Pacific Journal of Mathematics

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Vol. 74, No. 2 June 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 2, 1978

TENSOR PRODUCTS OF IDEAL SYSTEMS AND THEIR MODULES

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We prove the existence and furnish an explicit construction of the tensor product in various categories of ideal systems and module systems, structures previously introduced and studied by the first author as a setting for abstract commutative algebra.

1. Introduction. Among the various notions of commutative algebra which until recently had not been carried over to the general framework of ideal systems and module systems was the notion of tensor products. In [5] P. Ezust filled this gap for module systems by a fairly laborious categorical approach. He showed namely that the category of module systems (with zero element) over a given ideal system (with zero element) has the requisite properties, including the existence of an appropriate internal Hom-functor in order to secure the presence of a left adjoint to this functor—and hence of the tensor product. In spite of the fact that a direct construction of the tensor product of module systems is to some extent implicit in his work, he makes the comment that it is not clear how such a direct construction might be carried out.

It is the purpose of this paper to give some clarifications and complements to [5] which in particular lead to a direct construction of the tensor product for various categories of module systems and ideal systems. Among these is the tensor product whose existence is established indirectly in [5]. These constructions will be preceded by a general discussion of coinduced ("final") structures for the two basic categories. From the tensor product as constructed in these basic categories, tensor products in more special subcategories are derived by a process of reflection. What gives the present situation a somewhat unusual character is the fact that we are dealing not with purely algebraic systems which are equationally defined, but rather with systems which are similar to combined algebraic-topological structures, like topological monoids.

We would like to thank our good friends Paulo Ribenboim and Larry Cummings as well as Le Centre de Recherches Mathématiques for providing excellent material assistance which enabled us to carry out this work[1].

[1] Assistance from a Canadian National Research Council operating grant and a Québec-Ontario Exchange Grant is also acknowledged.

2. Ideal systems and module systems. Let S be a commutative monoid. We shall say that there is defined an *ideal system* (or x-system) (S, x) in S if to every subset A of S there corresponds a subset $A_x \subset S$ such that

$$A \subset A_x \tag{2.1}$$

$$B \subset A_x \Longrightarrow B_x \subset A_x \tag{2.2}$$

$$SA_x \subset A_x \tag{2.3}$$

$$BA_x \subset (BA)_x \,. \tag{2.4}$$

We say that A_x is the x-ideal generated by A and in case $A = A_x$ we say simply that A is an *x-ideal.* The crucial axiom of the theory is (2.4) which for obvious reasons is referred to as the *continuity axiom.* An equivalent way of formulating (2.4) is to require that the family $\mathscr{X}$ of all x-ideals in (S, x) is closed under the operation of taking residuals:

$$A_x \in \mathscr{X} \Longrightarrow A_x : b \in \mathscr{X} \quad \text{for all} \quad b \in S\,. \tag{2.4$'$}$$

In contradistinction to what seems to be a tacit assumption in some earlier papers on ideal systems we do not exclude the possibility $\varnothing \in \mathscr{X}$ which means that the intersection of all the x-ideals in S might be void.

By a *morphism of ideal systems* $f: (S_1, x_1) \to (S_2, x_2)$ we mean a mapping of S_1 into S_2 such that

$$f(ab) = f(a)f(b) \quad \text{for all} \quad a, b \in S_1 \tag{2.5}$$

$$f(A_{x_1}) \subset (f(A))_{x_2} \quad \text{for all} \quad A \subset S_1\,. \tag{2.6}$$

The condition (2.6) amounts to saying that the inverse image by f of an x_2-ideal in S_2 is an x_1-ideal in S_1. The category of ideal systems and morphisms of ideal systems will be denoted by IDS.

To define module systems we postulate that the elements of S in an ideal system (S, x) act on a set M which is equipped with a closure operation $U \to U_y$ $(U \subset M)$ such that the following conditions are satisfied

$$(ab)u = a(bu) \quad \text{whenever} \quad a, b \in S \quad \text{and} \quad u \in M \tag{2.7}$$

$$SU_y \subset U_y \quad \text{for all} \quad U \subset M \tag{2.8}$$

$$\left.\begin{aligned} &AU_y \subset (AU)_y && (2.9)\\ &A_xU \subset (AU)_y && (2.10)\end{aligned}\right\} \quad \text{for all sets} \quad A \subset S,\ U \subset M\,.$$

(In the case of an ordinary module over a ring (2.9) and (2.10) are the analogues of the two distributive laws one has in such a situation.) If $U=U_y$ we say that U is a *y-module.*

We call the whole set-up (S, x, M, y) or more shortly (M, y) — or just M — *a module system over the ideal system* (S, x) whenever the above requirements are fulfilled. Denoting the family of x-ideals in S by $\mathscr{X}$ and the family of y-modules in M by $\mathscr{Y}$ we can give the following equivalent forms of the two continuity axioms (2.9) and (2.10) which will be particularly pertinent in what follows

$$U_y \in \mathscr{Y} \Longrightarrow U_y : a \in \mathscr{Y} \quad \text{for all} \quad a \in S \tag{2.9'}$$

$$U_y \in \mathscr{Y} \Longrightarrow U_y : u \in \mathscr{X} \quad \text{for all} \quad u \in M\,. \tag{2.10'}$$

A *morphism of module systems* is a map $f: (M_1, y_1) \longrightarrow (M_2, y_2)$ such that the following two conditions are satisfied

$$f(au) = af(u) \quad \text{for all} \quad a \in S \quad \text{and all} \quad u \in M_1 \tag{2.11}$$

$$f(U_{y_1}) \subset (f(U))_{y_2} \quad \text{for all} \quad U \subset M_1\,. \tag{2.12}$$

Again (2.12) means that the inverse image of a y_2-module in M_2 is a y_1-module in M_1. The category of module systems over (S, x) and morphisms of such module systems will be denoted by MODS(S, x) or simply by MODS when there is no danger of confusion. For a more ample treatment of the fundamentals of the theory of ideal systems and module system one may refer to [1], [2], [3], and [5].

3. Coinduced structures for ideal systems and module systems.

The question of coinduced (or "final") structures in a certain category of module systems was treated in [5]. It is desirable, however, to have a more general and fuller treatment of the matter than that presented there.

Let us first look at the category MODS(S, x) without any restriction on the ideal system (S, x). If in a module system (S, x, M, y) we ignore the closure operations x and y and keep only the action of S on M in accordance with (2.7), we are left with what is called an *S-set*. Any subset U of M such that $SU \subset U$ is called an *S-set in* (or S-subset of) M. An S-set can thus be conceived of as a module system with respect to the generation processes $x_S: A \to SA \cup A$ $(A \subset S)$ in S and $y_S: U \to SU \cup U$ $(U \subset M)$ in M respectively. The morphisms, or *S-maps*, between such systems are those satisfying just (2.11) ((2.12) being a consequence in this case). This provides an isomorphism between the category of S-sets with their S-maps and the category of module systems (S, x_S, M, y_S).

PROPOSITION 1. *Let* $\{(M_i, y_i) \mid i \in I\}$ *be a family of module systems* *over (S, x) and let M be an S-set. Assume further that for each $i \in I$ there is given an S-map $g_i: M_i \to M$. Then there exists a unique finest closure system y on M such that (M, y) is a module system over (S, x) and such that all the g_i's become morphisms of module systems. The family $\mathscr{Y}$ of all the y-modules in (M, y) may be described explicitly as follows: Let $\mathscr{Y}^*$ be the family of all those S-sets in M which have the property that the inverse image by each g_i is a y_i-module. Then*

$$\mathscr{Y} = \{U_{y^*} \mid U_{y^*} \in \mathscr{Y}^* \quad \textit{and} \quad U_{y^*}\colon m \in \mathscr{X} \quad \textit{for all} \quad m \in M\}$$

where $\mathscr{X}$ denotes the family of x-ideals in (S, x).

The proof of this proposition is largely a routine matter and may be left to the reader. Just one point may deserve special mention, namely that $\mathscr{Y}$ verifies not only the continuity axiom (2.10′) (which is already built into its definition) but also verifies (2.9′). Indeed, let $U_y \in \mathscr{Y}$. In order to show that $U_y: s \in \mathscr{Y}$ for an $s \in S$ we must show that: (i) $U_y: s \in \mathscr{Y}^*$ and that (ii) $(U_y: s): m \in \mathscr{X}$ for all $m \in M$. Now $U_y: s$ is an S-set in M, and $g_i^{-1}(U_y: s) = g_i^{-1}(U_y): s \in \mathscr{Y}_i$ for all i, which shows (i). The condition (ii) follows from the identity $(U_y: s): m = U_y: sm$ together with the definition of $\mathscr{Y}$.

We shall say that y (or $\mathscr{Y}$) is *coinduced* by the S-maps g_i. In certain cases it is not necessary to require explicitly that the sets of $\mathscr{Y}^*$ are S-sets (i.e., that $SU_{y^*} \subset U_{y^*}$ for all $U_{y^*} \in \mathscr{Y}^*$), since this condition will automatically be fulfilled if the given family of maps $\{g_i\}$ satisfies a certain covering condition. It should be emphasized, however, that the existence of coinduced structures in the category MODS(S, x) is not dependent on any such covering condition, a fact which seems to have been overlooked in [5] where the existence of coinduced module systems is made to depend on a very strong covering condition. (It should also be noted that *if* a covering condition is imposed in order to make all the $U_{y^*} \in \mathscr{Y}^*$ S-sets it would suffice to replace the covering condition given in [5] by the following weaker and more easily applicable one: Any set U in M whose inverse images by the g_i's are y_i-modules for all i is contained in the union of the images $g_i(M_i)$ (and not necessarily in any *single* $g_i(M_i)$ as required in [5]). It is easily seen that this covering condition assures that such a set U is an S-set.)

The subject of coinduced ideal systems may be treated analogously and is even simpler than in the case of module systems.

PROPOSITION 2. *Let $\{(S_i, x_i)\}$ be a family of ideal systems and let $g_i: S_i \to S$ be a family of maps to a commutative monoid S such* *that* $g_i(ab) = g_i(a)g_i(b)$ *for all* $a, b \in S_i$ *and all* i. *Then there exists a unique finest ideal system* x *on* S *making* $g_i\colon (S_i, x_i) \to (S, x)$ *a morphism of ideal systems for all* i. *The family* $\mathscr{X}$ *of* x*-ideals in this coinduced ideal system in* S *is given by*

$$\mathscr{X} = \{A_{x^*} \mid A_{x^*} \in \mathscr{X}^* \quad \text{and} \quad A_{x^*}\colon s \in \mathscr{X}^* \quad \text{for all} \quad s \in S\}$$

where $\mathscr{X}^*$ *is the family of all* s*-ideals in* S *whose inverse images by the* g_i*'s are* x_i*-ideals for all* i.

The proof is quite simple and is essentially contained in the proof of Proposition 6 in [1].

4. Tensor products of module systems. In this and the following paragraph we turn to tensor products for module systems and ideal systems, i.e., within the categories MODS(S, x) and IDS. We construe these tensor products in the usual way as objects which provide a canonical factorization of bimorphisms from $M_1 \times M_2$: meaning that fixing either of the two arguments at any value $m_1 \in M_1$ or $m_2 \in M_2$ always results in a morphism in the other argument. In the case of module systems over a fixed ideal system (S, x) we consider commutative diagrams of the form

(4.1)
$$\begin{tikzcd} & M_1 \otimes M_2 \arrow[dr, "h"] & \\ M_1 \times M_2 \arrow[ur, "g"] \arrow[rr, "f"'] & & M_3 \end{tikzcd}$$

where M_1 and M_2 are given, and the task is to determine a unique module system $M_1 \otimes M_2$ equipped with a bimorphism g from $M_1 \times M_2$ to $M_1 \otimes M_2$ such that every bimorphism f from $M_1 \times M_2$ is a composition of the canonical bimorphism g and a unique morphism h from $M_1 \otimes M_2$. In this way all the bimorphisms from $M_1 \times M_2$ are obtained by letting the morphisms from $M_1 \otimes M_2$ "operate on" the fixed bimorphism g — which may thus be considered as a generator of the set of all bimorphisms which have $M_1 \times M_2$ as a domain of definition.

The categories we are dealing with here are of a mixed algebraic — topological kind. The algebraic part will be equational so that the tensor product within their poorer structure may be obtained as in the classical case via the free algebra on the (unstructured) set $M_1 \times M_2$ modulo the congruence generated by the minimal identifications which make the quotient map induce a bimorphism of $M_1 \times M_2$: More precisely it will be this factor algebra equipped with the restriction to $M_1 \times M_2$ of the quotient map as bimorphism. One will be able to convert this algebraic solution into one for the richer mixed category whenever the latter has coinduced structures. Indeed the mixed tensor product is then just the algebraic one equipped with the finest closure operation which makes each of the maps obtained by fixing an argument in the canonical bimorphism into a morphism of module systems (or ideal systems as the case may be).

THEOREM 1. *The category* MODS(S, x) *has tensor products.*

Proof. As already indicated we first consider the module systems M_1 and M_2 merely as S-sets and take the free S-set $\mathscr{F}_S(M_1 \times M_2)$ on the set $M_1 \times M_2$, which will be the disjoint union of copies of S indexed by $M_1 \times M_2$. Instead of using the cumbersome notation $s_{(m_1, m_2)}$ for an element in $\mathscr{F}_S(M_1 \times M_2)$ we shall denote this element by $s(m_1, m_2)$ and remark that the congruence in question is here generated by the relations

$$s(m_1, m_2) = (sm_1, m_2) = (m_1, sm_2) \, . \tag{4.2}$$

(Other relations such as $s_1 s_2(m_1, m_2) = s_1(m_1, s_2 m_2) = (s_1 m_1, s_2 m_2)$ etc. are easily seen to be derivable from 4.2). Denoting this congruence relation by $\sim$ we put

$$M_1 \otimes_S M_2 = \mathscr{F}_S(M_1 \times M_2)/\sim$$

and let $g\colon M_1 \times M_2 \to M_1 \otimes M_2$ be the restriction of the canonical quotient map. (This could just as well be obtained as the quotient map for the equivalence generated by (4.2) as in [4] or [9].)

Proposition 1 now tells us how to equip $M_1 \otimes M_2$ with a coinduced structure relative to the union of the two families of S-maps $\{{}_{m_1}g\}$ and $\{g_{m_2}\}$ defined by ${}_{m_1}g(x) = m_1 \otimes x$ for $m_1 \in M_1$ and $g_{m_2}(x) = x \otimes m_2$ for $m_2 \in M_2$, thus providing $M_1 \otimes M_2$ with the finest module system making g into a bimorphism. This module system on $M_1 \otimes M_2$ is denoted by $y_1 \otimes y_2$ and the corresponding family of $y_1 \otimes y_2$-modules by $\mathscr{Y}_1 \otimes \mathscr{Y}_2$.

Finally the map h defined by $h(m_1 \otimes m_2) = f(m_1, m_2)$ will now be a morphism of module systems making the diagram (4.1) commutative. Clearly, h is a well-defined S-map. To show that h is also a morphism it will thus suffice to show that $h^{-1}(U_{y_3}) \in \mathscr{Y}_1 \otimes \mathscr{Y}_2$ whenever $U_{y_3} \in \mathscr{Y}_3$. First of all $h^{-1}(U_{y_3})$ is an S-set since h is an S-map and U_{y_3} is an S-set. Using the notation ${}_{m_1}f$ and f_{m_2} analogously to ${}_{m_1}g$ and g_{m_2} above, we have ${}_{m_1}f = h \circ {}_{m_1}g$ and $f_{m_2} = h \circ g_{m_2}$. Since f is a bimorphism ${}_{m_1}f$ and f_{m_2} are morphisms of module systems i.e.,

$$ {}_{m_1}g^{-1}(h^{-1}(U_{y_3})) \in \mathscr{Y}_2 \quad \text{and} \quad g_{m_2}^{-1}(h^{-1}(U_{y_3})) \in \mathscr{Y}_1 \quad \text{for all} \quad m_1, m_2 \, . \tag{4.3}$$

Taking into account that the structure on $M_1 \otimes M_2$ is the coinduced one, this means that $h^{-1}(U_{y_3})$ belongs to the family which corresponds to $\mathscr{Y}^*$ in Proposition 1. But $h^{-1}(U_{y_3})$ also belongs to the "cut-down" family (designated by $\mathscr{Y}$ in Proposition 1) since $h^{-1}(U_{y_3})\colon m \in \mathscr{X}$ for all $m = m_1 \otimes m_2 \in M_1 \otimes M_2$. Indeed, from $f_{m_2}^{-1}(U_{y_3}) \in \mathscr{Y}_1$ for $U_{y_3} \in \mathscr{Y}_3$ it follows that $f_{m_2}^{-1}(U_{y_3})\colon m_1 \in \mathscr{X}$ for all $m_1 \in M_1$, and since $f_{m_2} = h \circ g_{m_2}$ this means that $g_{m_2}^{-1}(h^{-1}(U_{y_3}))\colon m_1 = h^{-1}(U_{y_3})\colon g_{m_2}(m_1) = h^{-1}(U_{y_3})\colon (m_1 \otimes m_2) \in \mathscr{X}$ for all $m_1 \otimes m_2 \in M_1 \otimes M_2$. This completes the proof of Theorem 1.

5. Tensor products of ideal systems. In the present paragraph we prove

THEOREM 2. *The category of ideal systems has tensor products.*

Proof. Let (S_1, x_1) and (S_2, x_2) be two ideal systems. In line with our general approach we first look for a monoid bimorphism g which factors every bimorphism f from $S_1 \times S_2$ the other factor being a unique monoid morphism h from $S_1 \otimes S_2$ as in the following diagram.

(5.1)
$$\begin{tikzcd} & S_1 \otimes S_2 \arrow[dr, "h"] & \\ S_1 \times S_2 \arrow[ur, "g"] \arrow[rr, "f"'] & & S_3 \end{tikzcd}$$

That f is a bimorphism means that

(5.2)
$$\begin{aligned} f(s_1, s_2t_2) &= f(s_1, s_2)f(s_1, t_2) \quad \text{and} \\ f(s_1t_1, s_2) &= f(s_1, s_2)f(t_1, s_2) \end{aligned}$$

for all s_1, t_1 in S_1 and s_2, t_2 in S_2. We first form the free commutative monoid $\mathscr{F}(S_1 \times S_2)$. In accordance with (5.2) one considers the congruence relation $\sim$ generated by the two relations $(s_1, s_2t_2) = (s_1, s_2)(s_1, t_2)$ and $(s_1t_1, s_2) = (s_1, s_2)(t_1, s_2)$ and puts

$$S_1 \otimes S_2 = \mathscr{F}(S_1 \times S_2)/\sim .$$

(See also [6] and [7].)

In complete analogy with the previous development for module systems, consider the coinduced structure on $S_1 \otimes S_2$ which results from the family of maps $\{{}_{s_1}g\}$ and $\{g_{s_2}\}$ in accordance with Proposition 2. This defines an ideal system $x_1 \otimes x_2$ on $S_1 \otimes S_2$ with a corresponding family of $x_1 \otimes x_2$-ideals denoted by $\mathscr{X}_1 \otimes \mathscr{X}_2$.

It is again easy to see that map h defined by $h(s_1 \otimes s_2) = f(s_1, s_2)$ will be a morphism of ideal systems from $(S_1 \otimes S_2, x_1 \otimes x_2)$ into (S_3, x_3). Since h is already a multiplicative (monoid) morphism we need only verify that $h^{-1}(A_{x_3}) \in \mathscr{X}_1 \otimes \mathscr{X}_2$ for every $A_{x_3} \in \mathscr{X}_3$ – which means according to Proposition 2 that the following conditions hold for all $s_1, t_1 \in S_1$, $s_2, t_2 \in S_2$ and all $A_{x_3} \in \mathscr{X}_3$.

$$h^{-1}(A_{x_3}) \text{ is an } s\text{-ideal in } S_1 \otimes S_2 \tag{5.3}$$

$$g_{s_2}^{-1}(h^{-1}(A_{x_3})) \in \mathscr{X}_1 \text{ and } {}_{s_1}g^{-1}(h^{-1}(A_{x_3})) \in \mathscr{X}_2 \tag{5.4}$$

$$\begin{gathered} g_{s_2}^{-1}(h^{-1}(A_{x_3})\colon (t_1 \otimes t_2)) \in \mathscr{X}_1 \text{ and} \\ {}_{s_1}g^{-1}(h^{-1}(A_{x_3})\colon (t_1 \otimes t_2)) \in \mathscr{X}_2 . \end{gathered} \tag{5.5}$$

To prove (5.3) let $b \in h^{-1}(A_{x_3})$, hence $h(b) \in A_{x_3}$ and $h(s)h(b) \in A_{x_3}$ for all $s \in S_1 \otimes S_2$. Thus $h(sb) \in A_{x_3}$ and $sb \in h^{-1}(A_{x_3})$ as required. The verification of (5.4) is similar to the verification of (4.3) above and may be left to the reader. Finally (5.5) results easily by using the identity $h^{-1}(A_{x_3})\colon (t_1 \otimes t_2) = h^{-1}(A_{x_3}\colon h(t_1 \otimes t_2))$ together with ${}_{s_1}f = h \circ {}_{s_1}g$ and $f_{s_2} = h \circ g_{s_2}$.

6. Tensor products in certain subcategories of IDS and MODS(S, x). Many of the ideal systems and module system which have been considered in the literature, e.g., those mentioned in [1], [2], and [3], are more restricted than those we have chosen here for the comprehensive categories IDS and MODS(S, x) in that they require the presence of zero elements. However, the notion of a zero generalizes slightly differently in the context of ideal systems and that of module systems.

A *multiplicative zero* in a commutative monoid S is an element $0 \in S$ such that $a0 = 0$ for all $a \in S$. Such a multiplicative zero in S, if it exists, is unique.

If (S, x) is an ideal system we shall say that the element $0 \in S$ is an *x-zero* if $\{0\}_x = \{0\}$. If 0 is an x-zero it is also a multiplicative zero. Any nonvoid x-ideal in an ideal system with a multiplicative zero will contain this zero element and the possibility $\varnothing \in \mathscr{X}$ may safely be dispensed with as uninteresting. Hence we postulate that the presence of a zero rules out the possibility that the void set be counted as an x-ideal. We define the *x-kernel* of an ideal system (S, x) as the x-ideal A_x^0 which is the intersection of all the x-ideals in S: thus the smallest x-ideal in (S, x). According to the above convention $A_x^0 \neq \varnothing$ whenever (S, x) has a multiplicative zero 0, and then $A_x^0 = \{0\}_x$.

The notion which in the "module" situation parallels that of a multiplicative zero is that of an element θ in an S-set M such that $a\theta = \theta$ for all $a \in S$. If (M, y) is a module system over (S, x) then θ is said to be a y^*-zero if $\{\theta\}_y = \{\theta\}$. Again $a\theta = \theta$ for all $a \in S$ when θ is a y^*-zero. Whereas an x-zero is uniquely determined by the requirement $\{0\}_x = \{0\}$ the corresponding requirement $\{\theta\}_y = \{\theta\}$ does not determine θ uniquely in M. In order to obtain uniqueness it is reasonable to define a *y-zero* in M as an element θ such that $\{\theta\}$ is equal to the intersection of all the nonvoid y-modules in (M, y). On the other hand if S has a multiplicative zero 0 such that $0m = \theta$ this requirement will of itself impose unicity on θ. In the case of module systems also we agree to discard the void set as a y-module in the presence of a y-zero. Furthermore the *y-kernel* U_y^0 of (M, y) is defined as the intersection of all the y-modules in (M, y).

Let IDS_0 denote the category of ideal systems with an x-zero and morphisms of ideal systems. Correspondingly $\mathrm{MODS}_0(S, x)$ will denote the category of module systems with a y-zero over (S, x) and morphisms of module systems. The categories IDS_0 and $\mathrm{MODS}_0(S, x)$ sit as full subcategories in IDS and $\mathrm{MODS}(S, x)$ respectively. In order to construct the tensor products in IDS_0 and $\mathrm{MODS}_0(S, x)$ we shall employ the notion of a Rees-congruence and the corresponding formation of factor systems. By means of this we can show that IDS_0 and $\mathrm{MODS}_0(S, x)$ are *reflective* subcategories in IDS and $\mathrm{MODS}(S, x)$ respectively and this categorical fact will tell us how to obtain the tensor product in the smaller category when we know it in the bigger. We shall return briefly to this general categorical viewpoint after we have treated the special cases of IDS_0 and $\mathrm{MODS}_0(S, x)$.

Given an ideal (s-ideal) in the monoid S (i.e., a subset A_s of S such that $SA_s \subset A_s$) the *Rees-congruence* modulo A_s is defined by declaring any two elements in A_s as congruent to each other whereas any element outside of A_s is only congruent to itself. A similar definition applies also in the "module" situation modulo S-subsets instead of s-ideals. In particular this applies to an x-ideal A_x in an ideal system (S, x) and to a y-module U_y in a module system (M, y). Denoting the factor systems modulo the Rees-congruence by a double bar, we obtain an ideal system $(S//A_x, \bar{\bar{x}})$ and a module system $(M//U_y, \bar{\bar{y}})$ by imposing the finest systems $\bar{\bar{x}}$ and $\bar{\bar{y}}$ which make the canonical quotient maps into morphisms (by Propositions 1 and 2 for a single i).

THEOREM 3. *The categories* $\mathrm{MODS}_0(S, x)$ *and* IDS_0 *have tensor products.*

Proof. We give the proof only in the case of $\mathrm{MODS}_0(S, x)$, the proof for IDS_0 being quite similar. We first note that the full subcategory $\mathrm{MODS}_*(S, x)$ consisting of those module systems for which the y-kernel is nonvoid is closed under the taking of tensor products in $\mathrm{MODS}(S, x)$. This follows from the definition of the tensor product together with the fact that $\mathrm{MODS}_*(S, x)$ is closed for coinduced structures since the inverse image of the void set is void. Moreover, this subcategory contains $\mathrm{MODS}_0(S, x)$. The advantage of restricting ourselves to the subcategory MODS_* is that the Rees factor system modulo the y-kernel in this subcategory will always yield a module system with a y-zero. Thus if we first form the tensor product in $\mathrm{MODS}(S, x)$ of systems M_1, M_2 in $\mathrm{MODS}_0(S, x)$ and then pass to the Rees quotient modulo its nonvoid y-kernel $U^0_{y_1\otimes y_2}$, we obtain in $M_1 \otimes M_2 // U^0_{y_1\otimes y_2}$ an object of $\mathrm{MODS}_0(S, x)$. This gives rise to the following extension of the Diagram (4.1) where also M_3 is now supposed to be in $\mathrm{MODS}_0(S, x)$.

(6.1)
$$\begin{tikzcd}
M_1 \otimes M_2 \arrow[rr, "\phi"] \arrow[drr, "h" near end] & & M_1 \otimes M_2 // U^0_{y_1\otimes Y_2} \arrow[d, "\bar{\bar{h}}"] \\
M_1 \times M_2 \arrow[u, "g"] \arrow[urr, "\psi" near start] \arrow[rr, "f"'] & & M_3
\end{tikzcd}$$

Here ϕ is the canonical quotient morphism, $\psi = \phi \circ g$ and $\bar{\bar{h}}$ is defined by $\bar{\bar{h}}(\overline{m_1 \otimes m_2}) = h(m_1 \otimes m_2)(=f(m_1, m_2))$ where $\overline{m_1 \otimes m_2}$ denotes the Rees residue class to which $m_1 \otimes m_2$ belongs modulo the $y_1 \otimes y_2$-kernel. The map $\bar{\bar{h}}$ is well-defined because of the definition of the Rees-congruence together with the fact that $U^0_{y_1\otimes y_2} \subset h^{-1}(\theta_3)$, h being a morphism and M_3 having a y_3-zero θ_3. Finally one easily verifies that $\bar{\bar{h}}$ is a morphism taking into account that the Rees factor system modulo $U^0_{y_1\otimes y_2}$ is coinduced by ϕ.

From the unique factorization $f = \bar{\bar{h}} \circ \psi$ it is clear that the bimorphism ψ solves the universal problem for bimorphisms in the category $\mathrm{MODS}_0(S, x)$ and establishes the module system

(6.2)
$$(M_1 \otimes M_2 // U^0_{y_1\otimes y_2}, \overline{y_1 \otimes y_2})$$

as the tensor product of (M_1, y_1) and (M_2, y_2) in the category $\mathrm{MODS}_0(S, x)$.

The Diagram (6.1) just represents the conjunction of the solutions to two different universal problems. Whereas the lower left triangle consisting of the arrows f, g, and h gives the canonical factorization of bimorphisms in $\mathrm{MODS}(s, x)$ the upper right triangle consisting of the arrows h, ϕ, and $\bar{\bar{h}}$ gives the canonical factorization of a morphism which goes into an object in $\mathrm{MODS}_0(S, x)$: Any morphism of the latter kind can be uniquely factored through the Rees factor system.

7. Tensor products in reflective subcategories. A subcategory $\mathscr{B}$ of $\mathscr{A}$ is said to be *reflective* in $\mathscr{A}$ [8] when the inclusion functor $\mathscr{B} \to \mathscr{A}$ has a left adjoint. This left adjoint functor R is sometimes called a *reflector* and the adjoint functor $\mathscr{A} \to \mathscr{B}$ a *reflection* of $\mathscr{A}$ in its subcategory $\mathscr{B}$. A reflection may be described in terms of universal morphisms: $\mathscr{B} \subset \mathscr{A}$ is reflective if and only if to each $A \in \mathscr{A}$ there is an object RA of the subcategory $\mathscr{B}$ and a morphism in $\mathscr{A}$ $g_A: A \to RA$ such that every morphism $f: A \to B$ where B is in $\mathscr{B}$ has the form $f = h \circ g_A$ for a unique morphism $h: RA \to B$ of $\mathscr{B}$. What we have made use of in constructing the tensor product in $\mathrm{MODS}_0(S, x)$ is that it is a reflective subcategory of $\mathrm{MODS}_*(S, x)$ with the functor $R: (M, y) \to (M//U_y^0, y)$ as reflector. In general, the tensor product in any reflective subcategory $\mathscr{B}$ of a concrete category $\mathscr{A}$ may be obtained from the tensor product in $\mathscr{A}$ according to the formula

$$B_1 \otimes_{\mathscr{B}} B_2 = R(B_1 \otimes_{\mathscr{A}} B_2) . \tag{7.1}$$

As a further illustration of this procedure consider the subcategory $\mathrm{MODSEP}_0(S, x)$ consisting of module systems over (S, x) with a separating y-zero. According to [2] and [5] a module system (M, y) is said to have a *separating* y*-zero* θ if θ is a y-zero and $(m_1)_y = (m_2)_y \Rightarrow m_1 = m_2(m_1, m_2 \in M)$. A y-zero may be made separating, and more generally a Rees factor system may be reduced to a MODSEP_0 system, by dividing out by a strengthened congruence: Given a y-module U_y in (M, y) one introduces as in [3] a y*-congruence* modulo U_y by putting $u \equiv v(U_y)$ whenever $(U_y, u)_y = (U_y, v)_y$. The resulting factor module system $(\bar{M}, \bar{y}) = (M/U_y, \bar{y})$ over (S, x) where $\bar{y}$ is the finest module system in $\bar{M}$ making the canonical map $(M, y) \to (\bar{M}, \bar{y})$ into a morphism of module systems, is in MODSEP_0 whenever U_y is nonvoid. In fact, that the y-zero θ is separating amounts to saying that the y-congruence modulo θ reduces to equality. Thus $\mathrm{MODSEP}_0(S, x)$ is reflective subcategory of $\mathrm{MODS}_*(S, x)$ with $R: (M, y) \to (M/U_y^0, \bar{y})$ as reflector. It is also possible to view $\mathrm{MODSEP}_0(S, x)$ as a reflective subcategory of $\mathrm{MODS}_0(S, x)$ with $(M, y) \to (M/(\theta)_y, \bar{y})$ as reflector. In either case the tensor product in $\mathrm{MODSEP}_0(S, x)$ may be obtained from the general formula (7.1). It goes without saying that the same procedure applies to the corresponding category IDSEP_0 in the case of ideal systems. In [6] this procedure has been used implicitly to construct tensor products of commutative semigroups as well as tensor products of semigroups with zero from an initial construction of tensor products for general (i.e., noncommutative) semigroups.

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Received February 24, 1977 and in revised form September 12, 1977.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 74, No. 2 June, 1978