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**COHERENT POLYNOMIAL RINGS OVER REGULAR RINGS
OF FINITE INDEX**

ANDREW CARSON

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ANDREW B. CARSON

It is shown that polynomial rings in finitely or infinitely many central indeterminates, over a regular ring of finite index, are right and left coherent.

In this paper all rings have unity and all ring homomorphisms preserve the unity.

DEFINITION 1. A ring R is:

(i) Regular, if it satisfies the sentence

$$(\forall r)(\exists s)[rsr = r];$$

(ii) Of index n , where $n \geq 1$ is an integer, if for all $m \geq n$, it satisfies the sentence

$$(\forall r)[r^m = 0 \longrightarrow r^n = 0];$$

(iii) Of finite index if it is of index n , for some integer $n \geq 1$.

DEFINITION 2. A ring R is left coherent if:

(i) $U \cap V$ is a finitely generated left ideal in R , whenever U and V are finitely generated left ideals in R , and

(ii) For each $r \in R$, the left annihilator of r in R is finitely generated, as a left ideal in R .

Right coherence for R is similarly defined.

DEFINITION 3. Let f be an element of and I a finite subset of a polynomial ring $T[X_1, \dots, X_q]$.

Then:

(i) $\deg(f)$ is the total degree of f ,

(ii) $\deg(I) = \text{Sup}\{\deg(f): f \in I\}$, and

(iii) $\langle I \rangle$ denotes the left ideal generated by I .

It is known (cf. [3, Theorem 2.2]) that a ring is left coherent iff each of its finitely generated left ideals is finitely presented. Thus, for certain homological applications, the left coherent rings seem to be a suitable generalization of the left Noetherian rings. In view of the Hilbert basis theorem (which states that $T[X]$ is left Noetherian if T is), this suggests the following conjecture: if R is a left coherent ring, then $R[X]$ is too. Soublin, in [11], disproved this conjecture, even when R is commutative. However he showed

that it does hold when R is commutative and regular. (All regular rings are right and left coherent and all commutative regular rings have index 1.)

The main result of this paper is:

THEOREM 1. *Let R be a regular ring of finite index. Then the polynomial ring $R[\{X_\alpha\}]$ is left and right coherent, for any finite or infinite set $\{X_\alpha\}$ of central indeterminates.*

In [1] we established this result in the special case when R is also a commutative algebraic algebra over some field. To do this we effectively showed that the result held for any regular ring that can be embedded in a ring S such that, for each $q \geq 1$, $S[X_1, \dots, X_q]$ is left and right coherent. We then showed that, in this case, suitable S actually exist.

For the rest of this paper let R be an arbitrary regular ring of finite index, and let $q \geq 1$ be any fixed integer. The following lemma yields Theorem 1:

LEMMA 1. *There exists a ring S containing R as a subring such that $S[X_1, \dots, X_q]$ is left and right coherent.*

Our proof of Lemma 1 hinges upon Lemma 2. Our approach to Lemma 2 is model theoretic.

Basic concepts of model theory, such as a (well formed) formula, a free variable, and a bound variable are found in [9]. A sentence is a formula in which all variables are bound. Let L denote the first order predicate calculus for rings.

The major obstacle in applying model theory to our problem is that many useful statements cannot be expressed in L . For example, there is no sentence in L which is satisfied by, and only by, polynomial rings. Further, the statement " $f \in U$ ", where U is an ideal in some ring, cannot be expressed in L . To overcome these difficulties, we note that certain formulae ϕ , concerning polynomial rings in X_1, \dots, X_q , can be translated as formulae Φ in L such that for any ring T , ϕ holds in $T[X_1, \dots, X_q]$ iff Φ holds in T .

Robinson observes (cf. [10, Chapter 5, §4]) that if r and n are fixed and we have bounds on the degrees of f and of each g_i , then the formula for polynomials (in X_1, \dots, X_q) over a division ring

$$(a) \quad (\exists h_1) \cdots (\exists h_r)[f = \sum_{i=1}^r h_i g_i \text{ and } \deg(h_j) \leq n, \text{ when } 1 \leq j \leq r]$$

can be translated into a formula of L , in the above sense. The translation is a conjunction of certain formulae involving the

coefficients of these polynomials. In this situation we shall always assume that $\text{deg}(f) \leq \max \{\text{deg}(h_i g_i) : 1 \leq i \leq r\}$. This bounds the number of variables required in the translation, in place of f . Further, there exists a function r such that for any $n \geq 1$, division ring D , and finite subset $K \subseteq D[X_1, \dots, X_q]$ such that $\text{deg}(K) \leq n$, $\{f \in \langle K \rangle : \text{deg}(f) \leq n\}$ is a vector space over D of dimension $\leq r(n)$. This is because $\{f \in D[X_1, \dots, X_q] : \text{deg}(f) \leq n\}$ is a vector space generated by those products π of indeterminates satisfying $\text{deg}(\pi) \leq n$. Thus K may (and always will) be identified with a set $A = \{g_1, \dots, g_{r(n)}\}$ (with repetitions if necessary) such that $\langle A \rangle = \langle K \rangle$.

For each m and $n \geq 1$ define predicates $\in_{m,n}$ and $\subseteq_{m,n}$, where f is a polynomial, $K = \{g_1, \dots, g_{r(m)}\}$ satisfies $\text{deg}(K) \leq m$, and $K' = \{g'_1, \dots, g'_{r(m+n)}\}$ satisfies $\text{deg}(K') \leq m+n$, by $f \in_{m,n} K$ iff (a) holds when $r = r(m)$; and $K' \subseteq_{m,n} K$ iff $g'_i \in_{m,n} K$, for all $g'_i \in K$.

Using Robinson's observation, identify these predicates with their translations into L . Let $f \in_{m,n}(K, K')$ be the conjunction

$$[(f \in_{m,n} K) \wedge (f \in_{m,n} K')],$$

where $\text{deg}(K') \leq m$ too. Similarly define $G \subseteq_{m,n}(K, K')$. Let $f \notin_{m,n} K$, $K' \not\subseteq_{m,n} K$, and $G \not\subseteq_{m,n}(K, K')$ be the negations of $f \in_{m,n} K$, $K' \subseteq_{m,n} K$, and $G \subseteq_{m,n}(K, K')$, respectively.

Although not themselves in the first order language L , the traditional \in and \subseteq are related to these predicates as follows, where (m, n) and (m', n') take values in $\{(m, n) : \text{deg}(K) \leq m \text{ and } \text{deg}(K') \leq m+n\}$ and in $\{(m', n') : \text{deg}(K) \leq m' \text{ and } \text{deg}(f) \leq m'+n'\}$, respectively:

$f \in \langle K \rangle$ iff $f \in_{m',n'} K$ for sufficiently large n' ,

$f \notin \langle K \rangle$ iff $f \notin_{m',n'} K$ for all n' ,

$\langle K' \rangle \subseteq \langle K \rangle$ iff $\langle K' \rangle \subseteq_{m,n} \langle K \rangle$ for sufficiently large n ,

and $\langle K' \rangle \not\subseteq \langle K \rangle$ iff $\langle K' \rangle \not\subseteq_{m,n} \langle K \rangle$ for all n .

The next result is crucial in establishing Theorem 1.

LEMMA 2. *There exist (for each q) integral valued functions $M(-)$ and $N(-, -)$ such that for any division ring D and finite subsets I and J of $D[X_1, \dots, X_q]$, having degrees \leq some m , there is a subset G such that:*

(i) $G \subseteq_{m, M(m)}(I, J)$ (so that $\text{deg}(G) \leq m + M(m)$);

(ii) Whenever $f \in_{m,n}(I, J)$, then $f \in_{m+M(m), N(m,n)} G$;

and thus

(iii) $\langle G \rangle = \langle I \rangle \cap \langle J \rangle$.

If the result were false, it would be obvious to any model theorist (cf. [4]) that ultraproducts could be used to construct a division ring E and finite subsets I and J of $E[X_1, \dots, X_q]$ such that

$\langle I \rangle \cap \langle J \rangle$ is not finitely generated. This would contradict the Hilbert basis theorem.

LEMMA 3. *If $S = \prod\{D_\alpha: \alpha \in A\}$ is a product of division rings and $T = S[X_1, \dots, X_q]$, then T is left and right coherent.*

Proof. By symmetry it suffices to show that T is left coherent.

An element $s \in S$ is a function such that $s(\alpha) \in D_\alpha$, for each $\alpha \in A$. For any $t = \sum s_i \pi_i \in T$, where each $s_i \in S$ and each π_i is a product of indeterminates, let $t(\alpha) = \sum s_i(\alpha) \pi_i$. For each subset U of T let $U_\alpha = \{u(\alpha): u \in U\}$. Clearly, if U is a left ideal in T , then U_α is a left ideal in T_α and $T_\alpha = D_\alpha[X_1, \dots, X_q]$, for each $\alpha \in A$.

To see that the left annihilator of any $t = \sum s_i \pi_i \in T$ is finitely generated (in fact generated by an idempotent) choose $e \in S$ such that $e(\alpha) = 1$ if $t(\alpha) = 0$, and $e(\alpha) = 0$ otherwise. Clearly $T \cdot e$ is the left annihilator of t .

Now let I and J be finite subsets of T and choose m such that $\deg(I) \leq m$ and $\deg(J) \leq m$. For each $\alpha \in A$, $\deg(I_\alpha) \leq m$ and $\deg(J_\alpha) \leq m$ so that there exists a subset $G_{(\alpha)} = \{g_{\alpha,1}, \dots, g_{\alpha,s}\} \subseteq T_\alpha$ (where $s = r(\mathbf{M}(m))$) such that conditions (i), (ii), and (iii) from Lemma 2 hold, when I, J, G , and D are replaced by $I_\alpha, J_\alpha, G_{(\alpha)}$, and D_α respectively. Define a finite subset $G = \{g_1, \dots, g_s\} \subseteq T$ by $g_i(\alpha) = g_{\alpha,i}$, for each $\alpha \in A$. We must now show that these g_i actually exist. For any i , $g_i \in T$ exists as defined iff $\{\deg(g_{\alpha,i}): \alpha \in A\}$ is bounded above. By Lemma 2 (i), $m + \mathbf{M}(m)$ is such a bound.

We shall establish that $\langle I \rangle \cap \langle J \rangle = \langle G \rangle$. To see that $\langle I \rangle \cap \langle J \rangle \subseteq \langle G \rangle$ let $f \in \langle I \rangle \cap \langle J \rangle$ and $n = \deg(f)$. Then, for each $\alpha \in A$, $f(\alpha) \in \langle I_\alpha \rangle \cap \langle J_\alpha \rangle = \langle G_\alpha \rangle \subseteq \langle T_\alpha \rangle$. Lemma 2 (ii) yields elements $h_{\alpha,i} \in T_\alpha$ such that $f(\alpha) = \sum_{i=1}^s h_{\alpha,i} g_i(\alpha)$, and an upper bound, $N(m, n)$, to $\{\deg(h_{\alpha,i}): 1 \leq i \leq s \text{ and } \alpha \in A\}$. Thus there are elements $h_i \in T$ satisfying $h_i(\alpha) = h_{\alpha,i}$. Therefore $f = \sum_{i=1}^s h_i g_i \in \langle G \rangle$. The proof that $\langle G \rangle \subseteq \langle I \rangle \cap \langle J \rangle$ is similar, and uses Lemma 2 (i).

Proof of Lemma 1. Let Q be the complete left quotient ring of R . By [2, Theorem A] there is an isomorphism

$$Q \cong \bigoplus_{i=1}^a (D_i)_{u(i)}$$

where a and the $u(i)$ are suitable integers, expressing Q as a direct sum of matrix rings over regular rings D_i , of index 1. For each i , let $S_i = \prod\{D_i/M: M \text{ is a maximal ideal in } D_i\}$. Kaplansky has shown (cf. [8, Theorem 2.3]) that each D_i/M is a division ring and there is an embedding $D_i \subseteq S_i$. Let $S = \bigoplus_{i=1}^a (S_i)_{u(i)}$. Clearly $R \subseteq S$ and

*
$$S[X_1, \dots, X_q] \cong \bigoplus \sum_{i=1}^q (S_i[X_1, \dots, X_q])_{u(i)} .$$

By Lemma 3, each $S_i[X_1, \dots, X_q]$ is left coherent. Thus [6, Corollary 2.2] establishes that each $(S_i[X_1, \dots, X_q])_{u(i)}$ is left coherent whence, by * and ([7, Corollary 2.1], $S[X_1, \dots, X_q]$ is too). Right coherence is established similarly.

REMARK 1. Let \mathcal{C} be any class of rings closed under elementary equivalence. (I.e., if $D \in \mathcal{C}$ satisfies the same sentences from L as D' , then $D' \in \mathcal{C}$.) The above methods may be used to show that if $D[X_1, \dots, X_q]$ is (left) coherent for each $D \in \mathcal{C}$ and S is a product of rings from \mathcal{C} , then $S[X_1, \dots, X_q]$ is (left) coherent too. However proving this is somewhat cumbersome since the number of elements in various $G \subseteq D[X_1, \dots, X_q]$ having degree $\leq m$, for various $D \in \mathcal{C}$, need not be bounded. This complicates the definitions of $\in_{m,n}, \subseteq_{m,n}$, and the statement and proof of Lemma 2. In addition, the $D[X_1, \dots, X_q]$ need not be integral domains. Thus another lemma is required stating that there exist integral valued functions $A(-)$ and $B(-, -)$ such that for each $D \in \mathcal{C}$ and $f \in D[X_1, \dots, X_q]$ having degree $\leq m$, there exists a subset

$$\{g_1, \dots, g_{A(m)}\} \subseteq D[X_1, \dots, X_q] \quad \text{of degree} \leq A(m)$$

such that $G \cdot f = 0$ and, if $k \cdot f = 0$ and $\text{deg}(k) \leq n$, then there exist $h_1, \dots, h_{A(m)}$ (each having degree $\leq B(m, n)$) such that

$$k = \sum_{i=1}^{A(m)} h_i g_i .$$

REMARK 2. In particular, for any fixed n , if $\mathcal{C} = \{(D)_m : m \leq n \text{ and } D \text{ is a division ring}\}$, then it is closed under elementary equivalence since, by [8, Theorem 2.3], it consists of regular rings of index n satisfying

$$(\forall e)([e^2 = e \wedge (\forall r)(re = er)] \rightarrow [e = 0 \vee e = 1]) .$$

This provides an alternate proof of Lemma 1. Suppose that R is a regular ring of index n . Let $S = R/M$: M is a maximal ideal in R . Since R is regular, it is standard that the natural map $R \rightarrow S$ is an embedding. By [8, Theorem 2.3] each $R/M \in \mathcal{C}$. As in the proof of Lemma 1, $T[X_1, \dots, X_q]$ is left coherent, for each $T \in \mathcal{C}$. Thus, by Remark 1, $S[X_1, \dots, X_q]$ is left coherent.

REMARK 3. Our approach to Theorem 1 uses the structure results for regular rings of finite index obtained in [2] and [8]. We do not know if Theorem 1 also holds for arbitrary regular rings.

REMARK 4. Eklof and Sabbagh have related coherence for rings to certain model theoretic concepts. They show (cf. [5, Theorem 3.16]) that a ring A is coherent iff each ultraproduct of \aleph_0 -injective A -modules is \aleph_0 -injective, and that (cf. [5, Theorems 4.1 and 4.8]) A is coherent iff the elementary theory of its modules has a model completion. (A ring A is \aleph_0 -injective if, for each finitely generated ideal U and $f \in \text{Hom}(U, A)$, there exists $g \in \text{Hom}(A, A)$ such that $g|_U = f$.)

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