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**A REMARK ABOUT GROUPS OF CHARACTERISTIC 2-TYPE
AND p -TYPE**

ROBERT L. GRIESS, JR.

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In the work of Klinger and Mason on groups of characteristic 2-type and p -type, a configuration for $p = 5$ emerges as a possibility in the conclusion to one of their theorems. In this note, we eliminate this possibility. Thus, the evidence that the only simple groups of characteristic 2-type and p -type are $G_2(3)$, $P\text{Sp}(4, 3)$ and $U_4(3)$ is strengthened.

Introduction. We consider the following recent theorem of Klinger and Mason [2]:

THEOREM D. *Let G be a finite group, p an odd prime, $P \in \text{Syl}_p(G)$. Assume $\mathcal{N}(P; 2) = \{1\}$, all 2-locals are 2-constrained with trivial core and all p -locals are p -constrained. Let A be an elementary abelian p -subgroup of G of maximal rank subject to $\mathcal{N}(A; 2) \neq \{1\}$. Assume $m(A) \geq 2$. Then $m(A) = 2$ and one of the following holds.*

- (a) *A contains every element of order p in $C_G(A)$.*
- (b) *$p = 3$ and we can choose A and $T \in \mathcal{N}^*(A; 2)$ so that T is the central product of A -invariant quaternion subgroups Q_1, \dots, Q_w , $w = 2, 3, 4$.*
- (c) *$p = 5$ and for every $T \in \mathcal{N}^*(A; 2)$, T is the central product of five copies of Q_8 and five copies of D_8 .*

It has been suspected that (c) never occurs. In fact, a conjecture of Gorenstein asserts that if G is a simple group, simultaneously of characteristic 2-type and p -type for some odd p (see [2] for definitions of these terms), then $p = 3$ and $G \cong G_2(3)$, $P\text{Sp}(4, 3)$ or $U_4(3)$. It is the purpose of this paper to eliminate (c).

For terminology and notation used in this paper, see [2]. The reader is referred to [1] or [5] for basic material about groups or Lie type. See [3] for further results on groups of characteristic 2-type and 3-type.

In this paper, we consider the following hypothesis.

Hypothesis (KM). (1) G is a finite simple group, all of whose 2-local subgroups are 2-constrained.

(2) A is a noncyclic elementary abelian 5-group of maximal rank lying in a 2-local subgroup.

(3) $A \cong Z_5 \times Z_5$.

(4) $T \in \mathcal{N}^*(A; 2)$ and T is an extraspecial 2-group of order 2^{21} ,

type -, i.e., T is the central product of 5 quaternion groups and 5 dihedral groups. Also, $T = O_2(C_G(Z(T)))$ and $N_G(T)$ is corefree and 2-constrained.

(5) Let A_i , $0 \leq i \leq 5$, be all the subgroups of order 5 in A . Set $T_i = C_T(A_i)$ for $0 \leq i \leq 5$. Then $T_0 = \langle z \rangle = Z(T)$ and $T_i \cong D_8 \circ Q_8$, for $1 \leq i \leq 5$. Also, $C_G(A_i)$ is 5-constrained, $0 \leq i \leq 5$.

(6) $\mathcal{N}(R; 2) = \{1\}$ for $R \in \text{Syl}_5(G)$.

(7) A is contained in an elementary abelian subgroup of order 5^3 in G .

(8) $|O_{5'}(C_G(A_i))|$ is odd for $0 \leq i \leq 5$.

We observe that this hypothesis holds whenever the hypotheses and conclusion (c) but not conclusion (a) of Theorem D holds. This is clear, except possibly for (5), for which we refer to the proof of Lemma 3.8 of [2].

We now state our main result, which eliminates conclusion (c) of Theorem D.

THEOREM. *No finite group exists satisfying hypothesis (KM).*

The proof proceeds by contradiction. Let G be a counterexample to our theorem. First some notation.

Notation. $C = C_G(A_1)$, $N = N_G(A_1)$, $\bar{N} = N/O_{5'}(N)$, P a Sylow 5-group of $O_{5',5}(N)$ normalized by T_1 .

For simplicity, we have singled out A_1 and T_1 . It will be obvious that the following results apply to A_i , T_i , $C_G(A_i)$, etc. for $1 \leq i \leq 5$.

(1) For every $x \in T_1^\#$, $C_{\bar{P}}(x)$ has rank at most 2.

Proof. (KM.2)

(2) z inverts \bar{P}/\bar{A}_1 and $\bar{P}/\bar{A}_1 \neq 1$.

Proof. $C_G(z)$ contains A . Let $R = C_{\bar{P}}(z)$ and assume $R > \bar{A}_1$. Then there is $U \leq PA_0 |U| = p^3$, $A \leq U$, $A_1 \leq Z(U)$ and $[U, z] = 1$. By (KM.4), $T = O_2(C_G(z))$ and U acts faithfully on T . Since $A \triangleleft U$ and A_0 is the unique A_i with $C_T(A_i) = \langle z \rangle$, we get $A \leq Z(U)$ and so U is abelian. Since U acts faithfully on T and leaves invariant each T_i , U is elementary of rank 3, against (KM.2). Therefore $R = \bar{A}_1$, which proves that z inverts \bar{P}/\bar{A}_1 . If $\bar{P} = \bar{A}_1$, then $T_1 \leq O_{5'}(C)$. A Frattini argument and (KM.7) then contradicts (KM.2).

(3) \bar{P} is extraspecial of order 5^{1+4m} , some $m \geq 1$.

Proof. Let R be a characteristic elementary abelian subgroup of P and suppose $|R| > 5$. Since z inverts R/A_1 , T_1 acts faithfully on R/A_1 , whence $|R/A_1| = 5^m$, $m \equiv 0 \pmod{4}$. But then, if $x \in T_1$, $|x| = 2$, $|C_R(x)| \geq 5^3$, against (KM.2). So, $|R| = 5$. Thus \bar{P} is of symplectic type, $P = P_0P_1$, P_0 cyclic, P_1 trivial or extraspecial. Since $P_0 = Z(P)$, z normalizes P_0 . Since z centralizes A_1 , (2) implies that $P_0 = A_1$. Since $P > A_1$, $P = P_1$ is extraspecial as required. Since T_1 operates faithfully on \bar{P} , $|\bar{P}| = 5^{1+2k}$ where k is even, because $T_1 \cong D_8 \circ Q_8$.

$$(4) \quad |\bar{P}| = 5^5, \exp(P) = 5.$$

Proof. By (3) and the fact that z inverts \bar{P}/\bar{A}_1 , $\exp(P) = 5$. Let $|\bar{P}| = 5^{1+4m}$, m an integer. By (KM.5), $m \geq 1$. Suppose $m \geq 2$. Let $x \in T_1$, $x \neq \langle z \rangle$, $|x| = 2$. Then $|C_{\bar{P}}(x)| = 5^{1+2m}$ and $C_{\bar{P}}(x)$ is extraspecial.

Since $C_{\bar{P}}(x)$ has exponent 5, $C_{\bar{P}}(x)$ contains an elementary group of rank $m + 1 \geq 3$, against (KM.2). Thus $m = 1$.

(5) \bar{P} is an indecomposable module for A_0 . Thus P is the unique group of its isomorphism type in PA_0 . Also, $\text{Scn}_3(PA_0)$ has one member.

Proof. The first statement follows from the fact that T_1A_0 acts faithfully on \bar{P} . The second follows from $A_1 = Z(PA_0)$ and the fact that P/A_1 is the unique maximal abelian subgroup of PA_0/A_1 . The third statement is obvious.

(6) Let $B > A_1$ satisfy $B/A = C_{P/A_1}(A_0)$. Then BA_0 is abelian of rank 3 and $||[BA_0, z]|| = 5$.

Proof. Suppose BA_0 were nonabelian. A Lie ring analysis of the action of z on BA_0 gives a contradiction. Thus, BA_0 is abelian and, furthermore, must be elementary because of the way z acts.

(7) Let p be an odd prime and let $S = \text{Sp}(4, p)$. Let $L \leq S$ and $R \in \text{Syl}_p(L)$. Assume $|R| \geq p^2$. Let M be the standard 4-dimensional module for $F_p S$. Then one of the following holds: (i) L contains a transvection on M ; (ii) $|R| = p^2$ and every element of $R^\#$ has quadratic minimal polynomial on M .

Proof. Let $\{a, b\}$ be a set of fundamental roots for a root system Σ of type C_2 with a short, b long (see [1] or [5] for the basic machinery about groups of Lie type). For $r \in \Sigma$, let X_r be the usual

one-parameter subgroup. On the standard 4-dimensional module M , elements of X_r , for r long, act as transvections.

We assume that L contains no transvection, then obtain (ii). Let $R \in \text{Syl}_p(L)$ and embed R in U , the standard Sylow p -group, $U = \langle X_r \mid r \in \Sigma, r \text{ positive} \rangle$. Let $s = 2a + b$ be the root of maximal height and let $K = N_G(X_s)$ be the parabolic subgroup associated with the set $\{a\}$ of fundamental roots. Then $O_p(K)$ is extraspecial of order p^3 , exponent p , and $K'/O_p(K) \cong \text{SL}(2, p)$. Now, $R_1 = R \cap O_p(K) \neq 1$, as $|R| \geq p^2$, so that $R_1 \cap X_s = 1$, $|R_1| = p$ and $|R| = p^2$. Since K is transitive on the nonidentity elements of $O_p(K)/X_s \cong \mathbf{Z}_p \times \mathbf{Z}_p$, we may assume $R_1 = X_{a+b}$, ($a + b$ is a short root). Then $N_S(R_1) = (X_b X_{a+b} X_{2a+b})H$, where H is the standard Cartan subgroup, so that $R \leq U_1 = X_b X_{a+b} X_{2a+b}$. But, $U_1 = O_p(K_1)$, where K_1 is the parabolic subgroup of S which stabilizes a maximal totally isotropic subspace, say N (i.e., K_1 is associated with the set $\{b\}$ of fundamental roots). Since $K_1'/U_1 \cong \text{SL}(2, p)$ is faithful on N and on M/N , U_1 stabilizes the chain: $M > N > 0$. Thus, (ii) holds, as required.

(8) $|G|_5 = 5^6$, $PA_0 \in \text{Syl}_5(G)$, A_i is G -conjugate to A_1 and lies in the center of a Sylow 5-group, $1 \leq i \leq 5$.

Proof. By (KM.5) and (3), A_1 is a Sylow 5-center. Since all of the preceding analysis could be applied to $N_G(A_i)$, $1 \leq i \leq 5$, we are done once we show $|N|_5 = 5^6$, i.e., $|\bar{N}/\bar{P}|_5 = 5$. Firstly, $|\bar{N}/\bar{C}|$ divides 4 and \bar{C}/\bar{P} is isomorphic to a subgroup of $\text{Sp}(4, 5)$. If $|\bar{C}/\bar{P}|_5 = 5$, we have $|N|_5 = 5^6$, as required. So, we may assume $|\bar{C}/\bar{P}|_5 \geq 5^2$. Let $C^* = C/O_{5',5}(C)$. Suppose first that C^* contains a transvection in its action on $M = \bar{P}/\bar{A}_1$. Let L be the subgroup of C^* generated by elements inducing transvections on M . Then T_1^* normalizes L and the structure of $S = \text{Sp}(4, 5)$ implies that L is not a 5-group. Therefore, $L \cong \text{SL}(2, 5)$ or $\text{Sp}(4, 5)$, by [4] and the structure of S . If $L \cong \text{SL}(2, 5)$, the fact that $(T_1 A_0)^*$ normalizes L forces $[T_1^*, L] = 1$, which is absurd since T_1^* is irreducible on M . So, $L \cong \text{Sp}(4, 5)$. Therefore, $C_G(Z)$ contains T_1 as a nonnormal subgroup, whereas $T_1 \leq O_2(C_G(Z)) = T$, by (KM.4) and (KM.5), contradiction.

We have that C^* contains no transvections. Therefore, (7) tells us that a Sylow 5-group of C^* consists of elements operating trivially or quadratically on M . This contradicts (5), since A_0 has minimal polynomial of degree 4. Therefore, we have $|N|_5 = 5^6$, as required.

(9) Let $W = BA_0$, as in (6). Then $W \in \text{Syl}_5(C_G(W))$. For each $i = 1, \dots, 5$, $A_G(W)$ contains a transvection t_i centralizing A_i . Also,

$1 \neq [A, t_1] \leq B$ but $A_1 \not\leq [A, t_1]$. If V is a conjugate of W for which $|N_{PA_0}(V)| \geq 5^5$, then $V \leq P$. Moreover, if $V \triangleleft P$, then $\{V\} = \text{Scn}_3(PA_0)$.

Proof. Choose $g \in G$ so that $V = W^g \leq PA_0$. If $V \leq P$, then $[V, P] = A_1$ and $V = C_{PA}(V)$ by (5). Say $|V \cap P| = 5^2$. Then $V_0 = V \cap P$ is centralized by $b \in PA_0 - P$. By (5), $V_1 = C_P(V_0)$ has index 5 in P and $V = C_{PA_0}(V)$.

Recall that an extremal conjugate of a subgroup S_1 of a p -Sylow group S of G is a G -conjugate S_2 of S_1 for which $N_S(S_2) \in \text{Syl}_p(N_G(S_2))$. If we now take V to be an extremal conjugate of W in PA_0 , we see from the previous paragraph that $V = C_{PA_0}(V)$, whence $W \in \text{Syl}_5(C_G(W))$. The existence of t_i follows from embedding W in a Sylow 5-group of $C_G(A_i)$. The further properties $1 \neq [A_1, t_1] \leq B$ and $A_1 \not\leq [A, t_1]$ are forced if we take $t_1 \in N_{PA_0}(W)$. The final statement follows from $V = C_{PA_0}(V)$ and (5).

(10) Let X be the subgroup of $A_G(W)$ generated by transvections. Then one of the following holds

- (i) X is a 5-group and $|X| \geq 5^2$.
- (ii) $X/O_5(X) \cong \text{SL}(2, 5)$.
- (iii) $X \cong \text{SL}(3, 5)$.

Proof. This follows from the classification of groups in odd characteristic generated by transvections [4]. To get $|X| \geq 5^2$ in Case (i), use (9).

(11) X is not isomorphic to $\text{SL}(2, 5)$.

Proof. Suppose false. Then $X \triangleleft A_G(W)$ implies that $A_G(W)$ is isomorphic to a subgroup of $Z_4 \times \text{GL}(2, 5)$ and that W is extremal in PA_0 . Write $W = W_0 \oplus W_1$ where $W_0 = C_W(X)$, $W_1 = [W, X]$. Now, $A \cap W_1 \neq 1$. Suppose $A_i \leq W_i$ for some i . Then, for a Sylow 5-group R of $N_G(W)$, $A_i = [W, R]$. If $i \neq 0$, this contradicts (9) and the action of $N_X(W)$ on W . Therefore, $[W, R] = A \cap W_1 = A_0$ and every group of order 5 in W_1 is fused in $N_G(W)$ to A_0 . The orbits of X on the 25 subgroups of order 5 in W which meet W_1 trivially have lengths 1 and 24. By (8), they lie in the G -conjugacy class of A_1 . On the other hand, the action of P on B implies that every group of order 5 in B distinct from A_1 is fused to the one in $B \cap W_1 \cong Z_5$. Therefore all subgroups of order 5 in W are fused in G to A_1 . Now embed R in a Sylow 5-group of $C_G(A_0)$. Since $R' = A_0$, the proof of (9) shows that $W \leq O_{5',5}(C_G(A_0))$. Since P is extraspecial, we get $|N_G(W)|_5 \geq 5^5$, a contradiction to our first remark.

(12) $|X|_5 = 5^3$ and the unique extremal conjugate V of W in PA_0 lies in P .

Proof. By (11), $|X|_5 \geq 5^2$, so by (9) any extremal conjugate V of W in PA_0 lies in P . Suppose $|X|_5 \leq 5^2$. Then (10.i) holds and we have $|X|_5 = 5^2$. Thus W is not extremal in PA_0 . The action of P on V and t_1 on W show that $|X|_5 = 5^3$, a contradiction.

(13) X is not a 5-group and $C_W(X) = 1$. Thus $X \cong \text{SL}(3, 5)$ or X stabilizes a unique hyperplane W_1 of W , in which case $X/O_5(X) \cong \text{SL}(2, 5)$ acts faithfully on W_1 .

Proof. By embedding W in a Sylow 5-group of $N_G(D)$, where $D \leq W$, $D \sim_G A_1$, we see that some Sylow 5-group of X centralizes D . Since $|C_W(R)| = 5$ for $R \in \text{Syl}_5(N_G(W))$, we take $D = A_i$, $i = 1, 2, 3, 4, 5$ to get the first statement. The second statement now follows, using (10) and (12).

(14) $A = W_1$ in case $X \not\cong \text{SL}(3, 5)$. Thus each group of order 5 in A is central in some Sylow 5-group of $N_G(W)$.

Proof. Suppose X normalizes the hyperplane W_1 . Then W_1 contains the center of every Sylow 5-group of $N_G(W)$. Therefore, using (9), $W_1 \cong A_1 A_2 = A$, as required.

(15) $X \cong \text{SL}(3, 5)$.

Proof. Suppose not. By (14), W is normal in a Sylow 5-group of C , so we may assume $W \leq P$. However, again by (14), $W \cong A_0 \not\leq P$, contradiction.

(16) $X \not\cong \text{SL}(3, 5)$.

Proof. Suppose $X \cong \text{SL}(3, 5)$. The action of z on W has determinant -1 (see (6)). So, there is $u \in N_G(W)$ inducing -1 on W . In fact $|u| = 2$, by (KM.6). Thus, $N = O_{s',s}(N) \cdot C_N(u)$, by the Frattini argument. It follows that V is complemented in PA_0 by $C_{PA_0}(u')$, for an appropriate involution u' . But then there are two Jordan blocks of degree 2 for the action of an element of order 5 in $C_{PA_0}(u) - P$ on P/A_1 . This contradicts (5).

Since evidently (15) and (16) are in conflict, the proof of our theorem is complete.

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Aharon Atzmon, <i>Spectral synthesis in some spaces of bounded continuous functions</i>	277
Karl Egil Aubert and Isidor Fleischer, <i>Tensor products of ideal systems and their modules</i>	285
Richard F. Basener, <i>Several dimensional properties of the spectrum of a uniform algebra</i>	297
R. H. Bing and Michael Peter Starbird, <i>Super triangulations</i>	307
Andrew Carson, <i>Coherent polynomial rings over regular rings of finite index</i>	327
Robert M. DeVos and Frederick W. Hartmann, <i>Sequences of bounded summability domains</i>	333
George Grätzer and R. Padmanabhan, <i>Symmetric difference in abelian groups</i>	339
Robert L. Griess, Jr., <i>A remark about groups of characteristic 2-type and p-type</i>	349
Emil Grosswald and F. J. Schnitzer, <i>A class of modified ζ and L-functions</i>	357
Jutta Hausen and Johnny Albert Johnson, <i>Ideals and radicals of some endomorphism rings</i>	365
Jean Ann Larson, <i>A solution for scattered order types of a problem of Haggendorf</i>	373
Peter A. McCoy, <i>Extremal properties of real biaxially symmetric potentials in $E^{2(\alpha+\beta+2)}$</i>	381
Héctor Alfredo Merklen, <i>Hereditary crossed product orders</i>	391
Hal G. Moore and Adil Mohamed Yaqub, <i>Equational definability of addition in certain rings</i>	407
Robert Laurens Moore, <i>Reductivity in C^*-algebras and essentially reductive operators</i>	419
Joseph Alvin Neisendorfer, <i>Lie algebras, coalgebras and rational homotopy theory for nilpotent spaces</i>	429
William Raymond Nico, <i>Bounded monoids</i>	461
Richard Paul Osborne, <i>Simplifying spines of 3-manifolds</i>	473
Richard Paul Osborne, <i>The simplest closed 3-manifolds. With an appendix by Osborne and J. Yelle</i>	481
Clayton Collier Sherman, <i>The K-theory of an equicharacteristic discrete valuation ring injects into the K-theory of its field of quotients</i>	497
Mitchell Herbert Taibleson, <i>The failure of even conjugate characterizations of H^1 on local fields</i>	501
Keti Tenenblat, <i>On characteristic hypersurfaces of submanifolds in Euclidean space</i>	507
Jeffrey L. Tollefson, <i>Involutions of Seifert fiber spaces</i>	519
Joel Larry Weiner, <i>An inequality involving the length, curvature, and torsions of a curve in Euclidean n-space</i>	531
Neyamat Zaheer, <i>On generalized polars of the product of abstract homogeneous polynomials</i>	535