LIE ALGEBRAS, COALGEBRAS AND RATIONAL HOMOTOPY THEORY FOR NILPOTENT SPACES

JOSEPH ALVIN NEISENDORFER

LIE ALGEBRAS, COALGEBRAS AND RATIONAL HOMOTOPY THEORY FOR NILPOTENT SPACES

JOSEPH NEISENDORFER

This paper establishes that the homotopy category of rational differential graded commutative coalgebras is equivalent to the homotopy category of rational differential graded Lie algebras which have a nilpotent completion as homology. This generalizes a result which Quillen proved in the simply connected case. When combined with Sullivan's work on rational homotopy theory, our result shows that the homotopy category of rational differential graded Lie algebras with nilpotent finite type homology is equivalent to the rational homotopy category of nilpotent topological spaces with finite type rational homology.

Our results include the construction of minimal Lie algebra models for simply connected spaces, and we show that the rational homotopy groups of a simply connected CW complex may be calculated from a free Lie algebra generated by the cells with a differential given on generators by the attaching maps.

0. **Introduction.** Sullivan [4] and Bousfield-Gugenheim [2] have demonstrated the equivalence of two categories: (a) the rational homotopy category of nilpotent spaces with rational homology of finite type, and (b) the homotopy category of commutative, associative, differential graded rational algebras with minimal models of finite type. Earlier, Quillen [23] had demonstrated the equivalence of the rational homotopy category of simply connected spaces with two categories (among others): (c) the homotopy category of commutative, associative, differential graded simply connected rational coalgebras, and (d) the homotopy category of differential graded connected rational Lie algebras. In this paper, we combine these two approaches and generalize the above portion of Quillen's work to nilpotent spaces with rational homology of finite type (Proposition 7.3).

Since we are developing a theory for nilpotent spaces, nilpotent Lie algebras play an important role. As a consequence of our work, we get the purely algebraic result that the homotopy category of commutative differential graded coalgebras is equivalent to the homotopy category of differential graded Lie algebras whose homology is a nilpotent completion (Definition 3.4 and Proposition 7.2).

We associate to each nilpotent space X with rational homology of finite type, three types of differential graded models, a minimal algebra M_x, a minimal coalgebra C_x, and a free Lie algebra $\mathscr{L}(C_x)$.

(The minimal algebra is due to Sullivan [4].) When X is simply connected, we can replace $\mathscr{L}(C_x)$ by a minimal Lie algebra L_x. All three minimal models are unique up to isomorphism and determine the rational homotopy type of X (Proposition 8.1).

John Moore (unpublished), Baues and Lemaire (*Minimal models in homotopy theory*, preprint), and Tim Miller [14] have given independent constructions of minimal Lie algebras.

If we forget the differential, the minimal Lie algebra model L_x is a free Lie algebra on the desuspension of the rational homology of X (Proposition 8.4). The homology of L_x, or of $\mathscr{L}(C_x)$ when X is not required to be simply connected, is $\pi_{n+1}(X) \otimes Q$ for $n \geqq 1$ and $H_0 \mathscr{L}(C_x)$ is the Lie algebra associated to the Malcev completion of the fundamental group $\pi_1(X)$. (Propositions 8.2 and 8.3.) These results are dual, in the sense of Hilton and Eckmann, to Sullivan's description of minimal algebras.

As a consequence of a Hirsch lemma (Proposition 8.11) for Lie algebras, we get a *CW* theory for rational homotopy (Proposition 8.12). More precisely, there is a differential on the free Lie algebra F generated by the desuspended cells of a simply connected *CW* complex X such that $HF = \pi(\Omega X) \otimes Q$.

After the basic definitions and lemmas in §1, §2 shows that there is a Lie algebra structure on the homology of the primitives of a symmetric coalgebra and a coalgebra structure on the homology of the generators of a free Lie algebra. When applied to the coalgebra and Lie algebra models, these structures correspond to Whitehead products in rational homotopy and comultiplication in rational homology, respectively.

Section 3 studies extension sequences of coalgebras. The main result is that a symmetric coalgebra is constructible by nonprimitive elementary extensions (Definition 3.9), modulo tensoring with an acyclic symmetric coalgebra, if and only if the Lie algebra structure defined in §2 is a nilpotent completion (Proposition 3.12). This is the algebraic analogue of representing a nilpotent space as an inverse limit of principal fibrations. What we do here was motivated by [4] and is a generalization of results that John Moore [18] proved for simply connected coalgebras.

Section 4 discusses certain adjoint functions $\mathscr{L}$ and $\mathscr{C}$ between differential graded coalgebras and Lie algebras. The main result is that the adjunction maps are homology isomorphisms with no restriction to simply connected coalgebras (Proposition 4.1). This result has been published [Moore, 17] but without proof, so we give one.

In §5, we introduce Quillen's closed model category concepts and discuss fibrant, cofibrant, and minimal models (Definitions 5.4 and 5.5), ending in proofs of existence, uniqueness, and a complete

algebraic characterization (Propositions 5.6 to 5.9).

Section 6 defines homotopy groups for algebras, coalgebras, and Lie algebras. This is essentially a combination of the structures defined in § 2 with the concepts introduced in § 5. Our definition of the homotopy Lie algebra of a coalgebra differs from that preferred by John Moore [19]. In particular, our homotopy Lie algebra is always a nilpotent completion (Proposition 6.2) and is invariant under homology isomorphism of coalgebras.

Sections 7 and 8 contain the results mentioned at the beginning of this introduction relating algebras, coalgebras, and Lie algebras to rational homotopy theory. These are the main results of the paper and the reader who seeks motivation should read those sections first.

The main point of this paper is to establish the equivalence between differential graded rational Lie algebras and rational homotopy theory as expounded by Sullivan. In other papers, [15] and [20], these results are applied to the study of Sullivan's formal spaces, i.e., those whose rational homotopy type is a formal consequence of their rational cohomology rings. Minimal Lie algebra models can be used to show that certain spaces are formal, e.g., every n connected compact manifold of dimension $\leq 3n+1$, $n \geq 1$ [15]. Galois cohomology applies to minimal Lie algebras to prove: Any simply connected space which is formal over an extension field of the rationals is necessarily rationally formal [15].

Most importantly, minimal Lie algebras can be used to give complete calculations of the rational homotopy groups of some quite complicated spaces. For example, in [20] I calculate the rational homotopy Lie algebra of all projective algebraic varieties which are complete intersections of complex dimension ≥ 2.

In Quillen's original approach to rational homotopy theory for simply connected spaces, there were no finite type restrictions on homology. We suspect that Quillen's approach may be generalized to nilpotent spaces with no finite type restrictions. But this removal of finite type restrictions would be achieved at the cost of the connection between Quillen's work and Sullivan's work. Many applications would be lost and the treatment would be significantly longer. So we chose to keep the connection.

Besides the obvious debts to authors already mentioned in this introduction, I owe much to Alison Beall, Tim Miller, John Moore, Jim Stasheff, and Larry Taylor for stimulating and encouraging conversations.

1. **Algebras, coalgebras, and Lie algebras.** This section presents basic definitions and lemmas concerning differential graded algebras, coalgebras, and Lie algebras. Good references are [16], [10], and [23].

Let k be a field of characteristic zero.

DGA is the category of differential graded, augmented, commutative, associative algebras defined over k which are concentrated in nonnegative degrees and have a cohomology differential (degree $+1$). We restrict DGA to algebras A which are homologically connected, $H^\circ(A) = k$. We call the objects in this category algebras and shall suppress the augmentations $A \to k$ in our notation.

The category DGA has coproducts $A \otimes B$ and products $A \vee B =$ the equalizer of the maps $A \oplus B \to A \to k$ and $A \oplus B \to B \to k$.

Recall that $\bar{A} =$ the kernel of $A \to k$ and $Q(A)(= \bar{A}/\bar{A} \cdot \bar{A})$ is the module of indecomposables.

DGC is the category of differential graded, connected, commutative, associative coalgebras defined over k which are concentrated in nonnegative degrees and have a homology differential (degree -1). We call the objects in this category coalgebras and note that connected coalgebras C have a unique augmentation $k \to C$.

The category DGC has products $C \otimes D$ and coproducts $C \vee D =$ the coequalizer of the maps $k \to C \to C \oplus D$ and $k \to D \to C \oplus D$.

Recall that $\bar{C} =$ the cokernel of $k \to D$ and $P(C)(=$ the kernel of $\bar{C} \to \bar{C} \otimes \bar{C})$ is the module of primitives.

$DGLA$ is the category of differential graded Lie algebras defined over k which are concentrated in nonnegative degrees and have a homology differential. We will call objects in this category Lie algebras.

For the convenience of the reader, we recall the identities for a graded Lie bracket [ ]: $L \otimes L \to L$ over a field of characteristic zero [11] [23].

(antisymmetry) $[x, y] = -(-1)^{\deg x \deg y}[y, x]$

(Jacobi identity) $[x, [y, z]] = [[x, y], z] + (-1)^{\deg x \deg y}[y, [x, z]]$

Notice that, whenever A is an associative algebra, defining $[x, y] = xy - (-1)^{\deg x \deg y}yx$ makes A into a Lie algebra.

The reader may dualize these identities to obtain the definition of a Lie coalgebra.

The category $DGLA$ has products $L \times L'$ and coproducts $L \vee L' =$ free product of L and L'.

Quillen [23] has shown:

PROPOSITION 1.1. *Lie algebras L have universal enveloping Hopf algebras $U(L)$ with natural isomorphisms $L \cong PU(L)$ and $HL \cong PHU(L)$.*

DEFINITION 1.2. For a Lie algebra L, we define $Q(L) = L/[L, L] =$ the abelianization of L.

Let V be a graded vector space with a differential. (The case of no differential is included, as always, by imposing the zero differential.) In DGA, DGC, or $DGLA$ we shall use various objects $S[V]$, $S'[V]$, or $F[V]$ which we shall now describe.

$T[V] = \bigoplus V^{\otimes n}$ is the tensor algebra with multiplication $V^{\otimes n} \otimes V^{\otimes m} = V^{\otimes(n+m)}$, $(x_1 \otimes \cdots \otimes x_n)(x_{n+1} \otimes \cdots \otimes x_{n+m}) = x_1 \otimes \cdots \otimes x_{n+m}$. This is not a commutative algebra. The quotient of $T[V]$ by the ideal generated by all $x \otimes y - (-1)^{\deg x \deg y} y \otimes x$, where x and y are in V, is a commutative algebra, denoted $S[V]$.

The symmetric group acts on each $V^{\otimes n}$ by permuting the tensors with the usual sign and the projection $T[V] \to S[V]$ has a canonical splitting $\sigma: S[V] \to T[V]$ onto the set of invariant elements, e.g., $\sigma(xy) = 1/2(x \otimes y + (-1)^{\deg x \deg y} y \otimes x)$. The map $\sigma: S[V] \to T[V]$ is not an algebra map.

$S[V]$ is called the free commutative or symmetric algebra generated by V and is characterized up to isomorphism by the universal property: Given any linear map $f: V \to B$, where B is a (commutative) algebra, there exists a unique algebra map $S[V] \to B$ such that $V \to S[V] \to B$ equals f.

$T'[V] = \bigoplus V^{\otimes n}$ is the tensor coalgebra with comultiplication $V^{\otimes r} \to \bigoplus_{m+n=r} (V^{\otimes n} \otimes V^{\otimes n})$, $x_1 \otimes \cdots \otimes x_r \to \Sigma(x_1 \otimes \cdots \otimes x_m) \otimes (x_{m+1} \otimes \cdots \otimes x_r)$.

This is not a commutative coalgebra. The subspace $S'[V]$ of invariant elements is a commutative subcoalgebra of $T'[V]$, and there is a canonical retraction $\sigma': T'[V] \to S'[V]$, e.g., $\sigma'(x \otimes y) = 1/2(x \otimes y + (-1)^{\deg x \deg y} y \otimes x)$. The map σ' is not a coalgebra map.

$S'[V]$ is called the injective commutative or symmetric coalgebra cogenerated by V and is characterized up to isomorphism by the universal property: Given any linear map $g: D \to V$, where D is a (commutative) coalgebra, there exists a unique coalgebra map $D \to S'[V]$ such that $D \to S'[V] \to V$ equals g.

REMARKS. As $S[V]$ was characterized by every map defined on V extending uniquely, so $S'[V]$ is characterized by every map into V lifting uniquely.

$F[V]$ denotes the free Lie algebra generated by V and is characterized up to isomorphism by the universal property: Given any linear map $f: V \to L$, where L is a Lie algebra, there exists a unique Lie algebra map $F[V] \to L$ such that $V \to F[V] \to L$ equals f.

$F[V]$ may also be described as follows: [23] Make the tensor algebra $T[V]$ into a Hopf algebra by defining the comultiplication on x in V by $x \to x \otimes 1 + 1 \otimes x$. Then $F[V] = PT[V]$, $T[V] = UF[V]$.

LEMMA 1.3. *If A, C, or L is a connected symmetric algebra, symmetric coalgebra, or free Lie algebra, respectively, then so is any retract.*

This lemma is a simple consequence of the definitions by universal mapping properties.

We close this section with a collection of technical lemmas for (connected) coalgebras. These lemmas are the duals of well known results for algebras and the proofs will be left as exercises.

If D is a coalgebra, we may pick a splitting $D \to PD$ for the inclusion $PD \to D$.

LEMMA 1.4. *Two coalgebra maps $C \to D$ are equal if and only if the two compositions $C \to D \to PD$ are equal.*

LEMMA 1.5. *Given a coalgebra map $C \to D$ and a surjective coalgebra map $C \to C'$, then $C \to D$ factors into $C \to C' \to D$ if and only if $C \to D' \to PD$ factors into $C \to C' \to PD$.*

If C is a coalgebra, a coderivation on C is a degree -1 linear map $d : C \to C$ such that the following diagrams commute.

$$
\begin{array}{ccc}
C & \xrightarrow{\ d\ } & C \\
\downarrow & & \downarrow \\
k & \xrightarrow{\ o\ } & k
\end{array}
\qquad\qquad
\begin{array}{ccc}
C & \xrightarrow{\qquad d \qquad} & C \\
\downarrow & & \downarrow \\
C \otimes C & \xrightarrow[\ 1 \otimes d + d \otimes 1\ ]{} & C \otimes C
\end{array}
$$

Note that a DG coalgebra is just a coalgebra with a coderivation d such that $d^2 = 0$.

LEMMA 1.6. *Two coderivations on a coalgebra C are equal if and only if the two compositions $C \to C \to PC$ are equal.*

LEMMA 1.7. *Given a coderivation d on a coalgebra C and a surjective coalgebra map $f : C \to C'$, then there exists a coderivation d' on C' such that $d'f = fd$ if and only if there exists a linear map $e : C' \to PC'$ such that $ef = \pi fd$ where $\pi : C' \to PC'$ is a splitting.*

LEMMA 1.8. *A coderivation d on C satisfies $d^2 = 0$ if and only if $\pi d^2 = 0$ where $\pi : C \to PC$ is a splitting.*

LEMMA 1.9. *Given a degree -1 linear map $\bar{d} : S'[V] \to V$, there exists a unique coderivation d on $S'[V]$ such that $\pi d = \bar{d}$.*

LEMMA 1.10. (a) *If $S'[V]$ is a symmetric coalgebra with $HV = 0$, then $HS'[V] = k$.*

(b) *If $S'[V] \to S'[W]$ is a map of symmetric coalgebras with $HV \to HW$ an isomorphism, then $HS'[V] \to HS'[W]$ is an isomorphism.*

Finally, we remark that John Moore (unpublished) has pointed out that the category DGC is closed under inverse limits but these inverse limits are not constructed in the obvious way. Hence, it is worth recording the following:

LEMMA 1.11. (a) *If V_α is an inverse system of graded vector spaces, then $\varprojlim S'[V_\alpha] = S'[\varprojlim V_\alpha]$.*

(b) *In $\widehat{DGC}$ an inverse limit of injective maps is the "intersection."*

2. Lie bracket and coproduct structures. Let C be a coalgebra (algebra, or Lie algebra). We shall denote by $C^\sharp$ the coalgebra (algebra, or Lie algebra) which is identical to C but has zero differential.

Suppose C is a coalgebra with $C^\sharp = $ a symmetric coalgebra $S'[V]$. Consider the primitive filtration on C [16], i.e., $F^\circ C = k$, $F^1(C) = k \oplus P(C)$, $\cdots$, $F^n(C) = k \oplus \ker(\bar{\Delta} : \bar{C} \to \bar{C}^{\otimes(n+1)})$, $\cdots$. This gives rise to a homology spectral sequence of coalgebras with $E^\circ(C) = C^\sharp = S'[PC]$, $E^1(C) = S'[HPC]$, $E^1_{1,n} = H_{n+1}PC$.

$d^1 : E^1_{2,n} \to E^1_{1,n}$ composes with $\sigma' : HPC \otimes HPC \to S'[HPC]$ to give a degree -1 map $\langle\ \rangle : HPC \otimes HPC \to HPC$.

Letting s^{-1} denote desuspension and following the standard sign convention that $f \otimes g(x \otimes y) = (-1)^{\deg x \deg g} fx \otimes gy$, we define $[\ ]: s^{-1}HPC \otimes s^{-1}HPC \to s^{-1}HPC$ by the commutative diagram:

$$
\begin{array}{ccc}
HPC \otimes HPC & \xrightarrow{\ \langle\ \rangle\ } & HPC \\
\Big\downarrow{\scriptstyle s^{-1} \otimes s^{-1}} & & \Big\downarrow{\scriptstyle s^{-1}} \\
s^{-1}HPC \otimes s^{-1}HPC & \xrightarrow{\ [\ ]\ } & s^{-1}HPC
\end{array}\ .
$$

PROPOSITION 2.1. *With the above bracket, $s^{-1}HPC$ is a Lie algebra.*

Proof. Since σ' is symmetric, $\langle\ \rangle$ is also, i.e., $\langle x, y \rangle = (-1)^{\deg x \deg y}\langle y, x \rangle$. Since $[s^{-1}x, s^{-1}y] = (-1)^{\deg x}s^{-1}\langle x, y \rangle$, if follows that $[\ ]$ is antisymmetric.

Since $(d^1 \otimes 1 + 1 \otimes d^1)^2\bar{\Delta} = 0$, where $\bar{\Delta}$ is the comultiplication, an elementary computation shows that:

$$\langle\langle x, y\rangle, z\rangle + \varepsilon\langle\langle y, z\rangle, x\rangle + \delta\langle\langle z, x\rangle, y\rangle = 0$$

where $\varepsilon = (-1)^{\deg x(\deg y + \deg z)}$, $\delta = (-1)^{\deg z(\deg x + \deg y)}$.

The Jacobi identity for [ ] follows.

REMARK. The binary operation $\langle\ \rangle: HPC \otimes HPC \to HPC$ satisfies the same identities as the Whitehead product $\pi_n(X) \otimes \pi_m(X) \to \pi_{n+m-1}(X)$ in homotopy theory [30]. HPC is a "Whitehead algebra" in the sense of [21].

Dually, let A be an algebra with $A^\sharp = $ a symmetric algebra $S[V]$. We may filter A by the powers of its augmentation ideal, $F_0(A)_n = A$, $F_1(A) = \bar{A}$, $\cdots$, $F_n(A) = (\bar{A})^n$, $\cdots$. This gives rise to a cohomology spectral sequence of algebras with $E_0(A) = A^\sharp = S[QA]$, $E_1(A) = S[HQA]$, $E_1^{1,n} = H^{n+1}QA$.

$d_1: E_1^{1,n} \to E_1^{2,n}$ composes with $\sigma: S[HQA] \to HQA \otimes HQA$ to give a degree $+1$ map $\langle\ \rangle^*: HQA \to HQA \otimes HQA$. Define $[\ ]^*: s^{-1}HQA \to s^{-1}HQA \otimes s^{-1}HQA$ by the commutative diagram.

$$
\begin{array}{ccc}
HQA & \xrightarrow{\ \langle\ \rangle^*\ } & HQA \otimes HQA \\
\downarrow{\scriptstyle s^{-1}} & & \downarrow{\scriptstyle s^{-1}\otimes s^{-1}} \\
s^{-1}HQA & \xrightarrow{\ [\ ]^*\ } & s^{-1}HQA \otimes s^{-1}HQA
\end{array}
$$

An argument similar to 2.1 shows [see 21]:

PROPOSITION 2.2. $s^{-1}HQA$ is a Lie coalgebra.

REMARK. If A is a finite type, then $(s^{-1}HQA)^* = s^{-1}HPA^*$ as Lie algebras.

REMARK. If the induced differential on $Q(A)$ is zero, i.e., if $d(\bar{A}) \subset \bar{A} \cdot \bar{A}$, then $QA = HQA$ and the Lie cobracket $s^{-1}QA \to s^{-1}QA \otimes s^{-1}QA$ can be read off from the quadratic term of the differential on generators [see 4 or 26].

Finally, let L be a Lie algebra with $L^\sharp = $ a free Lie algebra $F[V]$. We may filter L by the descending central series, $F^0(L) = L$, $F^{-1}(L) = [L, L]$, $\cdots$, $F^{-n}(L) = [L, F^{-n+1}(L)]$, $\cdots$. This gives rise to a homology spectral sequence of Lie algebras with $E^0(L) = L^\sharp = F[V]$, $E^1(L) = F[HV] = F[HQL]$, $E^1_{-1,n} = H_{n-1}QL$.

$d_1: E^1_{-1,n} \to E^1_{-2,n}$ may be composed with $F[HQL] \to T[HQL] \to HQL \otimes HQL$ to give a degree -1 map $HQL \to HQL \otimes HQL$. The commutative diagram below defines a degree 0 map $\bar{\Delta}: sHQL \to sHQL \otimes sHQL$, where s denotes suspension.

$$HQL \longrightarrow HQL \otimes HQL$$
$$\downarrow s \qquad\qquad \downarrow s \otimes s$$
$$sHQL \xrightarrow{\ \bar{\varDelta}\ } sHQL \otimes sHQL$$

We define $\varDelta$ on $k \oplus (sHQL)$ by $\varDelta(1) = 1 \otimes 1$, $\varDelta(sx) = sx \otimes 1 + 1 \otimes sx + \bar{\varDelta}(sx)$.

An argument similar to 2.1 shows:

PROPOSITION 3.3. $k \oplus (sHQL)$ *is a (commutative) coalgebra.*

A special case of 2.3 admits a nice description. Let L be a Lie algebra with $L^{\sharp} = $ a free Lie algebra $F[V]$ and suppose that dL is contained in $[L, L]$. Such Lie algebras are called minimal. (See 5.5 and 5.7c.) If x_α is a basis for V with $dx_\alpha = \Sigma c_{\beta\alpha}^\alpha [x_\beta, x_\alpha] +$ higher degree terms, then $\bar{\varDelta}(sx_\alpha) = \Sigma(-1)^{\deg x_\beta} c_{\beta\alpha}^\alpha (sx_\beta \otimes sx_\alpha + (-1)^{\deg x_\alpha \deg x_\beta} sx_\alpha \otimes sx_\beta)$. In other words, when $L^{\sharp}$ is free and the induced differential on $Q(L)$ is zero, then $\varDelta$ can be read off from the quadratic term of the differential on L. Combined with 4.2b, this gives a complete computation of the Lie algebra homology coalgebra of a minimal Lie algebra.

3. Nilpotence. Given a graded Lie algebra L, recall the descending central series $F^0 = L$, $F^{-1} = [L, L]$, $\cdots$ with $F^0/F^{-1} = Q(L)$.

We introduce another filtration $G^0 = L$, $G^{-1} = [L_0, L]$, $G^{-2} = [L_0, G^{-1}]$, $\cdots$ and set $\widetilde{Q}(L) = G^0/G^{-1}$.

DEFINITION 3.1. A graded Lie algebra L is nilpotent if either of the following equivalent conditions hold:
  (a) the descending central series terminates at 0 in each degree
  (b) the filtration G^0, G^{-1}, G^{-2}, $\cdots$ terminates at 0 in each degree.
Obviously, a connected Lie algebra, $L_0 = 0$, is nilpotent.
Recall the well known [16]:

LEMMA 3.2. *If $f: C \to D$ is a map between coalgebras with C connected, then f is monic if and only if $P(f): PC \to PD$ is monic. If $f: A \to B$ is a map between algebras with B connected, then f is epic if and only if $Q(f): QA \to QB$ is epic.*

For Lie algebras, we have:

LEMMA 3.3. *If $f: L \to L'$ is a map between Lie algebras with L' nilpotent, then f is epic if and only if either of the following conditions hold:*

(a) $Q(f): Q(L) \rightarrow Q(L')$ *is epic.*
(b) $\tilde{Q}(f): \tilde{Q}(L) \rightarrow \tilde{Q}(L')$ *is epic.*

The proof of 3.3 is a simple exercise in the use of the five lemmas.

DEFINITION 3.4.   A Lie algebra is a nilpotent completion if it is isomorphic to some $\varprojlim L/G^{-n}$ where L is a Lie algebra.

Clearly a nilpotent Lie algebra is a nilpotent completion.

We now relate nilpotence to extension sequences of coalgebras. We choose a quite restricted definition (compare [18]).

DEFINITION 3.5.   An extension sequence of coalgebras is a sequence of coalgebras and maps between them $C' \rightarrow C \rightarrow C''$ which is split when we forget the differential.

(A splitting is a coalgebra map $C \rightarrow C'$ such that $C' \rightarrow C \rightarrow C'$ is the identity and such that it defines an isomorphism $C \rightarrow C' \otimes C''$ of coalgebras.)

An extension sequence is called trivial if there is a splitting $C \rightarrow C'$ which is a map of differential coalgebras.

$C' \rightarrow C$ is called the fiber and $C \rightarrow C''$ the projection of the extension sequence.

Given a vector space V_n concentrated in dimension $n \geqq 1$, define W_n as follows:

If $n \geqq 2$, set $W_n = V \oplus s^{-1}V_n$ with $d(s^{-1}V_n) = 0$ and $d: V_n \rightarrow s^{-1}V_n$ an isomorphism.

If $n = 1$, set $W_n = 0$.   (In this section, we will not use this case but it plays a role in § 5.)

Let $E'[V_n] = S'[W_n]$ and for $n \geqq 2$, consider the extension sequence $S'[s^{-1}V_n] \rightarrow E'[V_n] \rightarrow S'[V_n]$.   If C'' is a coalgebra and f: $C'' \rightarrow V_n$ is a (chain) map, then f defines a unique map $C'' \rightarrow S'[V_n]$ by the universal property of symmetric coalgebras and we may form the pullback diagram:

$$
\begin{array}{ccc}
S'[s^{-1}V_n] & \xrightarrow{\ =\ } & S'[s^{-1}V_n] \\
\downarrow & & \downarrow \\
C & \longrightarrow & E'[V_n] \\
\downarrow & & \downarrow \\
C'' & \longrightarrow & S'[V_n] \ .
\end{array}
$$

LEMMA 3.6.   *The left hand side is an extension sequence.*

Proof.   The pullback is the cotensor product [16], hence $C = C'' \square_{S'[V_n]} E'[V_n] = C'' \otimes S'[s^{-1}V_n]$ as nondifferential coalgebras.

DEFINITION 3.7. An extension sequence of the type in 3.6 is called an elementary extension of dimension $n - 1$. An elementary extension is called nonprimitive if the map $PC'' \to C'' \to V_n$ is zero.

The reason for introducing nonprimitive elementary extensions is contained in the next proposition.

PROPOSITION 3.8. *Let $S'[s^{-1}V_n] \to C \to C''$ be an extension sequence of coalgebras with $(C'')^{\sharp} = a$ symmetric coalgebra. This sequence is a nonprimitive elementary extension if and only if the following sequence is short exact and a central extension of Lie algebras:*

$$0 \longrightarrow s^{-2}V_n \longrightarrow s^{-1}HPC \longrightarrow s^{-1}HPC'' \longrightarrow 0 \; .$$

Proof. First suppose that the sequence is a nonprimitive elementary extension. Recall that the cotensor product $C = C'' \square_{S'[V_n]} E'[V_n]$ is the kernel of $\varDelta \otimes 1 - 1 \otimes \varDelta : C'' \otimes E'[V_n] \to C'' \otimes S'[V_n] \otimes E'[V_n]$.

It follows from our nonprimitive hypothesis that $s^{-1}V_n \oplus PC'' \oplus (PC'' \otimes s^{-1}V_n)$ is a differential submodule of C. From this it follows that the sequence of Lie algebras is short exact and a central extension.

Now suppose that the Lie algebras are short exact and a central extension. Short exactness implies that $0 \to s^{-1}V_n \to PC \to PC'' \to 0$ is split exact as chain complexes. Let $PC \to s^{-1}V_n$ be a splitting. Consider the composition $C \to PC \to s^{-1}V_n$ where $C \to PC$ splits the inclusion $PC \to C$. (This is not a differential splitting.)

The commutative diagram

$$
\begin{array}{ccc}
C & \dashrightarrow & V_n \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
C & \longrightarrow & s^{-1}V_n
\end{array}
$$

defines a chain map $C \to W_n$ and hence a map of differential coalgebras $C \to E'[V_n] = S'[W_n]$.

The composition $C \to W_n \to V_n$ factors as $C \to C'' \to V_n$ by 1.5. The only possible obstruction is the composition $C_1 \otimes s^{-1}V_n \to C \xrightarrow{d} C \to PC \to s^{-1}V_n$ which is zero by the central extension hypothesis.

$C'' \to V_n$ defines a map of differential coalgebras $C'' \to S'[V_n]$ and the following diagram commutes by 1.4:

$$
\begin{array}{ccc}
C & \longrightarrow & E'[V_n] \\
\downarrow & & \downarrow \\
C'' & \longrightarrow & S'[V_n] \; .
\end{array}
$$

The horizontal maps are clearly isomorphisms, hence $S'[s^{-1}V_n] \to C \to C''$ is an elementary extension. That it is nonprimitive, i.e., that $PC'' \to C'' \to V_n$ is zero, follows from short exactness.

DEFINITION 3.9. Let D and C be coalgebras. D is called an inverse limit of nonprimitive elementary extensions starting from C if there is an inverse system $C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$ with each $C_n \to C_{n-1}$ the projection of a nonprimitive elementary extension, $C = C_0$, and $D = \varprojlim C_n$.

More generally, D is constructible by nonprimitive elementary extensions starting from C if there is an inverse system $D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots$ with $C = D_0$, $D = \varprojlim D_n$, and each D_n an inverse limit of nonprimitive elementary extensions of dimension n starting from D_{n-1}.

DEFINITION 3.10. A coalgebra C with $C^{\#} = $ a symmetric coalgebra is called a reduced symmetric coalgebra if $d = 0$ on PC.

The next proposition is the main result of this section.

PROPOSITION 3.11. *A reduced symmetric coalgebra C is constructible by nonprimitive elementary extensions starting from the ground field k if and only if the Lie algebra $s^{-1}HPC = s^{-1}PC$ is a nilpotent completion.*

Proof. If C is constructible by nonprimitive elementary extensions starting from k, then it follows from 1.11a and 3.8 that C is a reduced symmetric coalgebra with $s^{-1}PC$ a nilpotent completion.

Conversely, let C be a reduced symmetric coalgebra with $s^{-1}PC$ a nilpotent completion. Let $C^{\#} = S'[V]$ and denote by $V^{(n)}$ the graded vector space which is V truncated above dimension n. Use 1.7 to construct an inverse system $D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots$ with $D_n^{\#} = S'[V^{(n)}]$ and $C = \varprojlim D_n$.

It suffices to show that each D_n is an inverse limit of nonprimitive elementary extensions starting from D_{n-1}. Use 1.7 to construct an inverse system $C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$ with $(C_m)^{\#} = S'[X_m]$, $X_m = V^{(n-1)} \oplus (L_n/G^{-m} \cap L_n)$, where G^{-m} is the filtration introduced at the beginning of this section and $V = \varprojlim L/G^{-m}$. $C_0 = D_{n-1}$ and $D_n = \varprojlim C_m$ by 1.11a. Proposition 3.8 implies that each $C_m \to C_{m-1}$ is the projection of a nonprimitive elementary extension.

REMARK. A coalgebra constructible by nonprimitive elementary extensions starting from the ground field is automatically a reduced symmetric coalgebra.

Proposition 3.11 may be improved to:

PROPOSITION 3.12. *Let C be a coalgebra with $C^{\#} = a$ symmetric coalgebra. Then the following are equivalent.*

(a) *$s^{-1}HPC$ is a nilpotent completion,*

(b) *$C = D \otimes E$ where D is a reduced symmetric coalgebra with $s^{-1}PD$ a nilpotent completion and $D = S'[V]$ with $HV = 0$.*

Proof. That (b) implies (a) is trivial.

Let $s^{-1}HPC$ be a nilpotent completion and set $C^{\#} = S'[W]$. Then $W = PC = HPC \oplus V$, where V is acyclic. Since V is acyclic, it is injective in the category of chain complexes [5]. Hence, the inclusion $V \rightarrow C$ has a differential splitting $C \rightarrow V$.

The map $C \rightarrow V$ defines a coalgebra map $C \rightarrow S'[V]$. Let $D = C \square_{S'[V]} k =$ the kernel of this map $C \rightarrow S'[V]$. Then $D^{\#} = S'[HPC]$ and $HD \rightarrow HC$ is an isomorphism by 1.10.

It will follow from 5.8 that D is fibrant (see 5.4) and hence a homology isomorphism $D \rightarrow C$ can be split (5.11). It follows that there is an isomorphism $C \rightarrow D \otimes E$ with $E = S'[V]$.

We leave to the reader the obvious dualization of 3.5 to 3.12 to algebras. The algebra analogue of $E'[V_n]$ is denoted $E[V_n]$.

In particular, the dual of a nonprimitive elementary extension is just Sullivan's [4] notion of an elementary extension of algebras with decomposable differential. Just as the minimal algebras defined by Sullivan are constructible from the ground field by elementary extensions with decomposable differentials, so the minimal coalgebras (=reduced symmetric coalgebras C with $s^{-1}HPC$ a nilpotent completion) to be defined in § 5 are constructible by nonprimitive elementary extensions starting from the ground field. (See Proposition 5.8b.)

4. Adjoint equivalences. Quillen [23] has defined adjoint functors

$$DGC \underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}} DGLA$$

and shown that the adjunction maps $C \rightarrow \mathscr{C}\mathscr{L}(C)$ and $\mathscr{L}\mathscr{C}(L) \rightarrow L$ are homology isomorphisms when C is a simply connected coalgebra, $C_1 = 0$, and L is a connected Lie algebra, $L_0 = 0$. In this section we show that these assumptions are not necessary.

$\mathscr{L}$ may be described as follows: As a Lie algebra, $\mathscr{L}(C) = F[s^{-1}\bar{C}] =$ the free Lie algebra on the desuspension of $\bar{C} = \ker (C \rightarrow k)$. The differential on $\mathscr{L}(C)$ splits as a sum of two differentials $d = d_I + d_\Delta$, where on generators $s^{-1}c$, $c \in \bar{C}$,

$$d_I s^{-1}c = -s^{-1}dc$$
$$d_\Delta s^{-1}c = -\Sigma(-1)^{\deg c_i'}[s^{-1}c_i', \, s^{-1}c_i'']$$

if $\varDelta(c) = c \otimes 1 + 1 \otimes c + \Sigma(c_i' \otimes c_i'' + (-1)^{\deg c_i' \deg c_i''} c_i'' \otimes c_i')$.

$\mathscr{L}(C)$ has a natural bigraded structure with generators $s^{-1}c$ having bigrading = $(-1, \text{degree } c)$.

REMARKS. $\mathscr{L}(C) = P\varOmega C =$ primitive elements in the cobar construction $\varOmega C$ an C. [1], [10]. Hence, $H\mathscr{L}(C) = HP\varOmega C = PH\varOmega C = P$ $\mathrm{Cotor}_C(k, k)$ where Cotor is the differential Cotor of Eilenberg and Moore [5]. That $HP\varOmega C = PH\varOmega C$ follows from the fact that $\varOmega C$ is a cocommutative differential Hopf algebra. See Smith [25, p. 130] or Quillen [23, p. 280].

As a coalgebra, $\mathscr{C}(L) = S'[sL] =$ the symmetric coalgebra on the suspension of L. The differential on $\mathscr{C}(L)$ splits as a sum of two differentials $d = d_I + d_{[\,]}$. A differential d on $S'[sL]$ is uniquely characterized by the projection

$$\bar{d}: S'[sL] \longrightarrow S'[sL] \longrightarrow sL .$$

(See 1.6 and 1.9.)

With respect to the natural generators $s\ell$, $(s\ell'')(s\ell'') = s\ell' \otimes s\ell'' + (-1)^{\deg \ell' \deg \ell''} s\ell'' \otimes s\ell'$, $\cdots$ of $S'[sL] \subset T'[sL]$,

$$\bar{d}_I(s\ell) = -sd\ell$$
$$\bar{d}_{[\,]}(s\ell')(s\ell'') = s[\ell', \ell'']$$

and are zero on all other generators.

$\mathscr{C}(L)$ has a natural bigraded structure with elements $s\ell$ having bigrading = $(1, \text{degree } \ell)$.

REMARK. If L is concentrated in degree 0, then $\mathscr{C}(L)$ is just the standard complex for calculating Lie algebra homology. In general, $\mathscr{C}(L)$ is a subcoalgebra of $BU(L) =$ the bar construction on the universal enveloping algebra of L and the injection $\mathscr{C}(L) \to BU(L)$ is a homology isomorphism [17]. Hence, $H\mathscr{C}(L) = \mathrm{Tor}_{U[L]}(k, k)$ where this is the differential Tor of Eilenberg and Moore [25].

The adjunction maps $\alpha: \mathscr{L}\mathscr{C}(L) \to L$ and $\beta: C \to \mathscr{C}\mathscr{L}(C)$ may be described as follows:

    (a) α is the unique map extending $s^{-1}\overline{\mathscr{C}}(sL) \to \mathscr{C}(sL) \to sL \to L$,

    (b) β is the unique map lifting $C \to s^{-1}\bar{C} \to \mathscr{L}(C) \to s\mathscr{L}(C)$.

PROPOSITION 4.1. *The adjunction maps* $\mathscr{L}\mathscr{C}(L) \to L$ *and* $C \to \mathscr{C}\mathscr{L}(C)$ *are homology isomorphisms.*

This is an immediate consequence of our next theorem.

PROPOSITION 4.2. (a) *Let C be a coalgebra such that* $C^\sharp = S'[V]$,

then $s^{-1}HPC$ is isomorphic to $H\mathscr{L}(C)$ as Lie algebras.

(b) *Let L be a Lie algebra such that $L^{\sharp} = F[V]$, then $sHQL$ is isomorphic to $\bar{H}\mathscr{C}(L)$ as coalgebras.*

Proof. (a) Consider the map $s^{-1}PC \to \mathscr{L}(C)$. Filter $\mathscr{L}(C)$ by the second coordinate of its bidegree and give $s^{-1}PC$ a trivial one stage filtration. The map is a map of filtered objects and both filtrations are bicomplete [7]. Hence, a spectral sequence isomorphism implies a homology isomorphism $s^{-1}PC \to \mathscr{L}(C)$.

In the spectral sequence for $s^{-1}PC$ we have $E^1 = s^{-1}PC$, $E^2 = H(s^{-1}PC)$.

In the spectral sequence for $\mathscr{L}(C)$, we have $E^0 = \mathscr{L}(C^{\sharp}) = P\Omega(C^{\sharp})$. $E^1 = HP\Omega(C^{\sharp}) = PH\Omega(C^{\sharp}) = PS'[s^{-1}V] = s^{-1}PC$.

$E^2 = H(s^{-1}PC)$. Hence, $s^{-1}HPC \to H\mathscr{L}(C)$ is an isomorphism of graded vector spaces. That it is an isomorphism of Lie algebra follows from

$$ds^{-1}(x \otimes y + (-1)^{\deg x \deg y} y \otimes x)$$
$$= -s^{-1}d(x \otimes y + (-1)^{\deg x \deg y} y \otimes x)$$
$$-(-1)^{\deg x}[s^{-1}x, s^{-1}y]$$

if x and y are primitive in C.

(b) Consider the map $\mathscr{C}(L) \to sQL$. As before, filter $\mathscr{C}(L)$ by the second coordinate of its bidegree and give sQL a trivial one stage filtration. Both filtrations are bicomplete and it suffices to compute the spectral sequence for $\mathscr{C}(L)$.

$E^0 = \mathscr{C}(L^{\sharp})$. Since $L^{\sharp} = F[V]$ is free, $E^1 = H\mathscr{C}(L^{\sharp}) = \mathrm{Tor}_{U(L^{\sharp})}(k, k) = k \oplus sV = k \oplus sQL$.

Hence, $\bar{H}\mathscr{C}(L) \to sHQL$ is a isomorphism of vector spaces. That it is an isomorphism of coalgebras follows from the observation that: If $\ell \in L$ projects to a cycle in $Q(L)$ and if $s\ell + \Sigma(s\ell')(s\ell'') + \cdots$ is a cycle in $\mathscr{C}(L)$, then $-d\ell + \Sigma[\ell', \ell''] \equiv 0$ modulo boundaries and elements of length $\geqq 3$ in L.

The next proposition is an immediate consequence of the definitions of $\mathscr{C}$ and $\mathscr{L}$.

PROPOSITION 4.3. (a) *If L is a Lie algebra, then $s^{-1}HP\mathscr{C}(L) = HL$ as Lie algebras.*

(b) *If C is a coalgebra, then $sHQ\mathscr{L}(C) = \bar{H}C$ as coalgebras.*

We close this section with:

PROPOSITION 4.4. (a) *If $L \to L'$ is a homology isomorphism of Lie algebras, then $\mathscr{C}(L) \to \mathscr{C}(L')$ is a homology isomorphism.*

(b) *If $C \to C'$ is a homology isomorphism between simply connected coalgebras, then $\mathscr{L}(C) \to \mathscr{L}(C')$ is a homology isomorphism.*

Proof. Both (a) and (b) are consequences of respective Eilenberg-Moore spectral sequences. (Filter by the first coordinate of the bidegree.)

In (a), $E^2 = H\mathscr{C}(HL)$ and the filtration is bicomplete, so we have no problems [7].

In (b), $E^2 = H\mathscr{L}(HC)$ and the filtration is bicomplete if C is simply connected.

The simply connected hypothesis is necessary in 4.4b. For example, let $C = \mathscr{C}(\mathrm{so}\,(3))$ where $\mathrm{so}\,(3)$ is the real Lie algebra of the Lie group $SO\,(3)$. (We regard $\mathrm{so}\,(3)$ as being concentrated in dimension zero.) A classical computation of E. Cartan (see H. Weyl's book [29]) states that $HC = H\mathscr{C}(\mathrm{so}\,(3)) = H(SO\,(3); R) = S'[x]$ with degree $x = 3$. Hence, there is a homology isomorphism $C \to C'$ where $C' = S'[x]$. Then $H\mathscr{L}(C) = \mathrm{so}\,(3)$ by 4.1, but $H\mathscr{L}(C') = F[s^{-1}x]$ by 4.2. Therefore, $\mathscr{L}(C) \to \mathscr{L}(C')$ is not a homology isomorphism.

In 6.4, we shall draw the conclusion of 4.4b with an alternate hypothesis.

5. Cofibrant and fibrant models. We begin this section by observing that *DGLA*, *DGA*, and *DGC* are closed model categories in the sense of Quillen [22] [23]. Recall Quillen's definition.

DEFINITION 5.1. A closed model category is a category $\mathscr{C}$ endowed with three selected classes of morphisms, called fibrations, cofibrations, and weak equivalences and subject to the following axioms:

CM 1. $\mathscr{C}$ is closed under finite direct and inverse limits.

CM 2. If gf is defined and any two of f, g, gf are weak equivalences, then so is the third.

CM 3. A retract of a fibration, cofibration, or weak equivalence in one also.

CM 4. We can complete any commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle p} \\
B & \longrightarrow & Y
\end{array}
$$

whenever i is a cofibration, p is a fibration, and either p or i is a weak equivalence.

CM 5. Any map f may be factored in two ways: $f = pi$ where p is a fibration, i is a cofibration, and either p or i is a weak equivalence.

Following Quillen [23], *DLGA* is a closed model category with weak equivalences being maps which induce isomorphisms in homology, fibrations being maps which are surjective, and cofibrations being maps which satisfy axiom *CM* 4 for all fibrations which are weak equivalences. Although Quillen treats only those Lie algebras with $L_0 = 0$ and uses a slightly different notion of fibration, his proof that they form a closed model category remains valid in this more general context.

Bousfield and Gugenheim [2] have shown that *DGA* is a closed model category with weak equivalences being maps which induce isomorphisms in cohomology, fibrations being maps which are surjective, and cofibrations being maps which satisfy axiom *CM* 4 for all fibrations which are weak equivalences.

Again following Quillen [23], *DGC* is a closed model category with weak equivalences being maps which induce isomorphisms in homology, cofibrations being maps which are injective, and fibrations being maps which satisfy axiom *CM* 4 for all cofibrations which are weak equivalences. Unfortunately, Quillen's proof that this is a closed model category [23, p. 256] relies heavily upon the application of the comparison theorem to spectral sequences which happen to have a sufficiently nice E^2 term only when the coalgebras involved are simply connected, $C_1 = 0$. Fortunately, a proof can be given which avoids these technical difficulties.

PROPOSITION 5.2. *DGC (DGA, DGLA) is a closed model category.*

Proof. Only axioms *CM* 4 and *CM* 5 are nontrivial and half of *CM* 4 is true by definition.

In the course of this proof, we need two kinds of standard fibrations.

LEMMA 5.3. (a) *If V is a DG vector space, then $S'[V] \to k$ is a fibration.*

(b) *If V_n is concentrated in dimension $n \geq 1$, then $E'[V_n] \to S'[V_n]$, which is defined in §3, is a fibration.*

The proof of 5.3 is an exercise in the universal mapping property of symmetric coalgebras. Since pullbacks of fibrations are fibrations,

5.3b implies that elementary extensions (3.7) are fibrations.

Now let $f: C \to D$ be a map in DGC. We will verify CM 5 for this map:

(a) $f = pi$ where p is a weak equivalence.

(b) $f = pi$ where i is a weak equivalence.

(a) Pick an acyclic DG vector space V, $HV = 0$, and an injective DG map $C \to V$. By the universal property of S', there is a unique factorization $C \to S'[V] \to V$ and $S'[V]$ is a acyclic by 1.10. Since $\otimes$ is the product in DGC, there is an injective map $i: C \to D \otimes S'[V]$ and the projection $p: D \otimes S'[V] \to D$ is a weak equivalence. p is a fibration since it is the pullback of the fibration $S'[V] \to k$ under $D \to k$.

(b) By part (a), we can assume that $C \to D$ is an injection. Pick a DG map $C \to HC$ which is a homology isomorphism. This need not be a coalgebra map. This defines $C \to S'[HC]$ and, by replacing $C \to D$ by $C \to D \otimes S'[HC]$, we may assume that $C \to D$ is both injective and injective on homology.

Make the inductive assumption that we have constructed a factorization $C \xrightarrow{i_n} E_n \xrightarrow{p_n} D$ where p_n is a fibration, i_n is a cofibration, and $H(i_n)$ is an isomorphism in dimensions $\leq n$.

Pick a projection $E_n \to \operatorname{coker} H_{n+1}(i_n)$. This defines a map $E_n \to S'[\operatorname{coker} H_{n+1}(i_n)]$ and it is not hard to see that there is a commutative diagram
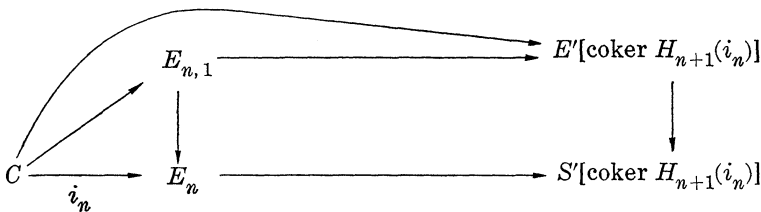
where $E_{n,1} \to E_n$ is the pullback fibration.

Repeating this process, construct an inverse system of fibrations $\cdots \to E_{n,2} \to E_{n,1} \to E_n$.

Set $E_{n+1} = \varprojlim_m E_{n,m}$ and let $i_{n+1}: C \to E_{n+1}$ be the natural map. Define p_{n+1} to be the composition $E_{n+1} \to E_n \to D$. Then i_{n+1} is a cofibration and p_{n+1} is a fibration since it is a composition of fibrations. Finally, $H(i_{n+1})$ is an isomorphism in dimensions $\leq n + 1$. This verifies the inductive assumption.

Set $E = \varprojlim E_n$. The factorization $C \to E \to D$ verifies the second half of CM 5.

To complete the verification that DGC is a closed model cate-

gory, it remains to demonstrate CM 4 when $p: X \to Y$ is a fibration
and a weak equivalence. In verifying the first half of CM 5 we
showed that p could be factored as $X \underset{i}{\to} Y \otimes S'[V] \underset{q}{\to} Y$ where i is
a cofibration and p is a fibration and weak equivalence. By CM 2,
i is a weak equivalence. By the definition of a fibration, we can
complete the commutative diagram below.

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ Y \otimes S'[V] & \underset{q}{\longrightarrow} & Y \end{array}$$

Hence, p is a retract of the product fibration q. It is easy to
verify CM 4 for q and hence for p.

This completes the proof that DGC is a closed model category.

Given a pointed closed model category $\mathscr{C}$ with point object $*$,
an object X is called cofibrant (respectively, fibrant) if the map
$* \to X$ is a cofibration (resp., if $X \to *$ is a fibration). Notice that
$DGLA$, DGC, and DGA are pointed with point objects $0, k,$ and k,
respectively. In $DGLA$ and DGA, every object is fibrant and in
DGC every object in cofibrant.

DEFINITION 5.4. Given an object X in a pointed closed model
category, a cofibrant model (respectively, fibrant model) for X is a
cofibrant (fibrant) object Y and a weak equivalence $Y \to X$ ($X \to Y$).

Cofibrant and fibrant models always exist because of axiom CM
5. We shall adopt the notation $cX \to X$ for cofibrant models and
$X \to fX$ for fibrant models. Notice that CM 5 implies that we may
take $cX \to X$ to be a fibration and $X \to fX$ to be a cofibration.
Using CM 4, it is easy to see that c (and f) give well defined func-
tors from $\mathscr{C}$ to the homotopy category ho $\mathscr{C}_c$ of cofibrant (ho $\mathscr{C}_f$
of fibrant) objects of $\mathscr{C}$ [22].

(Given a category $\mathscr{C}$ and a multiplicative class of morphisms S
in $\mathscr{C}$, in our case weak equivalences, the homotopy category ho $\mathscr{C} =$
$S^{-1}\mathscr{C}$ is the localization of $\mathscr{C}$ with respect to S [9]).

DEFINITION 5.5. An algebra A (respectively, Lie algebra L) is
called minimal if it is cofibrant and the induced differential on $Q(A)$
(respectively, $Q(L)$) is zero. A coalgebra C is called minimal if it is
fibrant and the differential is zero on $P(C)$.

A minimal model for an algebra A, denoted by $mA \to A$, or for
a Lie algebra L, denoted by $mL \to L$, is a minimal cofibrant model.
A minimal model for a coalgebra C, denoted by $C \to mC$, is a minimal

fibrant model.

The rest of this section will be devoted to the following five propositions, 5.6–5.10 Taken together, they give a rather complete characterization of cofibrant, fibrant, and minimal models.

PROPOSITION 5.6.   (a) *Every algebra A has a minimal model $mA \rightarrow A$ ond mA is unique up to isomorphism.*

(b) *Every coalgebra C has a minimal model $C \rightarrow mC$ and mC is unique up to isomorphism.*

(c) *Every Lie algebra L with $L_0 = 0$ has a minimal model $mL \rightarrow L$ and mL is unique up to isomorphism.*

Although it is not true that only those Lie algebras L with $L_0 = 0$ have minimal models, it is easy to see that some restriction is necessary. For example, let $L = \mathrm{so}\,(3) =$ the real Lie algebra of the Lie group $\mathrm{SO}\,(3)$. Suppose $mL \rightarrow L$ were a minimal model. Then $Q(mL)_0 \neq 0$ since $H_0 L = \mathrm{so}\,(3) \neq 0$. But 4.2b and 4.4a imply that $Q(mL)_0 = H_1\mathscr{C}(mL) = H_1\mathscr{C}(L) = 0$ since $H\mathscr{C}(L) = H(\mathrm{SO}\,(3); R) = H(S^3; R)$ [29]. Hence, no minimal model for $\mathrm{so}\,(3)$ exists.

Conjecture.   A Lie algebra L with HL nilpotent has a minimal model.

PROPOSITION 5.7.   (a) *An algebra A is cofibrant if and only if $A \approx mA \otimes S[V]$ where $HV = 0$.*

(b) *A coalgebra C is fibrant if and only if $C \approx mC \otimes S'[V]$ where $HV = 0$.*

(c) *A Lie algebra is cofibrant if and only if it is a retract of a Lie algebra L with $L^\ast = $ a free Lie algebra.*

PROPOSITION 5.8.   (a) *An algebra A is minimal if and only if A is constructible by decomposable elementary extensions starting from the ground field k.*

(b) *A coalgebra C is minimal if and only if C is constructible by nonprimitive elementary extensions starting from the ground field k.*

(See 3.9 for the definition of constructible by elementary extensions. For algebras, replace inverse limits by direct limits in the definition for coalgebras.)

Note that if we combine 5.8b with 3.11, we see that a coalgebra C is minimal if and only if C is a reduced symmetric coalgebra with the Lie algebra $s^{-1}HPC = s^{-1}PC$ a nilpotent completion.

PROPOSITION 5.9.   *A coalgebra C is fibrant if and only if $C^\# =$ a symmetric coalgebra and the Lie algebra $s^{-1}HPC$ is a nilpotent completion.*

PROPOSITION 5.10.   *If A is cofibrant (respectively, minimal) algebra of finite type, then A^* is a fibrant (reespctively, minimal) coalgebra.*

REMARK.   By 5.7, and 5.8, cofibrant algebras are symmetric algebras, fibrant coalgebras are symmetric coalgebras, and cofibrant Lie algebras are free Lie algebras.

The results 5.6, 5.7 and 5.8 are known for algebras. See [4] and [2]. Furthermore, 5.7 and 5.8 imply 5.10 since the dual of a decomposable elementary extension is a nonprimitive elementary extension and the dual of a symmetric algebra is a symmetric coalgebra. It remains to show 5.6, and 5.7, 5.8, and 5.9 for coalgebras and Lie algebras.

We treat coalgebras first. A slight modification of the proof of Cm 5b in 5.2 shows that we can factor $C \to k$ into $C \to D \to k$ where $C \to D$ is a weak equivalence (but not a cofibration) and $D \to k$ is a fibration. Given $C \to D_n \to k$ where $D_n \to k$ is a fibration and $HC \to HD_n$ is an isomorphism in dimensions $\leqq n$ and an epimorphism in dimension $n + 1$, construct a nonprimitive elementary extension $D'_n \to D_n$ of dimension $n + 1$ and a factorization $C \to D'_n \to D_n$ such that $HC \to HD'_n$ is an isomorphism in dimensions $\leqq n + 1$. Specifically, $D'_n = D_n \otimes S'[V]$ where $V =$ kernel of $H_{n+1}(C \to D_n)$. Following the proof of 5.2, construct an inverse limit of nonprimitive elementary extensions $D_{n+1} \to D'_n$ of dimension $n + 2$ and a factorization $C \to D_{n+1} \to D'_n$ such that $HC \to HD_{n+1}$ is an isomorphism in dimensions $\leqq n + 1$ and an epimorphism in dimension $n + 2$. Then $D = \lim D_n$ yields a factorization $C \to D \to k$ with $D \to k$ a fibration and $\overleftarrow{C \to D}$ a weak equivalence. D is therefore fibrant and, by the remark after 3.11, D is a reduced symmetric coalgebra, therefore minimal.

Hence, minimal models for coalgebras exist. Furthermore, the above minimal model $mC = D$ is constructible by nonprimitive elementary extensions starting from the ground field.

REMARK.   The above procedure is dual to Sullivan's construction of minimal models for algebras.

LEMMA 5.11.   (a)   *If $X \to Y$ is a cofibration and a weak equivalence and X is fibrant, then there is a splitting $Y \to X$ such that $X \to Y \to X$ is the identity.*

(b)   *If $X \to Y$ is a fibration and a weak equivalence and Y is*

cofibrant, then there is a splitting $Y \to X$ *such that* $X \to Y \to X$ *is the identity.*

This lemma is a consequence of axiom CM 4.

To show the uniqueness of minimal coalgebra models, let $C \to mC$ be any minimal model and let $C \to m_0C$ be the specific minimal model constructed above. We can find a fibrant model $C \to mC \otimes S'[V]$ such that this map is a cofibration and $HV = 0$. CM 4 implies that there is a map $mC \otimes S'[V] \to m_0C$ which is a weak equivalence. Hence, the composition $mC \to mC \otimes S'[V] \to m_0C$ is a weak equivalence between minimal coalgebras. We claim it is an isomorphism.

Factor $mC \to m_0C$ into $mC \to m_0C \otimes S'[W] \to m_0C$ where $mC \to m_0C \otimes S'[W]$ is a cofibration and $HW = 0$. Apply 5.11a to get a splitting $m_0C \otimes S'[W] \to mC$. mC is a retract of a symmetric coalgebra, therefore, it is a symmetric coalgebra. Hence, $P(mC) = HP(mC) \to HP(m_0C \otimes S'[V]) = P(m_0C)$ is a split monomorphism with splitting $P(m_0C) \to P(mC)$ induced by $m_0C \otimes S'[W] \to mC$.

Replacing $mC \to m_0C$ by $m_0C \otimes S'[W] \to mC$ in the above paragraph, we conclude that $P(m_0C) \to P(mC)$ is also a split monomorphism. Hence, $P(mC) \to P(m_0C)$ is an isomorphism. Since mC and m_0C are symmetric coalgebras, $mC \to m_0C$ is an isomorphism. This concludes the proof of 5.6 for coalgebras.

The minimal model m_0C is constructible by nonprimitive elementary extensions starting from the ground field. Any coalgebra so constructible is minimal. Together with 5.6, this proves 5.8 for coalgebras.

Since the minimal coalgebra m_0C satisfies 5.9, uniqueness proves 5.9 for all minimal coalgebras. An argument similar to the proof of 3.12 proves 5.7 for coalgebras. Proposition 5.9 for fibrant coalgebras follows from 5.7 and 5.9 for minimal coalgebras. This completes the proofs for coalgebras.

It remains to prove 5.6 and 5.7 for Lie algebras.

It is easy to see that any Lie algebra L with $L^\sharp = $ a free Lie algebra is cofibrant. Hence, so is any retract. Conversely, Quillen has shown [23, p. 256] that any cofibrant Lie algebra L is a retract of a Lie algebra L' with $(L')^\sharp = $ a free Lie algebra. His proof remains valid if $L_0 \neq 0$. This proves 5.7 for Lie algebras.

Let L be a Lie algebra with $L_0 = 0$. Pick a chain splitting $H_1L \to L$. This defines a map $F_1 = F[H_1L] \to L$ which is an epimorphism in one dimensional homology. F_1 is minimal.

Inductively, suppose we are given $F_n \to L$ which is a isomorphism (epimorphism) in homology in dimensions $<n(<n+1)$ and such that F_n is minimal and free on generators concentrated in dimensions $<n+1$. Furthermore, suppose $F_n = F[Z]$ with a splitting $Z_n =$

$W_n \oplus V_n$ where $d(W_n) = 0$, and $d: V_n \to HF_{n-1}$ and $W_n \to HF_n \to HL$ are monomorphisms. Construct $F_{n+1} \to L$ as follows. Pick a lift $V \to F_n$ where $V = \ker H_n(F_n \to L)$. Consider the composition $d: sV \to V \to F_n$ and use this to extend the differential on F_n to $G_{n+1} = F_n \vee F[sV]$. Then G_{n+1} is minimal and maps isomorphically in homology in dimension $< n + 1$.

Let $W = \operatorname{coker} H_{n+1}(G_{n+1} \to L)$ and pick a splitting $W \to L$. Define d to be zero on W and use this to extend the differential on G_{n+1} to $F_{n+1} = G_{n+1} \vee F[W]$. Then F_{n+1} is minimal and $W \to L$ defines and extension to a map $F_{n+1} \to L$ which is an isomorphism (epimorphism) in homology in dimensions $< n + 1$ ($< n + 2$). The required splitting of F_{n+1} is obvious from our construction.

Setting $F = \varinjlim F_n$, the map $F \to L$ is a weak equivalence and F is a minimal model for L.

Finally, minimal Lie algebra models are unique up to isomorphism by an argument similar to that for the uniqueness of minimal coalgebra models. This completes the proof of 5.6 for Lie algebras.

6. Homotopy groups. We now define "homotopy" groups for Lie algebras, coalgebras, and algebras.

DEFINITION 6.1. (a) If L is a Lie algebra, then $\pi L = HQ(cL)$.
(b) If C is a coalgebra, then $\pi C = HP(fc)$.
(c) If A is an algebra, then $\pi A = HQ(cA)$.
That πC and πA are well defined up to isomorphism follows from 5.6 and 5.7. That πL is well defined up to isomorphism will follow from 6.3. If we choose minimal models, then $\pi L = Q(mL)$, $\pi C = P(mC)$, and $\pi A = Q(mA)$.

By § 2, $s\pi L$ has a coalgebra structure, $s^{-1}\pi C$ has a Lie algebra structure, and $s^{-1}\pi A$ has a Lie coalgebra structure. As a consequence of 5.9, we have:

PROPOSITION 6.2. *For any coalgebra C, the Lie algebra $s^{-1}\pi C$ is a nilpotent completion.*

From 4.2 and 4.4 we get:

PROPOSITION 6.3. (a) *If L is a Lie algebra, then $s\pi L = H\mathscr{C}(L) = H\mathscr{C}(cL)$ as coalgebras.*
(b) *If C is a coalgebra, then $s^{-1}\pi C = H\mathscr{L}(fC)$ as Lie algebras. If $C_1 = 0$, then $s^{-1}\pi C = H\mathscr{L}(C)$.*

REMARK. This result shows that πL is a functor and πC is a functor if $C_1 = 0$. Bousfield and Gugenheim [2] have shown that πA

is a functor by constructing a natural cofibrant model $cA \to A$. It is possible to construct natural cofibrant (fibrant) models $cL \to L(C \to fC)$. Hence, πL, πC, πA are functors with no connectivity hypothesis.

Since πC is well defined, 6.3 implies:

PROPOSITION 6.4.   *If* $f: C \to C'$ *is a weak equivalence between fibrant coalgebras, then* $\mathscr{L}(f): \mathscr{L}(C) \to \mathscr{L}(C')$ *is a weak equivalence.*

Together with 4.4b, 6.3 implies:

PROPOSITION 6.5.   *If* C *is a coalgebra with* $C_1 = 0$, *then* $s^{-1}\pi C = H\mathscr{L}(C)$.

7.   Equivalence of homotopy categories.   Recall Quillen's equivalence theorem [23, p. 235] and [22, 4].

THEOREM 7.1.   *Let* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *be closed model categories and let*

$$\mathscr{C}_1 \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{C}_2$$

be a pair of adjoint functors such that
   (1)   *F preserves cofibrations and G preserves fibrations.*
   (2)   *If* $f: A \to B$ *is a weak equivalence between cofibrant objects in* $\mathscr{C}_1$, *then* $F(f)$ *is a weak equivalence in* $\mathscr{C}_2$. *If* $g: X \to Y$ *is a weak equivalence between fibrant objects in* $\mathscr{C}_2$, *then* $G(g)$ *is a weak equivalence in* $\mathscr{C}_1$.
   (3)   *If* A *is a cofibrant object in* $\mathscr{C}_1$ *and* X *is a fibrant object in* $\mathscr{C}_2$, *then a map* $f: A \to GX$ *is a weak equivalence if and only if the corresponding map* $f^\flat: FA \to X$ *is a weak equivalence.*

Then F and G are induce equivalences of homotopy categories. Moreover, if $\mathscr{C}_1$ and $\mathscr{C}_2$ are pointed, then this equivalence preserves the loop and suspension functors and the families of fibration and cofibration sequences.

A careful reading of Quillen's proof shows that F and G induce equivalences of homotopy categories even if hypothesis 1 is weakened to: F preserves cofibrant objects and G preserves fibrant objects. (See [22], 4.7, (3).)

Let $ncDGLA$ be the full subcategory of $DGLA$ consisting of Lie algebras L such that HL is a nilpotent completion.

Unfortunately, $ncDGLA$ is not a closed model category since it is not closed under finite coproducts. Hence, the homotopy category, $ho(ncDGLA)$, does not make sense by itself. What we mean by $ho(DGLA)$ is the full subcategory of $ho(DGLA)$ generated by objects

of $ncDGLA$. As in the paper [2] of Bousfield and Gugenheim, we shall use this convention whenever we are dealing with a full subcategory of a closed model category.

PROPOSITION 7.2. *The adjoint functors*

$$DGC \underset{\mathscr{C}}{\overset{\mathscr{L}}{\rightleftarrows}} DGLA$$

induce equivalence of homotopy categories $ho(DGC) \approx ho(ncDGLA)$.

Proof. Since $\mathscr{L}(C)^{\sharp} = $ a free Lie algebra, $\mathscr{L}(C)$ is always cofibrant by 5.7. Hence, $\mathscr{L}$ preserves cofibrant objects. If L is an object in $ncDGLA$, *then* $\mathscr{C}(L)^{\sharp} = $ a symmetric coalgebra with $s^{-1}HP\mathscr{C}(L) = HL$, a nilpotent completion. Therefore, $\mathscr{C}(L)$ is fibrant by 5.9 and $\mathscr{C}$ preserves fibrant objects when restricted to objects in $ncDGLA$.

Hypothesis 2 in 7.1 is a consequence of 4.4 and 6.4. Hypothesis 3 follows from hypotheses 1 and 2 and 4.1.

Hence, 7.1 implies 7.2. Specifically, the equivalence of homotopy categories is given on objects by $C \rightarrow \mathscr{L}(fC)$ and $L \rightarrow \mathscr{C}(L)$. Notice that $H\mathscr{L}(fC) = s^{-1}HP(fC)$ by 4.2 and is a nilpotent completion by 5.9.

In the remainder of this section, the ground field will be the rationals. We are going to relate the algebra developed in this paper to rational homotopy theory.

A connected pointed Kan complex, hereafter referred to as a space, is called nilpotent if π_1 is a nilpotent group and π_n for all $n \geq 2$ is a nilpotent π_1 module [2]. A nilpotent space X is called rational if the groups $H_n(X; Z)$, or equivalently the groups $\pi_n(X)$, are uniquely divisible. A nilpotent space X is of finite $\boldsymbol{Q}$-type if the vector spaces $H_n(X; Q)$, and hence $\pi_n(X) \otimes \boldsymbol{Q}$ for $n \geq 2$, are finite dimensional [2].

The homotopy category of rational nilpotent spaces is a retract of the homotopy category of nilpotent spaces. The retraction functor is denoted $X \rightarrow X \otimes \boldsymbol{Q}$ and if $X \otimes \boldsymbol{Q} \cong Y \otimes \boldsymbol{Q}$ we say that X and Y have the same rational homotopy type. (For details, see [8], [4], [3], or [23].)

Sullivan [4] [8] and, in a more functorial manner, Bousfield and Gugenheim [2] have shown that the homotopy category of rational finite $\boldsymbol{Q}$-type nilpotent spaces is equivalent to the homotopy category of rational algebras A with πA of finite type and $H^0(A) = \boldsymbol{Q}$.

We adopt the following notation:

(a) S is the category of connected spaces.

(b) nS is the category of connected finite Q-type nilpotent spaces.

(c) $S \otimes Q$ is the category of rational spaces in nS.

(d) $nDGA$ is the full subcategory of DGA of algebras A with πA of finite type.

(e) $nDGC$ is the full subcategory of DGC of coalgebras C with πC of finite type.

(f) $nDGLA$ is the full subcategory of $DGLA$ of Lie algebras L with HL nilpotent of finite type.

(g) $\mathscr{E}: S \rightarrow DGA$ is Sullivan's PL deRham functor (see [4] or [2]).

(h) $R: DGA \rightarrow S$ is Bousfield and Gugenheim's realization functor [2].

PROPOSITION 7.3.   *The following four homotopy categories are equivalent:* $ho(S \otimes Q)$, $ho(nDGA)$, $ho(nDGC)$, *and* $ho(nDGLA)$.

Proof.   The equivalence of $ho(S \otimes Q)$ and $ho(nDGA)$ is due to Bousfield and Gugenheim and is given by the contravariant functors $X \rightarrow \mathscr{E}(X)$ and $A \rightarrow R(mA)$ [2].

The equivalence of $ho(n(DGA)$ and $ho(nDGC)$ is given by the contravariant functors $A \rightarrow (mA)^*$ and $C \rightarrow (mC)^*$.

The equivalence of $ho(nDGC)$ and $ho(nDGLA)$ is given by the restriction of the covariant functors in 7.2, i.e., $C \rightarrow \mathscr{L}(fC)$, $L \rightarrow \mathscr{E}(L)$. Note that, if $s^{-1}\pi C$ is of finite type, it is nilpotent since it is a nilpotent completion 6.2. Since $s^{-1}\pi C = H\mathscr{L}(fC)$ by 6.3, $\mathscr{L}(fC)$ is in $nDGLA$. Conversely, given L in $nDGLA$, $\mathscr{E}(L)$ is in $nDGC$ since $s^{-1}\pi\mathscr{E}(L) = HL$ by 4.3.

8. **Minimal models for spaces.**   Let X be a connected finite Q-type nilpotent space. Using the equivalence in 7.3, we may associate to X a minimal algebra M_x, a minimal coalgebra C_x, and a cofibrant Lie algebra $\mathscr{L}(C_x)$. If X is simply connected, we may associate to X a minimal Lie algebra L_x. These three minimal objects are determined by X up to isomorphism. They are called the minimal algebra model, the minimal coalgebra model, and the minimal Lie algebra model, respectively.

Specifically, $M_x = m\mathscr{E}(X) =$ the minimal algebra model for the PL deRham forms. (This is contained in [4].) $C_x = (M_x)^* =$ the dual coalgebra to M_x and $L_x = m\mathscr{L}(C_x) =$ the minimal Lie algebra model for $\mathscr{L}(C_x)$. We do not know whether L_x always exists if X is not simply connected. The following proposition is immediate.

PROPOSITION 8.1.   *If X and Y are connected finite Q-type nilpotent spaces, then the following are equivalent:*

(a) $X \otimes Q \cong Y \otimes Q$.

(b) M_X and M_Y are isomorphic algebras.

(c) C_X and C_Y are isomorphic coalgebras.

(d) There is a weak equivalence $\mathscr{L}(C_X) \to \mathscr{L}(C_Y)$.

(e) If X and Y happen to be simply connected, L_X and L_Y are isomorphic Lie algebras.

Given a nilpotent group G, there is a universal morphism $G \to G \otimes Q$ of G into a nilpotent uniquely divisible group $G \otimes Q$. This is called the Malcev completion of G and agrees with the tensor product when G is abelian [13] [23].

There is an equivalence between the category of nilpotent uniquely divisible groups G and the category of nilpotent rational Lie algebras L (nongraded) given by $G \to L = P\hat{Q}[G]$ = the primitives of the completion of the rational group algebra with respect to its augmentation ideal [12] [23]. Given G denote the corresponding Lie algebra by $\mathscr{L}(G)$.

PROPOSITION 8.2. If X is a connected finite Q-type nilpotent space, then the following Lie algebras are isomorphic: $\mathscr{L}(\pi_1(X) \otimes Q) = (\pi^1 M_X)^* = \pi_1 C_X = H_0 \mathscr{L}(C_X)$.

Proof. The first isomorphism is in [4]. The rest follows from 4.2.

For $n, m \geqq 2$ the Whitehead product $\pi_n(X) \otimes \pi_m(X) \to \pi_{n+m-1}(X)$ corresponds to the Samelson product $\pi_{n-1}(\Omega X) \otimes \pi_{m-1}(\Omega X) \to \pi_{n+m-2}(\Omega X)$ under the isomorphism $\pi_n(X) = \pi_{n-1}(\Omega X)$. For $n \geqq 2$ the Samelson product makes the homotopy groups $\pi_{n-1}(\Omega X)$ into a graded Lie algebra. The next proposition is proved in the same way as 8.2 with a similar assist from [4].

PROPOSITION 8.3. If X is a connected finite Q-type nilpotent space, then for $n \geqq 2$ the following are isomorphisms of graded Whitehead algebras: $\pi_n(X) \otimes Q = (\pi^n M_X)^* = \pi_n C_X = H_{n-1} \mathscr{L}(C_X)$ $(= H_{n-1} L_X$ if X is simply connected).

From 4.2, we also get:

PROPOSITION 8.4. If X is a connected finite Q-type nilpotent space, the following are isomorphisms of graded coalgebras: $H_n(X; Q) = H^n(M_X)^* = H_n(C_X) = \pi_{n-1}(\mathscr{L}(C_X))$.

Propositions 8.2, 8.3, and 8.4 give the following description of

minimal models.

The minimal Lie algebra model L_X is the free Lie algebra generated by $s^{-1}\bar{H}(X; Q)$ with differential d given on generators $s^{-1}x$ by $ds^{-1}x = \Sigma[s^{-1}a_i, s^{-1}b] + P$, where P is a sum of brackets of length three or greater and $\Delta(x) = \Sigma a_i \otimes b_i + (-1)^{\deg a_i \deg b_i} b_i \otimes a_i$.

The minimal coalgebra model C_X is the symmetric coalgebra with primitives equal to $\angle(\pi_1(X) \otimes Q) + \sum_{n \geq 2} \pi_n(X) \otimes Q = sHL_X$.

The minimal algebra model M_X is the symmetric algebra generated by the dual of PC_X [4].

As exercises for the reader we leave the following lemmas.

LEMMA 8.5. $M_{X \times Y} = M_X \otimes M_Y$, $C_{X \times Y} = C_X \otimes C_Y$.

LEMMA 8.6. $L_{X \vee Y} = L_X \vee L_Y$ if X and Y are simply connected.

Recall Sullivan's definition of formality [4].

DEFINITION 8.7.   Let X be a finite Q-type nilpotent space. X is called formal if there is a weak equivalence $M_X \to H^*(X; Q)$.

The next proposition enables one to construct the minimal Lie algebra model for many examples.

PROPOSITION 8.8.   *If X is formal and simply connected, then L_X is isomorphic to $\mathscr{L}H(X; Q)$.*

Proof.   Dualize $M_X \to H^*(X; Q)$ to get a weak equivalence $H(X; Q) \to C_X$.   By 4.4b, $\mathscr{L}H(X; Q) \to \mathscr{L}(C_X)$ is a weak equivalence since C_X is simply connected.   Since $H(X; Q)$ has zero differential, $H(X; Q)$ has a decomposable differential and is a minimal Lie algebra. Since minimal models are unique up to isomorphism, L_X is isomorphic to $\mathscr{L}H(X; Q)$.

For example, let $X = \bigvee S^{n_\alpha}$, $n_\alpha \geq 2$, be a wedge of spheres. Then X is formal. (See, for example, [15].)   Hence, L_X is isomorphic to $\mathscr{L}H(X; Q)$, the free Lie algebra on generators x_α with degree $x_\alpha = n_\alpha - 1$.   Since all nontrivial cup products vanish, $dx_\alpha = 0$ for all α.

CP^n is formal.   In fact, any compact Kahler manifold is formal [4].   Hence, L_{CP^n} is isomorphic to $\mathscr{L}H(CP^n; Q)$, the free Lie algebra generated by $\gamma_1, \cdots, \gamma_n$ with degree $\gamma_k = 2k - 1$ and $d\gamma_k = -1/2 \sum_{i+j=k} [\gamma_i, \gamma_j]$.

To study nonformal examples, we need to study cofibration sequences.   Let $A \to X \to X/A$ be a cofibration sequence.   Recall that

$i^*\colon \mathscr{E}(X) \to \mathscr{E}(A)$ is an epimorphism [4] and let $\mathscr{E}(X, A) = \text{kernel } i^*$. Since the composition $\mathscr{E}(X, A) \to \mathscr{E}(X) \to \mathscr{E}(A)$ is trivial, there is a factorization $\mathscr{E}(X/A) \to \mathscr{E}(X, A) \to \mathscr{E}(X)$. From the long exact homology sequence and the five lemmas, it follows that $\mathscr{E}(X/A) \to \mathscr{E}(X, A)$ is a weak equivalence. Hence, we might as well use the sequence $\mathscr{E}(X, A) \to \mathscr{E}(X) \to \mathscr{E}(A)$ as the representative of $A \to X \to X/A$ in the category of algebras.

LEMMA 8.9. *If $A \to X \to X/A$ is a cofibration sequence of finite Q-type nilpotent spaces with A simply connected, then the sequence $\mathscr{E}(X, A) \to \mathscr{E}(X) \to \mathscr{E}(A)$ is weakly equivalent to a sequence $A_1 \to A_2 \to A_3$ of connected finite type algebras with $A_2 \to A_3$ surjective and $A_1 = \text{kernel}\,(A_2 \to A_3)$.*

Proof. By $CM\ 5$, we can choose a surjection $D \to \mathscr{E}(A)$ where D is cofibrant and acyclic. Hence, we get a map $M_A \otimes D \to \mathscr{E}(A)$ which is a fibration and a weak equivalence. By $CM\ 4$, there is a commutative diagram

$$
\begin{array}{ccc}
M_X & \longrightarrow & M_A \otimes D \\
\downarrow & & \downarrow \\
\mathscr{E}(X) & \longrightarrow & \mathscr{E}(A) \,.
\end{array}
$$

Pick a cofibrant acyclic algebra E and a surjection $E \to M_A \otimes D$. By $CM\ 4$, there is a commutative diagram

$$
\begin{array}{ccc}
E & \longrightarrow & M_A \otimes D \\
\downarrow & & \downarrow \\
\mathscr{E}(X) & \longrightarrow & \mathscr{E}(A)
\end{array}
\qquad \text{and hence} \qquad
\begin{array}{ccc}
M_X \otimes E & \longrightarrow & M_A \otimes D \\
\downarrow & & \downarrow \\
\mathscr{E}(X) & \longrightarrow & \mathscr{E}(A) \,.
\end{array}
$$

In the last commutative diagram, the vertical maps are weak equivalences and the horizontal maps are surjections. Now consider the composition $M_X \otimes E \to M_A \otimes D \to M_A$, where the second map is projection. If we use the same techniques that we applied to get E, we can find an acyclic F and a surjection $F \to M_A$ giving a commutative diagram

$$
\begin{array}{ccc}
M_X \otimes F & \longrightarrow & M_A \\
\downarrow & & \| \\
M_X \otimes E & \longrightarrow & M_A \,.
\end{array}
$$

But M_A is of finite type and $M_A^1 = 0$; hence, we can find an F which is of finite type and connected. Set $A_3 = M_A$, $A_2 = M_X \otimes F$, and $A_1 = \text{kernel}\,(A_2 \to A_3)$.

COROLLARY 8.10. *If $A \to X \to X/A$ is a cofibration sequence of finite Q-type simply connected spaces, then there is a sequence $L_1 \to L_2 \to L_3$ of cofibrant Lie algebras with L_1, L_2, L_3 weakly equivalent to $L_A, L_X, L_{X/A}$, respectively, and this sequence splits into a free product when we forget the differential, that is, $L_2^{\#} = L_1^{\#} \vee L_3^{\#}$.*

Proof. Apply the functor $\mathscr{L}$ to the sequence of coalgebras which is dual to the finite type sequence constructed in 8.9.

The next proposition is the Lie algebra version of the Hirsch lemma. It tells us how to attach a cell. Recall that $L_{S^n} = F[x]$, the free Lie algebra on a generator x of degree $n-1$.

PROPOSITION 8.11. *Let X be a finite Q-type simply connected space and let $L_X \to F$ be a weak equivalence with F cofibrant. Let $\alpha: S^n \to X$ and $Y = X \bigcup_\alpha e^{n+1}$ with $n \geq 1$. Then there is a weak equivalence $L_Y \to F'$ where $F' = F \vee F[y]$, the differential on F' extends that on F, and dy is a chain representative for the attaching map α, degree $y = n$.*

Proof. Apply 8.10 to the cofibration sequence $X \to Y \to S^{n+1}$ and let $L_1 \to L_2 \to L_3$ be the resulting sequence of Lie algebras.

Since F is cofibrant, it is not hard to use $CM\ 4$ to get a weak equivalence $F \to L_1$. Consider the map $F[x] = L_{S^n} \to L_X$ and form the compositions $F[x] \to L_X \to F \to L_1$. Let x_2 and x_1 denote the images of x in F and L_1, respectively.

Recall the "cofibration" spectral sequence in [23, p. 262, 6.7]. $L_2 = L_1 \vee L_3$ is filtered by the degree of the L_3 factor and $E^2_{p,q} = H_p L_3 \vee H_q L_1$ with E^r converging to HL.

Since $HL_3 = F[y]$, it follows that y transgresses to a generator of the kernel of $H_{n-1}L_1 \to H_{n-1}L_2$. Hence we can choose a representative y_1 for y in L_2 such that $dy_1 = x_1$. Set $F' = F \vee F[y]$ with $dy = x_2$. Map $F' \to L_2$ by the map $F \to L_1 \to L_2$ and $y \to y_1$. This map $F' \to L_2$ is a weak equivalence by the comparison theorem applied to the cofibration spectral sequences of $F \to F' \to F[y]$ and $L_1 \to L_2 \to L_3$.

Since $F' \to L_2$ is a weak equivalence, the minimal model for F' is isomorphic to L_Y. That is, there is a weak equivalence $L_Y \to F'$.

An immediate corollary of 8.11 is the following.

PROPOSITION 8.12. *Let X be a finite type CW complex with a single point for its one skeleton. Let $C(X)$ denote the rational CW chains on X. Then there is a differential on $L = F[s^{-1}\bar{C}(X)]$ such*

that there is a weak equivalence $L_X \to L$. In particular, HL is isomorphic to $\pi(\Omega X) \otimes Q$. If e is a cell of X, then $ds^{-1}e$ is a chain representative of the attaching map of e.

For example, let $X = S^2 \vee S^2 \bigcup_\alpha e^5$ where the attaching map α is the iterated Whitehead product $[[\beta, \gamma], \gamma]$ on generators β, γ of $\pi_2(S^2 \vee S^2)$. Let F be the free Lie algebra on generators x, y, z where degree $x = $ degree $y = 1$ and degree $z = 4$. Define $dx = dy = 0$ and $dz = [[x, y], y]$. By 8.12, there is a weak equivalence $L_X \to F$. Since F is a minimal Lie algebra, L_X is isomorphic to F.

As the above example shows, 8.12 gives the minimal Lie algebra model when applied to a minimal CW structure. Conversely, if the minimal model is known, then 8.12 gives representative cycles for the attaching maps in a minimal rational CW structure. For example, 8.8 implies that, for a simply connected formal space, the attaching maps for a rational CW structure can be read off from the rational homology coalgebra.

Minimal Lie algebra models can sometimes be effective in answering the following question. How many rational homotopy types are there with a fixed rational homology coalgebra C? By the remarks which follow 8.4, this is equivalent to looking at isomorphism classes of free Lie algebras $F[s^{-1}\bar{C}]$ with the quadratic term of the differential specified by C. Since this will be the subject of another paper, we will not go into any detail here. We leave the following straightforward exercises to the reader. There are two rational homotopy types that have the rational homology coalgebra $H(S^3 \vee S^3 \vee S^8; Q)$. There are countably infinite rational homotopy types that have the rational homology coalgebra $H(S^3 \vee S^3 \vee S^{12}; Q)$. We were motivated to look for this second example by Stasheff.

REFERENCES

1. J. F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A., **42** (1956), 409–412.
2. A. K. Bousfield and V. K. A. M. Gugenheim, *On PL deRham theory and rational homotopy type*, (preprint).
3. A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Berlin, Heidelburg, New York: Springer-Verlag, 1972.
4. P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math., **29** (1975), 245–274.
5. S. Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Amer. Math. Soc. Memoirs, **55** (1965).
6. ———, *Homology and fibrations* I., Comment. Math. Helv., **40** (1966), 199–236.
7. ———, *Limits and spectral sequences*, Topology, **1** (1962), 1–23.
8. E. Friedlander, P. A. Griffiths, and J. Morgan, *Homotopy theory and differential forms*, Seminorio di Geometria, Firenze, 1972 (mimeographed).
9. P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Berlin,

Heidelburg, New York: Springer-Verlag, 1966.

10. D. Husemoller, J. C. Moore, and J. Stasheff, *Differential homological algebra and homogeneous spaces*, J. Algebra, **5** (1974), 113–185.

11. J. M. Lemaire, *Algèbres Connexes et Homologie des Espaces de Lacets*, Berlin, Heidelburg, New York: Springer-Verlag, 1974.

12. M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Ec. Norm. Sup., **71** (1954), 101–190.

13. A. L. Malcev, *Nilpotent groups without torsion*, Izvest. Akad. Nauk SSSR, ser. Math., **13** (1949), 201–212.

14. T. Miller, *Minimal Lie algebras in rational homotopy theory*, University of Notre Dame dissertation, 1976.

15. T. Miller and J. Neisendorfer, *Formal and coformal spaces*, (preprint).

16. J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math., **81** (1965), 211–264.

17. J. C. Moore, *Differential homological algebra*, Actes, Congrès, Intern. Math., 1970. Tome 1, 335–339.

18. ———, *Some properties of the loop homology of commutative coalgebras, in the Steenrod algebra and its applications*, edited by Frank Peterson. Berlin, Heidelburg, New York: Springer-Verlag, 1970.

19. ———, *What it means for a coalgebra to be simply connected*, (preprint).

20. J. Neisendorfer, *Rational homotopy groups of complete intersections*, (preprint).

21. J. Neisendorfer and L. Taylor. *Dolbeault homotopy theory*, (preprint).

22. D. Quillen, *Homotopical Algebra*, Berlin, Heidelburg, New York: Springer-Verlag, 1967.

23. ———, *Rational homotopy theory*, Ann. of Math., **90** (1969), 205–295.

24. L. Smith, *Homological algebra and the Eilenberg-Moore spectral sequence*, Trans. Amer. Math. Soc., **129** (1967), 58–93.

25. ———, *Lectures on the Eilenberg-Moore Spectral Sequence*, Berlin, Heidelburg, New York: Springer-Verlag, 1970.

26. D. Sullivan, *Infinitesimal computations in topology*, (preprint).

27. ———, *Topology of manifolds and differential forms*, Proceedings of Conference on Manifolds, Tokyo, 1973.

28. R. G. Swan, *Thom's theory of differential forms of simplicial sets*, Topology, **14** (1975), 271–273.

29. H. Weyl, *The Classical Groups*, Princeton, 1946.

30. G. W. Whitlhead, *On mappings into group-like spaces*, Comment. Math., Helv. **28** (1954), 320–328.

SYRACUSE UNIVERSITY
SYRACUSE, NY 13210

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

Pacific Journal of Mathematics

Vol. 74, No. 2   June, 1978