SEQUENCES OF BOUNDED SUMMABILITY DOMAINS

ROBERT M. DEVOS AND FREDERICK W. HARTMANN
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R. M. DeVos and F. W. Hartmann

C. Goffman and G. N. Wollan conjectured that the bounded summability field of a regular matrix $A$ is so thin that the union of countably many such sets is not dense in $m$. G. M. Petersen proved this conjecture. This result is strengthened by showing if $A$ is a noncoercive matrix whose summability field contains all the finite sequences then its bounded summability field is so thin that the union of countably many such sets is not dense in $m$. An example is given to show that the condition of containing the finite sequences is necessary.

Preliminaries. Let $m$ and $c$ be respectively the Banach spaces of bounded and convergent sequences, $x = \{x_n\}$, of complex numbers with norm $\|x\|_\infty = \sup_n |x_n|$, $B(x, r) = \{z \in m: \|x + z\|_\infty < r\}$. Denote the $n$th section of $x$ by $P_n(x) = (x_1, \ldots, x_n, 0, 0, \ldots)$. For each infinite matrix $A$ the set of $x$ transformed by $A$ to convergent sequences is called the summability field of $A$ and denoted by $c_A$. The set of bounded sequences in $c_A$ is called the bounded summability field of $A$ and is denoted by $J^\infty_A$. $A$ is called conservative if and only if $c_A \supset c$, regular if and only if $A$ is conservative and limits are preserved, coercive if and only if $c_A \supset m$. If $A = \{a_{nk}\}$, then the $A$ transform of $x$ is designated by $Ax = \{(Ax)_n\} = \{\sum_k a_{nk} x_k\}$. $A$ is conservative if and only if $\|Ax\|_\infty = \sup_n \sum_k |a_{nk}| < \infty$, $a_k = \lim_n a_{nk}$ exists for each $k$ and $\lim_n \sum_k a_{nk}$ exists [5, p. 165]. $A$ is coercive if and only if $\sum_k |a_{nk}|$ converges uniformly in $n$ and $a_k$ exists for each $k$ [5, p. 169]. Define the essential norm of $A$ by $\|A\|_e = \limsup_n \sum_k |a_{nk} - a_k|$ whenever $a_k$ exists for each $k$. (Note $\| \|_e$ is not a true norm, since $\| \|_e$ may be infinite.)

Let $E^\infty$ be the set of all finite sequences and $N$ the set of all sequences of 0's and 1's. Using binary expansions there is a natural injective mapping of $(0, 1)$ onto all but a countable subset of $N$.

Main Results. C. Goffman and G. N. Wollan conjectured [4] that the bounded summability field of regular $A$ is so thin that the union of countably many such sets is not dense in $m$. G. M. Petersen proved this conjecture [6]. We strengthen that result and show that in a certain sense our result is best possible.

Theorem. Let $\{A_i\}$ be a countable collection of noncoercive matrices with $\mathcal{A} \supset E^\infty$, $i = 1, 2, \ldots$, then $\bigcup_{i=1}^\infty \mathcal{A}_i$ is not dense in $m$.

We prove the theorem through a series of lemmas. Since we
want $E^\infty \subset \mathcal{K}$, we shall assume all $A$ in the sequel have convergent columns.

**Lemma 1.** Let $\| A \|_\infty < \infty$ then $\| A \|_c = 0$ if and only if $A$ is coercive.

**Proof.** Suppose $A$ is coercive. Let $\varepsilon > 0$. There exists $k_0$ such that

$$\sum_{k=k_0+1}^{\infty} |a_{nk}| < \varepsilon/3$$

for all $n$. Since $\{a_k\} \in \ell^1$, there is a $k_1$ such that $k > k_1$ implies

$$\sum_{k=k_1+1}^{\infty} |a_k| < \varepsilon/3.$$

Let $k_2 = \max(k_1, k_0)$. There exists $n_0 = n_0(k_2)$ such that $n > n_0$ implies

$$\sum_{k=1}^{k_2} |a_{nk} - a_k| < \varepsilon/3.$$ Let $n > n_0$ then

$$\sum_{k=1}^{\infty} |a_{nk} - a_k| = \sum_{k=1}^{k_2} |a_{nk} - a_k| + \sum_{k=k_2+1}^{\infty} |a_{nk} - a_k|$$

$$\leq \sum_{k=1}^{k_2} |a_{nk} - a_k| + \sum_{k=k_2+1}^{\infty} |a_{nk}| + \sum_{k=k_2+1}^{\infty} |a_k|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$ Conversely assume $A$ is noncoercive. There exists $\varepsilon > 0$ and an increasing sequence of positive integers $\{n(p)\}_{p=1}^{\infty}$ such that $\sum_{k=p+1}^{\infty} |a_{n(p), k}| > \varepsilon$. There exists $k_0$ such that $\sum_{k=k_0+1}^{\infty} |a_k| < \varepsilon/2$. Pick $p$ with $p > k_0$ then

$$\sum_{k=1}^{\infty} |a_{n(p), k} - a_k| \geq \sum_{k=k_0+1}^{\infty} |a_{n(p), k} - a_k|$$

$$\geq \sum_{k=k_0+1}^{\infty} |a_{n(p), k}| - \sum_{k=k_0+1}^{\infty} |a_k|$$

$$\geq \varepsilon - \varepsilon/2 = \varepsilon/2.$$ Therefore $\| A \|_c > 0$.

Let $\Gamma(c, c)$ be the Banach algebra of conservative matrices and $\mathcal{K}$ be the ideal of compact operators. It is well known [8] that $A \in \mathcal{K}$ if and only if $A$ is coercive. $\Gamma(c, c)/\mathcal{K}$ is a Banach algebra and is called a Calkin algebra [2]. It is easily seen that $\| \|_c$ is the norm in the Calkin algebra.

**Lemma 2.** Let $\| A \|_c < \infty$ and $a$ and $b$ be cluster points of $Ax$,
Proof. Let \( a \) and \( b \) be cluster points of \( Ax \) and \( \varepsilon > 0 \). There exist increasing sequences of positive integers \( \{n(i)\}, \{m(j)\} \) and \( N_0 \) such that for \( n(i), m(j) > N_0 \)

\[
\left| \sum_k a_{n(i), k} x_k - a \right| < \varepsilon
\]

and

\[
\left| \sum_k a_{m(j), k} x_k - b \right| < \varepsilon.
\]

There exists \( N_i \) such that \( n > N_i \) implies

\[
\sum_k |a_{nk} - a_k| < ||A||_e + \varepsilon.
\]

Let \( n(i), m(j) > \max(N_0, N_i) \) then

\[
\left| a - b \right| \leq \left| \sum_k a_{n(i), k} x_k - \sum_k a_{m(j), k} x_k \right| + 2\varepsilon
\]

\[
\leq \sum_k \left| a_{n(i), k} - a_{m(j), k} \right| |x_k| + 2\varepsilon
\]

\[
\leq ||x||_\infty \sum_k \left| (a_{n(i), k} - a_k) - (a_{m(j), k} - a_k) \right| + 2\varepsilon
\]

\[
\leq ||x||_\infty (||A||_e + \varepsilon + ||A||_e + \varepsilon) + 2\varepsilon.
\]

Since \( \varepsilon \) is arbitrary the conclusion follows.

The next lemma is due to Bennett and Kalton and appears as Lemma 7 of [1, p. 577].

**Lemma 3.** (Bennett and Kalton). If \( z_1, z_2, \ldots, z_n \) is any finite collection of complex numbers then there exists a subset \( J(n) \) of \( \{1, \ldots, n\} \) such that

\[
\left| \sum_{j \in J(n)} z_j \right| \geq \frac{1}{4} \sum_{i=1}^n |z_i|.
\]

**Lemma 4.** If \( ||A|| = \infty \), then there exists \( E(A) \) with \( E(A) \subset N_0 \), \( N_0 \setminus E(A) \) of first category and if \( u \in E(A) \) then \( B(u, 1/32) \cap \mathcal{D} = \emptyset. \)

Proof. Case 1. Assume all the rows of \( A \) are in \( \mathcal{D} \). Let \( ||A|| = \infty \). Pick sequences \( n(k) \) and \( q(k) \) inductively such that \( n(1) = 1 \) and

(i) \( \sum_{i=q(k)+1}^{q(k)+1} |a_{n(k), i}| < 2^{-k} \)

(ii) \( \sum_{i=q(k)+1}^{q(k)+1} |a_{n(k), i}| > (65/7) \sup_j \{\sum_{i=1}^{q(k)-1} |a_{j, i}|\}. \)

By Lemma 3 select \( J(k) \subset \{q(k-1) + 1, \ldots, q(k)\} \) with
For each natural number \( k \) define the sequence \( u^k \) by \( u^k_i = 1 \) if \( i \in J(k) \), \( u^k_i = 0 \) if \( i \notin J(k) \). Let

\[
O_k = \{ u \in N_0; (P_q(k) - P_{q(k-1)})(u - u^k) = 0 \}.
\]

If \( E(A) = \bigcap_{k=1}^{\infty} \bigcup_{k=n}^{\infty} O_k \), then \( E(A) \) is of second category. [\( \bigcup_{k=n}^{\infty} O_k \) is open and dense, hence by the Baire theorem \( E(A) \) is of second category.] Let \( u \in E(A) \) and \( \| z \|_\infty < 1/32 \). \( u \) is in an infinite number of the \( O_k \). Let \( u \in O_r \). Then

\[
| (A(u + z))_{n(r)} | \geq | (A u)_{n(r)} | - | (A z)_{n(r)} | \\
\geq \left| \sum_{i=q(r-1)+1}^{q(r)} a_{n(r), i} u_i \right| - \sum_{i=q(r)+1}^{\infty} a_{n(r), i} u_i \left| \\
- \frac{1}{32} \sum_{i=1}^{\infty} a_{n(r), i} \right| \\
\geq \frac{1}{4} \sum_{i=q(r)+1}^{q(r-1)} \left| a_{n(r), i} \right| - \frac{33}{32} \sum_{i=1}^{q(r-1)} \left| a_{n(r), i} \right| \\
- \frac{33}{32} \sum_{i=q(r)-1}^{\infty} \left| a_{n(r), i} \right| \left| \\
\geq \frac{7}{32} \sum_{i=q(r)+1}^{\infty} \left| a_{n(r), i} \right| - \frac{33}{32} \sum_{i=1}^{q(r-1)} \left| a_{n(r), i} \right| - \frac{33}{32} 2^{-r} \left| \\
\geq \frac{7}{32} \sup_j \left\{ \left| \sum_{i=1}^{q(r-1)} a_{j,i} \right| \right\} - \frac{33}{32} \sup_j \left\{ \left| \sum_{i=1}^{q(r-1)} a_{j,i} \right| \right\} - 2^{-r} \left| \\
\geq \sup_j \left\{ \left| \sum_{i=1}^{q(r-1)} a_{j,i} \right| \right\} - 2^{-r} \rightarrow \infty \text{ as } r \rightarrow \infty.
\]

Hence the \( A \) transform of \( u + z \) is unbounded.

**Case 2.** Let \( A \) have one row, \( x \), not in \( \mathcal{C} \). Let \( B = (b_{nk}) \) where \( b_{nk} = P_n(x) \), \( n = 1, 2, \ldots \). Then \( \mathcal{A} \subset \mathcal{B} \) and \( B \) satisfies the hypothesis of Case 1. Let \( E(A) = E(B) \) then \( E(A) \cap \mathcal{A} = \emptyset \) and \( E(A) \) satisfies the other conditions of the lemma’s conclusion.

**Lemma 5.** If \( \| A \| < \infty \), and \( A \) is noncoercive then there is \( E(A) \) with \( E(A) \subseteq N_0 \), \( N_0 \backslash E(A) \) is of first category and if \( u \in E(A) \), then \( B(u, 1/32) \cap \mathcal{A} = \emptyset \).

**Proof.** Case 1. Assume \( a_k = 0 \), \( k = 1, 2, \ldots \). Let \( \alpha^n \) be the \( n \)th row of \( A \). Using an argument similar to that of Petersen and Baker [6] (see also the construction of Lemma 4) it can be shown that without lose of generality one may assume that the rows and columns of \( A \) are in \( E^\infty \) and moving to the right, (if \( P_x \alpha^n = 0 \) then
\(P_\alpha a_m = 0\) for \(m \geq n\). By Lemma 1 \(\|A\|_\epsilon > 0\). Hence there exists increasing sequences \(n(j)\) and \(r(j)\) of positive integers such that

\[(i) \sum_{k=0}^{r(j)-1} |a_{n(j),k}| \geq \frac{1}{4} \left( \sum_{k=r(2j)-1}^{r(2j)} |a_{n(j),k}| \right) \geq \frac{\|A\|_\epsilon}{8}\]

Let \(J(2j)\) be a subset of \(r(2j-1)\) to \(r(2j) - 1\) with \(\sum_{k \in J(2j)} a_{n(j),k} \geq \frac{1}{4} \sum_{k=r(2j)-1}^{r(2j)} |a_{n(j),k}| \geq \frac{\|A\|_\epsilon}{8}\) (see Lemma 3). Define \(O_j = \{u \in N^0_\epsilon: u_k = 1\) if \(k \in J(2j)\), \(u_k = 0\) if \(r(2j-2) + 1 \leq k \leq r(2j), k \notin J(2j)\}\). Since only a finite number of coordinates are specified for elements of \(O_j\), \(O_j\) is open. For each \(k\), \(U_{j=0}^\infty O_j\) is open and dense, hence by the Baire category theorem, \(\bigcap_{j=0}^\infty \bigcup_{j=m}^\infty O_j\) is of second category. Let \(E(A) = \{u \in N^0_\epsilon: Au\) has cluster points, \(a, b\), with \(|a - b| \geq \|A\|_\epsilon/8\}\). By construction each element of \(\bigcap_{j=0}^\infty \bigcup_{j=m}^\infty O_j\) has 0 and \(|a| > \|A\|_\epsilon/8\) as cluster points thus \(E(A)\) is of second category. Let \(u \in E(A)\) and \(\|z\|_\infty < 1/32\) and consider \(A(u + z)\). \(Au\) has two cluster points separated in distance by at least \(\|A\|_\epsilon/8\), and \(A(z)\) has cluster points separated by at most \(2(1/32)\|A\|_\epsilon\) (Lemma 2). Therefore \(A(u + z)\) has at least two cluster points; hence \(u \neq z \notin \mathcal{A}\).

**Case 2.** Let \(a_k \neq 0\) for some \(k\). Define \(B = (b_{nk})\) where \(b_{nk} = a_k\), \(n = 1, 2, \ldots\). \(B\) transforms every bounded sequence to a constant sequence, thus the cluster points of \((A - B)u, u \in m\), are a shift of those of \(Au\), and \(A - B\) satisfies the hypothesis of Case 1. Thus the conclusion follows in a manner similar to Case 1.

**Proof of Theorem.** Let \(A_i\) be a countable collection of non-coercive matrices with \(\mathcal{A}_i \supset E^\infty\), \(i = 1, 2, \ldots\). By Lemmas 4 and 5 for each \(i\) there exists \(E(A_i) \supset N_\epsilon, E(A_i)\) of second category, and if \(u \in E(A_i), B(u, 1/32) \cap \mathcal{A}_i = \emptyset\). Thus \(\bigcap_{i=1}^\infty E(A_i) \neq \emptyset\) and if \(u \in \bigcap_{i=1}^\infty E(A_i)\), then \(B(u, 1/32) \cap (\bigcup_{i=1}^\infty \mathcal{A}_i) = \emptyset\). Hence \(U_{i=1}^\infty \mathcal{A}_i\) is not dense in \(m\).

Goffman and Wollan in [4] gave an example of a countable family of FK spaces contained in \(m\) whose union is dense in \(m\). They can be realized as summability domains in the following manner. Let \(\{r_i\}\) be a denumeration of the nonzero rationals. Define \(A_i = (a_{nk}^{(i)})\) by

- \((i) a_{n1}^{(i)} = r_i, a_{n2}^{(i)} = -1, n = 1, 3, 5, \ldots\)
- \((ii) a_{n1}^{(i)} = -1, a_{n2}^{(i)} = r_i^{-1}, n = 2, 4, 6, \ldots\)
- \(a_{nk}^{(i)} = 0, k \geq 3, n = 1, 2, 3, \ldots\)

Then \(\mathcal{A}_i = \{x_k^{(i)}: x_1 = x, x_2 = r_i x, x_k \text{ arbitrary for } k \geq 3 \text{ and } x \text{ complex}\} \cap m\). Each \(\mathcal{A}_i\) is nowhere dense in \(m\), but \(\bigcup_{i=1}^\infty \mathcal{A}_i\) is dense. Note, however, that \(\mathcal{A}_i \not\supset E^\infty\). Hence the hypothesis that each
\( \mathcal{X} \supseteq E^\infty \) cannot be removed and our result is in some sense best possible.

Although we have proved our result only for \( \mathcal{X} \), we conjecture that the following more general result holds:

Conjecture. If \( \{F_i\} \) is a countable collection of FK-spaces each containing \( E^\infty \) but not \( m \), then \( \bigcup_{i=1}^\infty F_i \) is not dense in \( m \). (See [8] for definitions and basic results.)

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Aharon Atzmon, *Spectral synthesis in some spaces of bounded continuous functions* ................................................................. 277
Karl Egil Aubert and Isidor Fleischer, *Tensor products of ideal systems and their modules* .......................................................... 285
Richard F. Basener, *Several dimensional properties of the spectrum of a uniform algebra* ............................................................. 297
R. H. Bing and Michael Peter Starbird, *Super triangulations* ........... 307
Andrew Carson, *Coherent polynomial rings over regular rings of finite index* .... 327
Robert M. DeVos and Frederick W. Hartmann, *Sequences of bounded summability domains* ...................................................... 333
George Grätzer and R. Padmanabhan, *Symmetric difference in abelian groups* ................................................................. 339
Robert L. Griess, Jr., *A remark about groups of characteristic 2-type and p-type* ................................................................. 349
Emil Grosswald and F. J. Schnitzer, *A class of modified ζ and L-functions* ........ 357
Jutta Hausen and Johnny Albert Johnson, *Ideals and radicals of some endomorphism rings* .................................................. 365
Jean Ann Larson, *A solution for scattered order types of a problem of Hagendorf* ................................................................. 373
Peter A. McCoy, *Extremal properties of real biaxially symmetric potentials in \( E^2(\alpha+\beta+2) \) ................................................................. 381
Héctor Alfredo Merklen, *Hereditary crossed product orders* .................. 391
Hal G. Moore and Adil Mohamed Yaqub, *Equational definability of addition in certain rings* .................................................. 407
Robert Laurens Moore, *Reducitivity in \( C^* \)-algebras and essentially reductive operators* ............................................................. 419
Joseph Alvin Neisendorfer, *Lie algebras, coalgebras and rational homotopy theory for nilpotent spaces* ..................................... 429
William Raymond Nico, *Bounded monoids* ........................................ 461
Richard Paul Osborne, *Simplifying spines of 3-manifolds* ................... 473
Richard Paul Osborne, *The simplest closed 3-manifolds. With an appendix by Osborne and J. Yelle* ........................................... 481
Clayton Collier Sherman, *The K-theory of an equicharacteristic discrete valuation ring injects into the K-theory of its field of quotients* ................. 497
Mitchell Herbert Taibleson, *The failure of even conjugate characterizations of \( H^1 \) on local fields* .................................................. 501
Keti Tenenblat, *On characteristic hypersurfaces of submanifolds in Euclidean space* ............................................................. 507
Jeffrey L. Tollefson, *Involutions of Seifert fiber spaces* ....................... 519
Joel Larry Weiner, *An inequality involving the length, curvature, and torsions of a curve in Euclidean n-space* ......................... 531
Neyamat Zaheer, *On generalized polars of the product of abstract homogeneous polynomials* ............................................................. 535