lemma 2.1 may be found in Lemma 10.5 of [4].

**Lemma 2.1.** Assume $N \leq G$, $H \leq G$, $NH = G$, and $N \cap H = M$. Assume $\phi \in \text{IRR}(N)$ is invariant in $G$ and $\phi_M \in \text{IRR}(M)$. Then $\chi \mapsto \chi_H$ defines a one-to-one correspondence between $\text{IRR}(G | \phi)$ and $\text{IRR}(H | \phi_M)$. 

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Theorem 2.2 is a generalization of a result of Dade. He proves the theorem when $E$ is an extra-special $p$-group and when $p + |L|$ (see Theorems 1.2 and 1.4 of [1]). We use his result to prove this.

**THEOREM 2.2.** Assume (i) $G$ is the semi-direct product $EH$, $E \unlhd G$.

(ii) $1 < Z(E) \leq Z(G)$ and $Z(E)$ is cyclic;

(iii) $E/Z(E)$ is an elementary abelian $p$-group for some prime $p$;

(iv) $[L, E/Z(E)] = E/Z(E)$ for some $L/C_H(E) \trianglelefteq H/C_H(E)$ such that $p + |L/C_H(E)|$; and

(v) $\lambda \in IRR(E)$ is faithful.

Then $\Lambda$ extends to an irreducible character $\psi$ of $G$ such that $C_G(\psi) \trianglelefteq \ker(\psi)$.

**Proof.** We may extend $\Lambda$ to an irreducible character of $E \times C_H(E)$ with kernel $C_H(E)$. It is no loss to assume $C_H(E) = 1$. If $E' = Z(E)$, we finish by Dade's result. We assume $E' < Z(E)$.

Fitting lemma (Theorem 5.2.3 of [3]) implies $E/E' = F/E' \times C_{E/E'}(L)$ where $F/E' = [E/E', L]$. As $p + |L|$, the hypotheses yield $Z(E)/E' = C_{E/E'}(L)$. Note $E' = Z(F)$ and $E/Z(E)$ is isomorphic to $F/E'$.

Let $\phi$ be the irreducible constituent of $\Lambda_{Z(E)}$. As $\phi_{E'} \in IRR(F')$, Lemma 2.1 yields $\Lambda_{F'} \in IRR(F')$. By induction on $|G|$, $\Lambda_{F'}$ extends to some $\beta \in IRR(FH)$. If $I_0(\Lambda) = G$, we have by Lemma 2.1 that $\beta = \psi_{FH}$ for some $\psi \in IRR(G/A)$. Furthermore, $\psi(1) = \Lambda(1)$. We are done as long as $I_0(\Lambda) = G$. Note that $\Lambda_{F'}$ and $\phi$ are $H$-invariant. So, if $h \in H$, $\Lambda^h = \alpha \Lambda$ for a linear $\alpha \in IRR(E/F')$. This implies $\phi^h = \alpha_{Z(E)} \phi$ and $\alpha_{Z(E)} = 1_{Z(E)}$. So $\alpha = 1_{E}$, completing the proof.

The following theorem also generalizes a result of Dade (see Theorem 5.10 of [1]).

**THEOREM 2.3.** Assume (i) $G = EH$, $E \unlhd G$, $E \cap H = Z(E)$ is in $Z(G)$;

(ii) $1 \neq Z(E)$ is cyclic;

(iii) $E/Z(E)$ is an elementary abelian $p$-group for a prime $p$;

(iv) $[L, E/Z(E)] = E/Z(E)$ for some $C_H(E) \leq L \leq H$ such that $p + |L/C_H(E)|$; and

(v) $\lambda$ is a faithful character of $Z(E)$.

Then there exists a one-to-one correspondence $T: IRR(G|\lambda) \rightarrow IRR(H|\lambda)$ such that for $\chi \in IRR(G|\lambda)$, $\chi(1) = e[\chi T](1)$ where $e = |E: Z(E)|^{1/2} \in Z$.

**Proof.** Let $\Lambda \in IRR(E|\lambda)$. As $E$ is nilpotent and $\lambda$ is faithful, $\Lambda$ is faithful. If $Z(E) < T < E$ with $|T: Z(E)|$ prime, $\Lambda_T$ has each extension of $\lambda$ to $T$ as a constituent. It follows that $\Lambda$ vanishes on
$E - Z(E)$. So $A$ and $\lambda$ are fully ramified with respect to $E/Z(E)$ and $I_G(\Lambda) = G$.

Let $H_i$ be an isomorphic copy of $H_i$; say $\sigma: H_i \to H_i$ is an isomorphism. Say $Z(E) = \langle x \rangle$ and $\sigma(x) = x_i$. From the semidirect product $G = E \cdot H_i$. Note, by Theorem 2.2, $\Lambda$ extends to $\psi \in IRR(G_i)$.

Let $Z_0 = \langle x \rangle \times \langle x_i \rangle \leq G_i$. Define $\lambda_i \in IRR(\langle x_i \rangle)$ by $\lambda_i(x_i) = \lambda(x)$. Define $\tau: G_i \to G$ by $\tau(g) = t \cdot \sigma^{-1}(g)$ for $t \in E$, $g \in H_i$. Then $\tau$ is a homomorphism onto $G$ with kernel $Z_1 < Z_0$. So $\tau: G/Z_1 \to G$ is an isomorphism, $\tau(\langle x \rangle \times \langle x_i \rangle) = E/Z(E)$, and $(\lambda \times \lambda_i)^\tau = \lambda$, viewing $\tau$ as mapping $IRR(Z_0/Z)$ to $IRR(Z(E))$.

Hence, we need just show there is a one-to-one correspondence $T: IRR(G_i|\lambda \times \lambda_i) \to IRR(H_i|\lambda_i)$ such that $\chi(1) = e[(\chi T)(1)]$.

If $\beta \in IRR(H_i)$, then $\beta$ is $\beta^* \vert H_i$ for a unique $\beta^* \in IRR(G_i/E)$. Now $\beta \to \beta^* \psi$ defines a one-to-one correspondence from $IRR(H_i)$ onto $IRR(G_i|\lambda) = IRR(G_i|\lambda_i)$. As $\psi(1) = e$, it suffices to show for $\beta \in IRR(H_i)$ that $\beta \in IRR(H_i|\lambda_i)$ if and only if $Z_1 \leq \ker (\beta^* \psi)$. If $\mu$ is the irreducible constituent of $\beta$ restricted to $\langle x_i \rangle$, then $\beta^* \psi(x, x_i^{-1}) = e\beta(1)\lambda(x)\mu^{-1}(x)$. So $Z_1 \leq \ker (\beta^* \psi)$ if and only if $\mu = \lambda_i$, completing the proof.

3. The McKay conjecture. If $\pi$ is a set of primes, let $I_\pi(G) = \{\chi \in IRR(G) | (p, \chi(1)) = 1 \text{ for all } p \in \pi\}$. Now $G$ is $\pi$-solvable if $G$ has a normal series where each factor is either a $\pi'$-group or a solvable $\pi$-group. If $G$ is $\pi$-solvable or $\pi'$-solvable, the Schur-Zassenhaus theorem implies $G$ has a Hall-$\pi$-subgroup and that any two Hall-$\pi$-subgroups are conjugate in $G$ (see 6.3.5 and 6.3.6 in [3]). Proof of the following lemma, due to Glauberman [2], requires the conjugacy part of the Schur-Zassenhaus theorem and thus uses the Odd-Order theorem to ensure the solvability of either $A$ or $G$.

**Lemma 3.1.** Assume $A$ acts on $G$ by automorphisms and $(|A|, |G|) = 1$. Assume $A$ and $G$ act on a set $T$ such that $G$ is transitive on $T$ and $(t \cdot g) \cdot a = (t \cdot a) \cdot g^a$ for all $t \in T$, $g \in G$, $a \in A$. Then

(a) $A$ fixes an element of $T$, and

(b) $C_\sigma(A)$ acts transitively on the fixed points in $T$ of $A$.

**Proof.** See [2] or 13.8 and 13.9 of [5].

**Corollary 3.2.** Assume $A$ acts on $G$ by automorphisms, $N \leq G$ is $A$-invariant, $(|G:N|, |A|) = 1$, and $C_{G/N}(A) = 1$. Let $\chi \in IRR(G)$ and $\phi \in IRR(N)$ be $A$-invariant. Then

(a) $\chi_N$ has a unique $A$-invariant irreducible constituent; and
(b) If $G/N$ is abelian, $\phi^G$ has a unique $A$-invariant irreducible constituent.

Proof. Now $A$ and $G/N$ act on the irreducible constituents of $\chi_N$ and $G/N$ is transitive. Thus, part (a) follows from Lemma 3.1.

For (b), note $A$ and $\text{IRR}(G/N)$ act on the irreducible constituents of $\phi^G$ and $\text{IRR}(G/N)$ is transitive in this action. We are done by Lemma 3.1 if $A$ acts fix point free on $\text{IRR}(G/N)$. If $\psi \in \text{IRR}(G/N)$ is $A$-fixed, then $A$ centralizes $G/\text{Ker}(\psi)$ and $\text{Ker}(\psi) = G$. This completes the proof.

**Theorem 3.3.** Assume that $G$ is $\pi'$-solvable with a Hall-$\pi$-subgroup $S$; $N = N_0(S)$; $K, L \subseteq G$; $H = LN$; $K/L$ is an abelian $\pi'$-group; $KH = G$; and $K \cap H = L$. Let $\theta \in \text{IRR}(K)$ such that $S \unlhd I_G(\theta)$. Then

(a) $\theta_L$ has a unique $S$-invariant irreducible constituent $\phi$; and

(b) There is a one-to-one and onto map $T: \text{IRR}(G|\theta) \rightarrow \text{IRR}(H|\phi)$ such that $\chi(l)/(\chi_T(l))$ is an integer dividing $|G:H|$.

Proof. As $C_{K/L}(S) = 1$, part (a) is a consequence of Corollary 3.2. To prove (b), induct on $|G|$. By induction, it is no loss to assume $K/L$ is chief in $G$ and $H$ is maximal in $G$. Note $KN = G$. For $n \in N$, $\theta^n$ and $\phi^n$ are $S$-invariant. If $R = I_G(\theta)$, it then follows from Corollary 3.2 that $R \cap H = I_H(\phi)$. Now character induction yields one-to-one maps from $\text{IRR}(R|\theta)$ onto $\text{IRR}(G|\theta)$ and from $\text{IRR}(R \cap H|\phi)$ onto $\text{IRR}(H|\phi)$. As $|G:R| = |H:H \cap R|$, we finish by induction on $|G|$ if $R < G$.

So, we assume $I_G(\theta) = G$ and $I_H(\phi) = H$. If $I_G(\phi) = H$, $\phi^G = \theta$ and character induction defines a one-to-one map from $\text{IRR}(H|\phi)$ onto $\text{IRR}(G|\phi) = \text{IRR}(G|\theta)$. As $H$ is maximal in $G$; we assume $I_G(\phi) = G$.

If $\theta_L = \phi$, we are done by Lemma 2.1. With no loss, we assume $\theta_L = e\phi$ and $e^2 = [K:L]$. Replace $(G, L, \phi)$ by an isomorphic character triple $(G^*, L^*, \phi^*)$ where $\phi^*$ is faithful and linear (8.2 of [4]). Now $\theta^*$ is fully ramified with respect to $K^*/L^*$ and consequently vanishes off $L^*$. So $Z(K^*) = L^* \leq Z(G^*)$. Note $SL \subseteq H$ and that Fitting's lemma (5.2.3 of [3]) implies $[K/L, S] = K/L$. Also, $G^*/L^* \cong G/L$. For $\chi \in \text{IRR}(G|\phi)$ and $\psi \in \text{IRR}(H|\phi)$; $\chi^*(1)/\psi^*(1) = (\chi^*(1)/\phi^*(1)) \times (\phi^*(1)/\psi^*(1)) = \chi(1)/\psi(1)$. As $\text{IRR}(G|\theta) = \text{IRR}(G|\phi)$; the character triple isomorphism and Lemma 2.3 yield here a one-to-one and onto map $F: \text{IRR}(G|\theta) \rightarrow \text{IRR}(H|\phi)$ such that $\chi(1) = e(\chi^F)(1)$. This completes the proof.

**Theorem 3.4.** Let $G$ be $\pi'$-solvable and let $P$ be a Hall-$\pi$-subgroup of $G$. Then $|I_\pi(G)| = |I_\pi(N_0(P))|$. 

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Proof. Induct on $|G|$. Let $N = N_G(P)$ and $K = O^{\pi'}(G)$. We assume $K \neq 1$, else $N = G$. The Frattini argument yields $KN = G$. Let $K/L$ be a chief factor, so that $K/L$ is an elementary abelian $q$-group for a prime $q \in \pi'$. Let $H = LN$, so that $G = KH$. By definition of $K$, $C_{K/L}(P) = 1$. So $H \cap K = L$. It suffices via induction to show $|I_\pi(G)| = |I_\pi(H)|$.

Corollary 3.2 gives us a one-to-one correspondence between all $P$-invariant irreducible characters $\theta$ of $K$ and all $P$-invariant irreducible characters $\phi$ of $L$, in which $\theta$ and $\phi$ correspond if and only if $[\theta_L, \phi] \neq 0$ or, equivalently $[\theta, \phi^K] \neq 0$. Furthermore, this correspondence is invariant under conjugation by $N$. Since $G = KN$ and $H = LN$, we conclude that this correspondence carries $G$-conjugacy classes of $\theta$'s one-to-one and onto the $H$-conjugacy classes of $\phi$'s.

Let $S_1 = \{\chi \in IRR(G) | \chi_K \text{ has a } P \text{-invariant irreducible constituent}\}$ and $S_2 = \{\psi \in IRR(H) | \psi_L \text{ has a } P \text{-invariant irreducible constituent}\}$. The last paragraph and Theorem 3.3 yield a one-to-one and onto map $F: S_1 \rightarrow S_2$ such that $\chi(1)/(\chi F)(1)$ is an integer dividing $|G: H| = |K:L|$. If $\chi \in IRR(G)$ (or $\chi \in IRR(H)$) and $p\chi(1)$ for all $p \in \pi$; then $\chi \in S_1$ (respectively, $\chi \in S_2$). Hence $\chi \in I_\pi(G)$ if and only if $\chi \in S_1$ and $(\chi F) \in I_\pi(H)$. The proof is complete.

Actually the above results yield a one-to-one map $T: I_\pi(G) \rightarrow I_\pi(N)$ such that $\chi(1)/(\chi T)(1)$ divides $|G: N|$. In the case $\pi = \{p\}$, the above theorem states precisely that $|I_p(G)| = |I_p(N(P))|$ for $G$ solvable, where $P \in Syl_p(G)$.

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MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824
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