SCHUR INDICES OVER THE 2-ADIC FIELD

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In this paper it is proved that if \( G \) is a finite group with abelian Sylow 2-subgroups, then the Schur index of any character of \( G \) over the 2-adic numbers \( \mathbb{Q}_2 \) is equal to 1. Examples are given so as to show that this statement is false for each odd prime \( p \).

The problem of determining the Schur index of a character of a finite group was reduced by R. Brauer and E. Witt to the case of handling hyper-elementary groups at \( q \), \( q \) being a prime. Each of these groups has a cyclic normal subgroup with a factor group which is a \( q \)-group. Let \( p \) be a prime and \( \mathbb{Q}_p \) the \( p \)-adic numbers. Let \( G \) be a hyper-elementary group at \( q \) and \( \chi \) an irreducible character of \( G \). It follows from a result of Witt [1] that if \( p = q \neq 2 \) then the Schur index \( m_{\mathbb{Q}_p}(\chi) \) of \( \chi \) over \( \mathbb{Q}_p \) is equal to 1. This statement is false for the case \( p = q = 2 \), because the quaternion group of order \( 2^3 \) has an irreducible character \( \chi \) with \( m_{\mathbb{Q}_2}(\chi) = 2 \).

The purpose of this paper is to show that the above statement also holds for the case \( p = q = 2 \), provided the Sylow 2-subgroups of a hyper-elementary group at 2 are abelian. In fact, we will prove more generally the following theorem.

**Theorem.** Let \( G \) be a finite group with abelian Sylow 2-subgroups. Let \( \chi \) be any irreducible character of \( G \). Then \( m_{\mathbb{Q}_2}(\chi) = 1 \), that is the Schur index of \( \chi \) over the 2-adic numbers \( \mathbb{Q}_2 \) is equal to 1.

**Proof.** It is well-known that \( m_{\mathbb{Q}_2}(\chi) = 1 \) or 2 (cf. [1]), so \( m_{\mathbb{Q}_2}(\chi) \) equals its 2-part. Let \( n \) be the exponent of \( G \) and let \( L \) be the subfield of \( \mathbb{Q}_2(\zeta_n) \), \( \zeta_n \) a primitive \( n \)-th root of unity, such that \( L \supset \mathbb{Q}_2(\chi) \), \( 2 \nmid [L: \mathbb{Q}_2(\chi)] \) and \( [\mathbb{Q}_2(\zeta_n): L] \) is a power of 2. By the Brauer-Witt theorem [3, p. 31] there is an \( L \)-elementary subgroup \( H \) of \( G \) with respect to 2 and an irreducible character \( \theta \) of \( H \) with the following properties: (1) there is a normal subgroup \( N \) of \( H \) such that \( \theta = \psi^H \); (2) \( H/N \cong \text{Gal}(L(\psi)/L) \), in particular, \( H/N \) is a 2-group; (3) \( L(\theta) = L \); (4) \( m_L(\theta) = m_L(\chi) = m_{\mathbb{Q}_2}(\chi) = m_{\mathbb{Q}_2}(\chi) \); (5) for every \( h \in H \) there is a \( \tau(h) \in \text{Gal}(L(\psi)/L) \) such that \( \psi(hnh^{-1}) = \tau(h)(\psi(n)) \) for all \( n \in N \); (6) \( m_L(\theta) \) is the index of the crossed product \( (\beta, L(\psi)/L) \) where, if \( D \) is a complete set of coset representatives of \( N \) in \( H \) \( (1 \in D) \) with \( hh' = n(h, h')h'' \) for \( h, h', h'' \in D, n(h, h') \in N \), then \( \beta(\tau(h), \tau(h')) = \psi(n(h, h')) \). Since \( \psi \) is
a linear character of $N$, the values of the factor set $\beta$ are roots of unity.

Denote by $N_0$ the kernel of $\psi$. Then the factor group $N/N_0$ is cyclic. Put $2^rt = |N/N_0|$, $(2, t) = 1$. It is easy to see that there exist elements $a, b$ of $N$ such that $N/N_0 = \langle aN_0 \rangle \times \langle bN_0 \rangle$, $a^t \in N_0$, $b^t \in N_0$ and that the order of $a$ is a power of 2. We have $\psi(a) = \zeta_{2^r}$, $\psi(b) = \zeta_t$, and $Q_4(\psi) = Q_4(\zeta_{2^r}, \zeta_t)$, where $\zeta_{2^r}$ and $\zeta_t$ are some primitive $2^r$th and $t$th roots of unity, respectively. Let $P$ be a Sylow 2-subgroup of $H$, which contains $a$. Since $H/N$ is a 2-group, we may clearly assume that $D \subset P$. By assumption, $P$ is abelian. Hence for each $x \in D$, $xax^{-1} = a$, and so

$$\theta(a) = \psi^H(a) = \sum_{x \in D} \psi(xax^{-1}) = |D| \psi(a) = |D| \zeta_{2^r}.$$  

Consequently, $\zeta_{2^r} \in L = L(\theta)$.

Since $L(\psi) = L(\zeta_{2^r}, \zeta_t) = L(\zeta_t)$, $(2, t) = 1$, it follows that the extension $L(\psi)/L$ is unramified. Recall that the values of the factor set $\beta$ are roots of unity. Hence the crossed product $(\beta, L(\psi)/L)$ is similar to $L$, i.e., $(\beta, L(\psi)/L) \sim L$ (cf. [3, Lemma 4.2]). This implies $m_{\psi}(\chi) = m_L(\theta) = 1$, and the theorem is proved.

If $p$ is an odd prime, then Witt [1] determined that $m_{\psi}(\chi)$ divides $p - 1$ for an irreducible character $\chi$ of a finite group $G$. Let $d$ be a natural number that divides $p - 1$. We now give an irreducible character $\chi$ of a finite group $G$ with abelian Sylow $p$-subgroups such that $m_{\psi}(\chi) = d$: The group $G$ is generated by the elements $x, y$ with defining relations

$$x^p = 1, \quad y^{d(p - 1)} = 1, \quad xyx^{-1} = x^r,$$

where $r$ is a primitive root modulo $p$. (This group was dealt with in Appendix of [2].)

Now put $H = \langle x \rangle \times \langle y^{p-1} \rangle$. Then $H$ is a normal, cyclic subgroup of $G$ of order $pd$, the factor group $G/H$ is cyclic of order $p - 1$, and $G = H \cup Hx \cup \cdots \cup Hx^{p-2}$. Let $\psi$ be the faithful linear character of $H$ given by $\psi(x) = \zeta_p$, $\psi(y^{p-1}) = \zeta_d$. For each $i = 1, \ldots, p - 2$, the character $\psi^{si}$ of $H$ defined by $\psi^{si}(z) = \psi(y^izy^{-i})$, $z \in H$, is algebraically conjugate to $\psi$ over the field $Q_p(\zeta_d)$, and $\psi^{si} \neq \psi$. It follows that the induced character $\chi = \psi^G$ is irreducible and that the simple component of the group algebra $Q_p[G]$ which corresponds to $\chi$ is isomorphic to the cyclic algebra $B = (\zeta_d, Q_p(\zeta_d), \zeta_p)/Q_p(\zeta_d)$, where $\langle \sigma \rangle = Gal(Q_p(\zeta_d), \zeta_p)$, $\sigma(\zeta_p) = \zeta_p$, $\sigma(\zeta_d) = \zeta_d$ (cf. Propositions 3.4, 3.5 of [3]). Since $p \equiv 1 \pmod{d}$, then $Q_p(\zeta_d) = Q_p$, so $B = (\zeta_d, Q_p(\zeta_d))/Q_p$, $\sigma$. It is easy to see that the index of this cyclic algebra is equal to $d$ (see also Theorem 4.3 of [3]). Thus we conclude that $m_{\psi}(\chi) = d$. 
The above example shows that the similar statement to the theorem for each odd prime $p$ does not hold.

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