Pacific Journal of Mathematics

EXTREMAL PROPERTIES OF REAL BIAXIALLY SYMMETRIC POTENTIALS IN $E^{2(\alpha+\beta+2)}$

PETER A. MCCOY

Vol. 74, No. 2

June 1978

EXTREMAL PROPERTIES OF REAL BIAXIALLY SYMMETRIC POTENTIALS IN $E^{2(\alpha+\beta+2)}$

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The set \mathscr{B} consists of all real biaxially symmetric potentials $U^{(\alpha,\beta)}(x,y) = \sum_{n=0}^{\infty} a_n (x^2 + y^2)^n P_n^{(\alpha,\beta)}(x^2 - y^2/x^2 + y^2) / P_n^{(\alpha,\beta)}(1)$, $\alpha > \beta > -1/2$ which are regular in the open unit sphere Σ about the origin in $E^{2(\alpha+\beta+2)}$. Three problems appear regarding \mathscr{B} and subset \mathscr{B}_* whose members have the first m+1 coefficients a_0, \dots, a_m specified. (1) For $U^{(\alpha,\beta)} \in \mathscr{B}$, determine $I(U^{(\alpha,\beta)}) = \inf \{ U^{(\alpha,\beta)}(x,y) | (x,y) \in \Sigma \}$ as limit of a monotone sequence of constants $\{\lambda_{2n}(a_0, \dots, a_n)\}_{n=0}^{\infty}$ which can be computed algebraically. (2) Find $U_0^{(\alpha,\beta)} \in \mathscr{B}_*$ and the constant $\lambda_{2m}(a_0, \dots, a_m) = \sup \{I(U^{(\alpha,\beta)}) | U^{(\alpha,\beta)} \in \mathscr{B}_*\} = I(U^{(\alpha,\beta)}_0).$ (3) Determine necessary and sufficient conditions from the Fourier coefficients so that $U^{(\alpha,\beta)} \in \mathscr{B}$ and $U^{(\alpha,\beta)}$ is nonnegative in Σ . We develop solutions using operators based on Koornwinder's Laplace type integral for Jacobi polynomials, along with applications of the methods of ascent and descent to the Caratheodory-Fejer and Caratheodory-Toeplitz problems which focus on the properties of harmonic functions in E^2 .

1. Introduction. Real biaxially symmetric potentials (BASP) $U^{(\alpha,\beta)}$ which are regular in some domain Ω about the origin in $E^{2(\alpha+\beta+2)}$ may be expanded uniquely as a series

$$(1)$$
 $U^{(lpha,eta)}(x,y)=a_{\scriptscriptstyle 0}+2\sum\limits_{n=1}^{\infty}a_nU^{(lpha,eta)}_n(x,y)$, $\ lpha,eta>-1/2$

in terms of the complete set of biaxially symmetric harmonic polynomials

$$(\,2\,) \qquad U_n^{\scriptscriptstyle(lpha,\,eta)}(x,\,y) = (x^2+y^2)^n P_n^{\scriptscriptstyle(lpha,\,eta)}(x^2-y^2\!/x^2+y^2)/P_n^{\scriptscriptstyle(lpha,\,eta)}(1)$$
 ,

defined from the Jacobi polynomials [1, p. 9]. These functions are necessarily even, satisfying the Cauchy data

$$(3) U_{x}^{(\alpha,\beta)}(0, y) = U_{y}^{(\alpha,\beta)}(x, 0) = 0$$

along the singular lines x = 0, y = 0 in Ω .

Symmetry about one axis reduces $U_n^{(\alpha,\beta)}$ to zonal harmonics $(\alpha=\beta)$, identifying $U^{(\alpha,\beta)}$ as a generalized axially symmetric potential (GASP) [1, p. 10; 5, p. 167] which corresponds to the real part of an analytic function of one complex variable when $\alpha = \beta = -1/2$. This simple correspondence provides characterizations of the fundamental properties of harmonic functions in E^2 from their Fourier coefficients in circular harmonics as they are determined by those of the associated analytic functions; serving as a point of reference in seeking the singularities, zeros, and extremal values for other parameters α , β .

R. P. Gilbert [5, 6] employed properties of the Jacobi polynomials to represent each (complex valued) BASP as the integral transform of a unique associated analytic function of one complex variable and conversely. Then reasoning as in the "Envelope Method" [4, 5], a generalization of the Hadamard argument in the Singularities Theorem [3, 5], he showed that the classical criterion of Hadamard and Mandelbrojt [3] for determining the location and structure of the singularities of harmonic functions in E^2 from their Fourier coefficients provides analogous information for BASP in $E^{2(\alpha+\beta+2)}$, $\alpha, \beta > -1/2$.

M. Marden [11] and P. McCoy [12, 13] applied convexity arguments and conformed mapping techniques to the Bergman \mathscr{R}_3 [2, 5] and Gilbert \mathscr{M}_{μ} [4, 5] integral representations of GASP, describing their value distribution as in the classical Cauchy [10, p. 123], Caratheordory-Toeplitz [16, p. 153] and Schur [17, p. 159] coefficient theorems for harmonic functions in E^2 . T. Koornwinder's [1, 9] new Laplace type integral for Jacobi polynomials was used by P. McCoy and J. D'Archangelo [15] to extend properties developed by Marden for the zeros of axially symmetric harmonic polynomials to harmonic polynomials with biaxial symmetry.

Further applications of Koornwinder's integral by McCoy [16] produced operators mapping analytic functions of one complex variable onto (complex valued) BASP and conversely. These operators, valid for limited ranges of the parameters, permitted a partial extension of the Caratheodory-Toeplitz and Schur theorems. Moreover, a new aspect of the coefficient problem—that of the extremal properties of the real axially symmetric potentials of the Caratheodory-Fejer [8, p. 145ff] type—was introduced by operators related to \mathcal{M}_3 and \mathcal{M}_{μ} .

This article provides a unified treatment of the above mentioned theorems and properties, extending them to real BASP without restriction beyond Koornwinder's on the parameters. Taken in union with Gilbert's theory of singularities, it completes the generalization of the classical coefficient theorems pertaining to real harmonic (or analytic) functions in E^2 . These may be also viewed as a means of calculating the infimum (supremum) of solutions to the biaxially symmetric potential equation [5, 9] from the Fourier coefficients which taken with the methods of ascent and descent [4] indicates similiar possibilities for solutions to more general partial differential equation generated by operators whose properties are analogous to those found in 2. Basic formulas and definitions. Koornwinder's formula [9, p. 130] represents the biaxially symmetric harmonic polynomials as

$$(4) \qquad \qquad U_n^{(\alpha,\beta)}(x, y) = \int_0^1 \int_0^\pi \zeta^n d\mu_{\alpha,\beta}(t, s) , \quad \alpha > \beta > -1/2$$

$$(5)$$
 $\zeta = x^2 - y^2 t^2 + i 2xyt \cos s$

with nonnegative measure

$$(6) \qquad \qquad \frac{d\mu_{\alpha,\beta}(t,s) = \gamma_{\alpha,\beta}(1-t^2)^{\alpha-\beta-1}t^{2\beta+1}(\sin s)^{2\alpha}dtds}{\gamma_{\alpha,\beta} = 2\Gamma(\alpha+1)/\Gamma(1/2)\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}$$

normalized so that

(7)
$$\int_{0}^{1}\int_{0}^{\pi}d\mu_{lpha,eta}(t,s)=1$$
.

The real harmonic polynomials

$$(8) \qquad u_n(x, y) \equiv u_n(x^2 - y^2, 2xy) \\ v_n(x, y) \equiv v_n(x^2 - y^2, 2xy)$$

are defined by

$$u_n(x, y) + iv_n(x, y) = (x + iy)^{2n}$$
 .

Expanding the vector ζ^n in terms of these as

$$(9) \qquad \zeta^n = u_n(x^2 - y^2 t^2, 2xyt\cos s) + iv_n(x^2 - y^2 t^2, 2xyt\cos s)$$

and transforming according to Koornwinder's formula establishes that v_n , the harmonic conjugate of u_n , is in the null space of (4). This suggests the relation

(10)
$$U_n^{(\alpha,\beta)}(x, y) = \int_0^1 \int_0^\pi u_n(x^2 - y^2 t^2, 2xyt \cos s) d\mu_{\alpha,\beta}(t, s)$$

associating the real (even) BASP (1) and the real (even) harmonic function

(11)
$$u(x, y) = a_0 + 2\sum_{n=1}^{\infty} a_n u_n(x, y)$$

viz.

$$u(x^2 - y^2, 2xy) = a_0 + 2\sum_{n=1}^{\infty} a_n u_n(x^2 - y^2, 2xy)$$

by the operator

(12)
$$U^{(\alpha,\beta)} = A_{\alpha,\beta}(u)$$

whose definition is

Evidently, if u is harmonic in the open unit disk D_{ρ} of radius ρ about the origin in E^2 , $U^{(\alpha,\beta)}$ is a BASP in the open unit sphere Σ_{ρ} of radius ρ about the origin in $E^{2(\alpha+\beta+2)}$.

The inverse operator (related to $\mathscr{G}_{\alpha,\beta}^{-1}$ [see 6]) uses orthogonality of the Jacobi polynomials

$$\int_{-1}^{+1} P_n^{(\alpha,\beta)}(\tau) P_m^{(\alpha,\beta)}(\tau) (1-\tau)^{\alpha} (1+\tau)^{\beta} d\tau = h_n^{(\alpha,\beta)} \delta_{nm}, \, \alpha, \, \beta > -1$$

to define the measure

$$d oldsymbol{
u}_{lpha,eta}(\xi,\,\eta,\, au) = \mathop{S_{lpha,eta}}_{\scriptscriptstyle{lpha}}(\xi,\,\eta,\, au)(1- au)^{lpha}(1+ au)^{eta}d au$$
 ,

 $S_{\alpha,\beta}(\xi,\eta,\tau) = \sum_{n=0}^{\infty} u_n(\xi,\eta) P_n^{(\alpha,\beta)}(\tau) P_n^{(\alpha,\beta)}(1) / h_n^{(\alpha,\beta)}$

inverting the relation (10) as

$$u_n(x, y) = \int_{-1}^{+1} U_n^{(lpha,eta)}\Big(r\sqrt{rac{1- au}{2}}, r\sqrt{rac{1+ au}{2}}\Big) d
u_{lpha,eta}(xr^{-1}, yr^{-1}, au) \ ,$$

determining the inverse operator

$$u = A_{lpha,eta}^{-1}(U^{(lpha,eta)})$$

as

(14)

(15)
$$u(x, y) = \int_{-1}^{+1} U^{(\alpha, \beta)} \left(r \sqrt{\frac{1-\tau}{2}}, r \sqrt{\frac{1+\tau}{2}} \right) d\nu_{\alpha, \beta}(xr^{-1}, yr^{-1}, \tau) .$$

An absolutely and uniformly convergent dominant of $S_{\alpha,\beta}$ for (ξ, η, τ) on compact subsets of $[0, 1) \times [0, 1) \times [-1, +1]$ is the Poisson kernel [1, p. 11]. By construction of the operators, it follows directly that $A_{\alpha,\beta}$ and $A_{\alpha,\beta}^{-1}$ are one-one onto maps between the families

$${\mathscr H}_{_{
ho}}^{(lpha,\,eta)}=\{U^{(lpha,\,eta)}|\, {
m expansion}\,\, (1)\,\, {
m regular}\,\, {
m in}\,\, {\varSigma}_{
ho}\}\,,\,\,\,\, lpha>\beta>-1/2$$

and

$$\varkappa_{
ho} = \{u \, | \, ext{expansion (11) regular in } D_{
ho} \}$$

which share the normalization

$$A_{lpha,eta}(1)=A_{lpha,eta}^{-1}(1)=1$$
 .

A principle interest is in the values of the functionals

$$egin{aligned} I(U^{(lpha,eta)}) &= \inf_{\Sigma} \, U^{(lpha,eta)} ext{ , } & U^{(lpha,eta)} \in \mathscr{H}^{(lpha,eta)} \ i(u) &= \inf_{p} u ext{ , } & u \in \mathscr{L} \end{aligned}$$

(subscripts $\rho = 1$ are dropped) as they are determined by the minimal eigenvalues $\lambda_{2k}(a_0, \dots, a_k)$ of the Toeplitz matrices

(16)
$$T_{2k}(a_0, \dots, a_k) = \begin{pmatrix} a_0 & 0 & a_1 & 0 & a_2 & 0 & a_3 \dots & a_k \\ 0 & a_0 & 0 & a_1 & 0 & a_2 & 0 & \dots & 0 \\ a_1 & 0 & a_0 & 0 & a_1 & 0 & a_2 & \dots & a_{k-1} \\ \vdots & & & & \vdots \\ a_k & 0 & \dots & & & a_0 \end{pmatrix}$$

found by applying theorem (a) [8, p. 147] to the function $F(z) = f(z^2)$. We now turn to

3. Extremal properties. The following is an extension of theorem (a) [8, p. 147] referred to in an equivalent form [7, p. 499ff] as the Caratheodory-Fejer theorem which is how we identify it.

THEOREM 1. Let $U^{(\alpha,\beta)}(x, y) = a_0 + 2 \sum_{n=1}^{\infty} a_n U_n^{(\alpha,\beta)}(x, y)$ be a real BASP regular in the sphere Σ and $\{\lambda_{2k}(a_0, \dots, a_k)\}_{k=0}^{\infty}$ be the sequence of smallest eigenvalues associated with the Toeplitz matrices $\{T_{2k}(a_0, \dots, a_k)\}_{k=0}^{\infty}$. Then

(17)
$$I(U^{(\alpha,\beta)}) = \lim_k \lambda_{2k}(a_0, \cdots, a_k), \quad \alpha > \beta > -1/2.$$

Proof. For the nonnegativity of the measure (6) and the normalization (7), it is immediate that

$$U^{\scriptscriptstyle (lpha,eta)}(x,\,y)=A_{lpha,eta}(u)\geqq i(u)$$
 , $\ lpha>eta>-1/2$

and

(18)
$$I(U^{(\alpha,\beta)}) \geq i(u) = \lim_{k} \lambda_{2k}(a_0, \cdots, a_k).$$

The smaller functional is evaluated by the Caratheodory-Fejer theorem [8, p. 147]. Anticipating the reverse inequality, we define the functionals

$$egin{aligned} &I_{
ho_0}(U^{(lpha,eta)}) = \inf_{arsigma_{
ho_0}} U^{(lpha,eta)}\ &i_{
ho_0}(u) = \inf_{
ho_{
ho_0}} u \end{aligned}$$

with

$$ho_{_0} = \sup \left\{
ho \, | \, S_{\!lpha, \, eta}(\xi, \, \eta, \, au) > 0
ight.$$
 , ' $\xi^{_2} + \eta^{_2} <
ho^{_2} < 1$, $au \, \epsilon \, [\, -1, \, 1]
brace$.

The number ρ_0 exists since $S_{\alpha,\beta}$ is continuous in a cylinder of small enough radius with center on the τ -axis, $\tau \in [-1, +1]$, and ρ_0 is positive as $S_{\alpha,\beta}(0, 0, \tau) = 1$ there.

Now, if $U_*^{(\alpha,\beta)}(x_1, y_1)$ is a BASP which is nonnegative for $x_1^2 + y_1^2 \leq \rho_0^2$ and the $A_{\alpha,\beta}^{-1}$ associate is u_* , then

$$egin{aligned} &u_st(x_{\scriptscriptstyle 1},\,y_{\scriptscriptstyle 1})=A_{lpha,eta}^{-1}(U^{(lpha,eta)}_st)\ &\geqq A_{lpha,eta}^{-1}(I_{
ho_0}(U^{(lpha,eta)}_st)))=I_{
ho_0}(U^{(lpha,eta)}) \end{aligned}$$

so that

(19)
$$i_{\rho_0}(u_*) \ge I_{\rho_0}(U_*^{(\alpha,\beta)})$$
.

The homothetic transformations $x_1 = \rho_0 x$, $y_1 = \rho_0 y$, and the homogeneity of the harmonic polynomials

$$u_n(x, y) = \rho_0^{-2n} u_n(x \rho_0, y \rho_0)$$

and

$$U_n^{\scriptscriptstyle(lpha,\,eta)}(x,\,y)=
ho_{\scriptscriptstyle 0}^{-2n}U_n^{\scriptscriptstyle(lpha,\,eta)}(x
ho_{\scriptscriptstyle 0},\,y
ho_{\scriptscriptstyle 0})$$

produce harmonic functions

$$u_*(x_1, y_1) = a_0 + 2\sum_{n=1}^{\infty} a_n \rho_0^{-2n} u_n(x_1, y_1)$$

and

$$U^{\scriptscriptstyle(lpha,\,eta)}_{\,m{\star}}(x_{\scriptscriptstyle 1},\,y_{\scriptscriptstyle 1})=a_{\scriptscriptstyle 0}+\,2\sum_{n=1}^\infty a_n
ho_{\scriptscriptstyle 0}^{_{-2n}}U^{\scriptscriptstyle(lpha,\,eta)}_{\,n}(x_{\scriptscriptstyle 1},\,y_{\scriptscriptstyle 1})$$

regular for $x_1^2 + y_1^2 < \rho_0^2$ corresponding to the regular functions (1) and (11) in $x^2 + y^2 < 1$. Evidently,

$$egin{aligned} &I_{
ho_0}(U^{\scriptscriptstyle(lpha,eta)}_*)=I(U^{\scriptscriptstyle(lpha,eta)})\ &i_{
ho_0}\!(u_*)=I(U^{\scriptscriptstyle(lpha,eta)}) \end{aligned}$$

and because of inequality (19),

(20)
$$i(u) \ge I(U^{(\alpha,\beta)})$$
.

Thus,

(21)
$$i(u) = I(U^{(\alpha,\beta)})$$

and because of (18) the theorem is proved.

We next define the set $\mathscr{H}_{*}^{(\alpha,\beta)} = \mathscr{H}^{(\alpha,\beta)}(a_{0}, \dots, a_{m})$ as the subset of $\mathscr{H}^{(\alpha,\beta)}$ whose members have their first m + 1 coefficients a_{0}, \dots, a_{m} fixed and turn to the analogy of the second classical theorem [8, p. 151].

THEOREM 2. Let $U^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)}(\dot{a}_0, \dots, a_m)$ be expanded as in (1) and $\lambda_{2m}(a_0, \dots, a_m)$ be the smallest eigenvalue of the Toeplitz matrix $T_{2m}(a_0, \dots, a_m)$. Then

$$I\!\left(U^{\left(lpha ,eta
ight) }
ight) \leq \lambda _{2m}$$
 , $lpha >eta >-1/2$

and

$$\sup \{I(U_0^{(\alpha,\beta)}) | U^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)}(a_0, \cdots, a_m)\} \\ = I(U_0^{(\alpha,\beta)}) = \lambda_{2m}$$

for unique $U_0^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)}(a_0, \cdots, a_m)$ expanded as

(22)

$$U_{0}^{(\alpha,\beta)}(x, y) = \lambda_{2m} + \sum_{k=1}^{j} \sigma_{k} W_{k}^{(\alpha,\beta)}(x, y) ,$$

$$W_{k}^{(\alpha,\beta)}(x, y) = A_{\alpha,\beta}(w_{k}) ,$$

$$w_{k}(x, y) = [1 - (x^{2} + y^{2})][g_{k}(x, y)]^{-1} ,$$

$$g_{k}(x, y) = 1 - 2(x^{2} + y^{2}) \cos \{2 \arccos x/\sqrt{x^{2} + y^{2}} - \phi_{k}\} + (x^{2} + y^{2})^{2}$$

for unique $1 \leq j \leq m$, $\phi_k \in [0, 2\pi)$, $\sigma_k > 0$ provided $c_1^2 + \cdots + c_m^2 \neq 0$, otherwise if and only if

$$(23) U_0^{(\alpha,\beta)}(x, y) = c_0 = \lambda_{2m}$$

Proof. For the subfamily $\mathscr{H}^{(\alpha,\beta)}_*$ we associate the subfamily $\mathscr{H}_* = A_{\alpha,\beta}^{-1}\{(U^{(\alpha,\beta)} | U^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)}_*)\}$ whose members u satisfy the requisite inequality [8, p. 151],

$$(24) i(u) \leq \lambda_{2m}(a_0, \cdots, a_m) .$$

The matrix $T_{2m}(a_0, \dots, a_m)$ is identified from [8, p. 146] by $u(x, y) = \text{Re } f((x + iy)^2)$. Because of the relation (21), we find

$$I(U^{(lpha,\,eta)}) \leq \lambda_{2m}(a_0,\,\cdots,\,a_m)$$
.

The $A_{\alpha,\beta}^{-1}$ associate of the extremal function $U_0^{(\alpha,\beta)}$ is

Because of (12), u_0 transforms onto the required extremal function (22) as defined.

The final result is the generalization of the Caratheodory-Toeplitz theorem [8, p. 152; 17, p. 157] which classified nonnegative harmonic functions in D from their coefficients.

THEOREM 3. Necessary and sufficient conditions for the expansion $U^{(\alpha,\beta)} \in \mathscr{H}^{(\alpha,\beta)}$ and

$$U^{(lpha,eta)}(x,\,y) \geqq 0$$
 , $x^2+y^2 < 1$, $lpha > eta > -1/2$

specify that the determinants

 $\Delta_n(a_0, \cdots, a_n) = \det T_{2n}(a_0, \cdots, a_n)$

generated from the coefficients of the expansion are either

(i)
$$arDelta_n(a_0, \ \cdots, \ a_n) > 0$$
 , $n = 0, 1, \ \cdots$

or in case

(ii)
$$\begin{aligned} & & \Delta_n(a_0,\,\cdots,\,a_n)>0\;, \quad n=0,\,\cdots,\,m\;, \\ & & & \Delta_n(a_0,\,\cdots,\,a_n)=0\;, \quad n=m+1,\,\cdots\,, \\ & & & where \;\; U^{(lpha,\,eta)}(x,\,y)=\,U_0^{(lpha,\,eta)}(x,\,y)-\lambda_{2m}\;. \end{aligned}$$

Proof. When u, the associate of $U^{(\alpha,\beta)}$, is nonnegative and regular in D so must $U^{(\alpha,\beta)}$ be nonnegative and regular in Σ because the measure of the transform is nonnegative. This is indeed the case [see 8, 18] if (i) or (ii), establishing the sufficiency. Conversely,

$$u_*(x_1, y_1) = A_{lpha, eta}^{-1}(U_*^{(lpha, eta)}) \geqq 0$$

when

$$U^{\scriptscriptstyle(lpha,eta)}_{\,*}(x_{\scriptscriptstyle 1},\,y_{\scriptscriptstyle 1})\geqq 0$$
 , $x_{\scriptscriptstyle 1}^{\scriptscriptstyle 2}+y_{\scriptscriptstyle 1}^{\scriptscriptstyle 2}<
ho_{\scriptscriptstyle 0}^{\scriptscriptstyle 2}$.

However,

$$ext{sgn} \ u_*(x_{\scriptscriptstyle 1},\ y_{\scriptscriptstyle 1}) = ext{sgn} \ u(x,\ y) \ ext{sgn} \ U_*^{\scriptscriptstyle (lpha,eta)}(x_{\scriptscriptstyle 1},\ y_{\scriptscriptstyle 1}) = ext{sgn} \ U^{\scriptscriptstyle (lpha,eta)}(x,\ y)$$

so that $U^{(\alpha,\beta)}$ nonnegative and regular in Σ implies u nonnegative and regular in D which asserts (i) or (ii).

4. Generalizations. For $\beta > \alpha$, the symmetry relations found from [1, p. 8]

$$U_n^{(\alpha,\beta)}(x, y) = (-1)^n U_n^{(\beta,\alpha)}(-x, y)$$

may be employed with the proper interpretation of the biaxially symmetric potential equation [6, 9]. The axisymmetric case $\alpha = \beta$, may be interpreted with $\alpha \downarrow \beta$ in Koornwinder's formula which becomes Gegenbauer's integral for the Jacobi polynomials.

The classical theorems of Caratheodory-Fejer and Caratheodory-Toeplitz have analogous calculations for the supremum [see 8] and bounds on the maximum modulus (Caratheodory-Schur [see 17]) which generalize directly by the methods contained here in. Domains $\Omega \subset E^{2(\alpha+\beta+2)}$ about the origin which are not spheres are defined by their projections into $\omega \subset E^2$ as

$$arOmega=\{(x,\,y)\,|\,\zeta^{\scriptscriptstyle 2}\,\in\,\omega,\,0\leq s\leq\pi,\,-1\leq t\leq+1\}$$
 ,

 ω being a simply connected domain about the origin. To consider extensions of theorems and 1 and 2, ω is mapped conformally onto D. The required connection coefficients between $U^{(\alpha,\beta)}$ a regular BASP in Ω and the "associated" BASP regular in Σ may be found as in [13, 14]. The methods of ascent and descent may be utilized to extend the above properties.

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Received January 25, 1977. Based on a presentation to the AMS at the Annual meetings in St. Louis, January 1977.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).

8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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Pacific Journal of Mathematics Vol. 74, No. 2 June, 1978

Aharon Atzmon, Spectral synthesis in some spaces of bounded continuous	
functions	277
Karl Egil Aubert and Isidor Fleischer, <i>Tensor products of ideal systems and their modules</i>	285
Richard F. Basener. Several dimensional properties of the spectrum of a uniform	
algebra	297
R. H. Bing and Michael Peter Starbird, <i>Super triangulations</i>	307
Andrew Carson, <i>Coherent polynomial rings over regular rings of finite index</i>	327
Robert M. DeVos and Frederick W. Hartmann, <i>Sequences of bounded summability</i>	
domains	333
George Grätzer and R. Padmanabhan, <i>Symmetric difference in abelian groups</i>	339
Robert L. Griess, Jr., A remark about groups of characteristic 2-type and	
<i>p-type</i>	349
Emil Grosswald and F. J. Schnitzer, A class of modified ζ and L-functions	357
Jutta Hausen and Johnny Albert Johnson, <i>Ideals and radicals of some</i>	
endomorphism rings	365
Jean Ann Larson, A solution for scattered order types of a problem of	
Hagendorf	373
Peter A. McCoy, <i>Extremal properties of real biaxially symmetric potentials in</i> $r^{2(\alpha+\beta+2)}$	201
$E^{2(\alpha+p+2)}$	381
Hector Alfredo Merklen, <i>Hereditary crossed product orders</i>	391
Hal G. Moore and Adıl Mohamed Yaqub, Equational definability of addition in	407
certain rings	407
Robert Laurens Moore, Reductivity in C*-algebras and essentially reductive	410
operators	419
Joseph Alvin Neisendorfer, Lie algebras, coalgebras and rational homotopy	420
theory for nupotent spaces	429
William Raymond Nico, <i>Bounded monoids</i>	461
Richard Paul Osborne, Simplifying spines of 3-manifolds	473
Richard Paul Osborne, <i>The simplest closed 3-manifolds</i> . With an appendix by	101
Osborne and J. Yelle	481
Clayton Collier Sherman, <i>The K-theory of an equicharacteristic discrete valuation</i>	407
ring injects into the K-theory of its field of quotients	497
Mitchell Herbert Taibleson, The failure of even conjugate characterizations of H ⁺	501
on local fields	501
Keti Tenenblat, On characteristic hypersurfaces of submanifolds in Euclidean	
<i>space</i>	507
Jeffrey L. Tollefson, <i>Involutions of Seifert fiber spaces</i>	519
Joel Larry Weiner, An inequality involving the length, curvature, and torsions of a	
curve in Euclidean n-space	531
Neyamat Zaheer, On generalized polars of the product of abstract homogeneous	
polynomials	535