Pacific Journal of Mathematics

THE FAILURE OF EVEN CONJUGATE CHARACTERIZATIONS OF H^1 ON LOCAL FIELDS

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Vol. 74, No. 2 June 1978

PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 2, 1978

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If K is a local field, the Hardy space $H^1(K)$ is defined as follows: If f is a distribution on K let $f(x, k)$ (defined on $K \times \boldsymbol{Z}$) be its regularization. Let $f^*(x) = \sup_k |f(x, k)|$. Then $f \in H^1$ iff the maximal function f^* is integrable. Chao has given the following conjugate function characterization of H^1. Let π be a multiplicative character on K that is homogeneous of degree zero, ramified of degree 1, and is odd. Then $f \in L^1$ is in H^1 iff $(\pi\hat{f})^\vee \in L^1$. He also shows that if μ is a finite (Borel) measure then μ is absolutely continuous whenever $(\hat{\mu}\pi)^\vee$ is also a finite measure. In this paper proofs are given that these results fail if π is not odd.

It is shown that if π is even (but otherwise satisfies the conditions above) then there is a singular measure μ and an integrable function f. $f \notin H^1$ such that $\pi\hat{\mu} = \hat{\mu}$ and $\pi\hat{f} = \hat{f}$. These results were announced earlier [Gandulfo, Garcia-Cuerva, and Taibleson, Bull. Amer. Math. Soc., 82 (1976), 83–85].

A basic reference for this paper is [4]; in particular, Chapters I, II, and IV. Regularizations are discussed in detail in IV §1. The results proven here are [3; Thm. 1 and Lemma 1]. The theorem of Chao can be found in [4; IV § 3] or in [1]. Other characterizations of H^1 can be found in [2].

A local field is a locally compact field that is not connected and not discrete. A complete list of such fields is: the p-adic number fields and finite algebraic extensions of p-adic fields (these are of characteristic zero), and fields of formal Laurent series over a finite field, $GF(p^n)$, the so-called p^n-series fields (these are of characteristic p). We note that there is a "natural" ring multiplication for the dyadic group, 2^ω, so that the field of quotients of 2^ω is the 2-series field.

There is a norm, $|\cdot|$, on K that is ultrametric ($|x+y| \leqq \max[|x|, |y|]$ and so if $|x| \neq |y|$, $|x+y| = \max(|x|, |y|)$. If $x \in K$, $x \neq 0$, then $|x| = q^k$ for some $k \in \boldsymbol{Z}$. The fractional ideals $\{\mathfrak{P}^k\}$ are the balls: $\mathfrak{P}^k = \{|x| \leqq q^{-k}\}$. We fix a character χ on the additive group of K such that χ is trivial (identically 1) on $\mathfrak{D} = \mathfrak{P}^0$ (the ring of integers in K) and is nontrivial on $\mathfrak{P}^{-1}$. We choose $\mathfrak{p}$ to be a generator of the prime ideal $\mathfrak{P} = \mathfrak{P}^1$ (in $\mathfrak{D}$). $|\mathfrak{p}| = q^{-1}$, and $\mathfrak{D}/\mathfrak{P} \cong GF(q)$ (the local class field of K) where $q = p^n$, p a prime. The measure of a set E is denoted

$|E|$. $|\mathfrak{P}^k| = q^{-k}$, so $|\mathfrak{D}| = 1$. For $u \in K$, we set $\chi_u(x) = \chi(ux)$, $\tau_u f(x) = f(x-u)$. Φ_k denotes the characteristic function of $\mathfrak{P}^k$.

DEFINITION. If K is of finite characteristic let $h_k = \chi_{\mathfrak{p}^{-k}}\Phi_0$. If K is of characteristic zero let $h_k = \sum_{l=1}^{q^{k-1}} \tau_{c_l^{k-1}}(\chi_{\mathfrak{p}^{-k}}\Phi_{k-1})$ where $\{c_l^k\}$ is a complete set of coset representatives of $\mathfrak{P}^k$ in $\mathfrak{D}$.

Note. (1) If K is of finite characteristic the two definitions essentially agree. (2) If $q = 2$, $\{h_k\}$ is the sequence of Rademacher functions.

LEMMA 1. *$\{h_k\}_{k \geq 1}$ is a sequence of independent, identically distributed random variables on $\mathfrak{D}$.*

Proof. Each h_k is supported on $\mathfrak{D}$ and we identify h_k with its restriction to $\mathfrak{D}$. The values of h_k are pth roots of unity. h_k is constant on the $q^k = p^{nk}$ cosets of $\mathfrak{P}^k$ in $\mathfrak{D}$. On each of the $q = p^n$ cosets of $\mathfrak{P}^{k-1}$ in cosets of $\mathfrak{P}^k \subset \mathfrak{D}$ it takes on each of its p possible values exactly p^{n-1} times. Thus, if ε is a pth root of unity $|\{h_k = \varepsilon\}| = p^{-1}$. We see that the h_k are identically distributed. To show independence we need to observe that if $\{k_j\}_{j=1}^t$ is a finite collection of distinct positive integers and $\{\varepsilon_j\}$ a set of pth roots of unity then $|\{h_{k_j} = \varepsilon_j, j = 1, \cdots, t\}| = p^t$. Using the facts above we get this result by systematically counting. This completes the proof.

The Fourier transform of a distribution f is denoted $\hat{f}$ and for $f \in L^1$, $\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi}(x)dx$. If μ is a finite Borel measure, $\hat{\mu}(\xi) = \int_K \overline{\chi_\xi}(x)d\mu(x)$. We note that $\overline{\chi_u} = \chi_{-u}$, $(\chi_u f)\hat{} = \tau_u\hat{f}$, $(\tau_u f)\hat{} = \overline{\chi_u}\hat{f}$, and $\hat{\Phi}_k = q^{-k}\Phi_{-k}$.

LEMMA 2. *Let $g_k = Re\, h_k$ and*

$$\mu(x, k) = \begin{cases} g_1(x)\prod_{l=2}^{-k}(1 - g_l(x)), & x \in \mathfrak{D},\ k \leq -1 \\ 0, & \textit{otherwise}. \end{cases}$$

Then $\mu(x, k)$ is the regularization on K of a nontrivial, real-valued, finite Borel measure μ, that is singular, supported on $\mathfrak{D}$, $|\mu| < 1$, $\mu(\mathfrak{D}) = 0$, and $\hat{\mu}$ is supported on $C = \bigcup_{k\geq 1}\{(\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}) \cup (-\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1})\}$. If $q = 2$, μ is supported on a two point set. If $q > 2$, μ is continuous.

Proof. From Lemma 1 we see that $\{g_k\}$ is a sequence of independent, identically distributed random variables on $\mathfrak{D}$ (are i.i.d. on $\mathfrak{D}$). Observe that if J is a coset of $\mathfrak{P}^l$, $l < k$, then $\int_J g_k = 0$. We break the proof into smaller steps.

I. $\mu(x, k)$ is regular. Since g_k is constant on cosets of $\mathfrak{P}^k$, $\mu(x, k)$ is constant on cosets of $\mathfrak{P}^{-k}$. Next we see that $\int_{\mathfrak{D}} \mu(x, -1) = \int g_1 = 0 = \mu(x, 0)$. Finally we need to show that if $J = y + \mathfrak{P}^{-(k+1)} \subset \mathfrak{D}$, $k < -1$, then $\int_J \mu(x, k) = \int_J \mu(x, k+1) = q^{k+1}\mu(y, k+1)$. But, $\mu(x, k) = \mu(x, k+1)(1 - g_{-k})$, so $\int_J \mu(x, k) = \mu(y, k+1)\int_J (1 - g_{-k}) = \mu(y, k+1)|J| = q^{k+1}\mu(y, k+1)$.

II. $|\mu(x, k)|$ is regular on the domain, $\mathfrak{D} \times \{k \leqq -1\}$. The proof for I works since $(1 - g_l(x)) \geqq 0$ for all x.

III. $\mu(x, k)$ is regularization of a nontrivial, real-valued, finite Borel measure, that is supported on $\mathfrak{D}$, and $\mu(\mathfrak{D}) = 0$. Using [4; IV (1.8)(e) and (1.9)(b)] we only need observe that $\mu(x, k)$ is real-valued; $\mu(x, k) = 0$ if $x \notin \mathfrak{D}$; and show that $\int \mu(x, k)dx \equiv 0$, $k \in \boldsymbol{Z}$; and $\int |\mu(x, k)|dx = \int |g_1| > 0$, $k \geqq -1$. $\int \mu(x, k)dx \equiv 0$ follows I. For $k \geqq 0$ it is trivial, for $k \leqq -1$, $\mu(x, k)$ is regular so

$$\int_K \mu(x, k)dx = \int_{\mathfrak{D}} \mu(x, k)dx = \int_{\mathfrak{D}} \mu(x, 0)dx = 0 .$$

That $\int \mu(x, k) = \int |g_1|$ follows from II. $|\mu(x, k)|$ is regular for $k \leqq -1$, so if $k \leqq -1$,

$$\int_K |\mu(x, k)|dx = \int_{\mathfrak{D}} |\mu(x, k)|dx = \int_{\mathfrak{D}} |\mu(x, -1)|dx = \int_K |g_1| > 0 .$$

IV. μ is a singular measure. To see that μ is not absolutely continuous we use [4; (1.8)(d)]. This implies that the regularization of an absolutely continuous measure is Cauchy in L^1. We use the fact that $\{g_k\}$ is i.i.d. on $\mathfrak{D}$. Then for $k \geqq -1$,

$$\begin{aligned}\int_K |\mu(x, k) - \mu(x, k-1)| &= \int_{\mathfrak{D}} |g_1|(1 - g_2) \cdots (1 - g_{-k})|g_{-k+1}| \\ &= \int_{\mathfrak{D}} |g_1| \int_{\mathfrak{D}} (1 - g_2) \cdots \int_{\mathfrak{D}} (1 - g_{-k}) \int_{\mathfrak{D}} |g_{-k+1}| = \left[\int |g_1|\right]^2 > 0 .\end{aligned}$$

Note that $|\{(1 - g_k(x)) = 0\}| = p^{-1}$, so $|\{\mu(x, k)\} \neq 0\}| = (1 - p^{-1})^{-(k+1)}$ and so $\mu(x, k) \to 0$ a.e. From which it follows that $\mu^*(x) < \infty$ a.e. [4; V (2.3)]. Let $E_N = \{\mu^*(x) < N\}$. By the dominated convergence theorem $|\mu|(E_N) = 0$ (use II) and so $|\mu|(\bigcup_N E_N) = 0$, but $\bigcup_N E_N$ is a set of full measure, so μ is supported on a set of measure zero.

Actually we can do the whole thing in one simple step if we carefully analyse the set in $\mathfrak{D}$ on which $\mu(x, k) = 0$. That set, call it F_k, is a union of cosets of $\mathfrak{P}^{-k}$, $|F_k| \to 1$, $\{F_k\}$ is increasing. Thus μ is supported on the set $\sim(\bigcup_k F_k)$ which is a closed set of measure zero.

V. If $q = 2$ μ is a 2-point measure. If $q > 2$, μ is continuous. For $q = 2$ a little computation shows that there are decreasing sequences of cosets $\{I_k^i\}$, $i = 1, 2$, such that I_k^i is a coset of $\mathfrak{P}^{-k}$ and

$$\mu(x, k) = \begin{cases} 2^{-k-1}, & x \in I_k^1 \\ -2^{-k-1}, & x \in I_k^1 \\ 0, & \text{otherwise.} \end{cases}$$

Since $|I_k^i| = 2^k$, we see that $\mu(\cdot, k)$ converges W^* to a 2-point measure with mass $1/2$ at one point and mass $-1/2$ at the other. More generally we note that $|\mu(x, k)| \leqq 2^{-k-1}$ for all x, so that if I_k is a coset of $\mathfrak{P}^{-k}$, then

$$\begin{aligned} |\mu|(I_k) &= \lim_{l \to -\infty} \int_{I_k} |\mu(x, l)|\, dx \\ &= \int_{I_k} |\mu(x, k)| \leqq |I_k| 2^{-k-1} = (1/2)(q/2)^{-k} \longrightarrow 0 \end{aligned}$$

as $k \to -\infty$ if $q > 2$. Thus, if $\{I_k\}$ is a decreasing sequence of cosets, $|\mu|(I_k) \to 0$ and so μ has no atomic component.

VI. $\hat{\mu}$ is supported on C. It will suffice to show that each $\hat{\mu}(\cdot, k)$ is supported on C. Note also that for $q = 2$, this is an uninteresting statement since $C = K \sim \mathfrak{D}$. To show that $\mu(\cdot, k)$ is supported on C it will be sufficient to show that if $\{k_j\}$ is a finite set of distinct positive integers with $k_s = \max_j k_j$ then $(g_{k_1} \cdots g_{k_s})^\wedge$ is supported on

$$\{(\mathfrak{p}^{-k_s} + \mathfrak{P}^{-k_s+1}) \cup (-\mathfrak{p}^{-k_s} + \mathfrak{P}^{-k_s+1})\}\,.$$

We consider two cases. If K is of finite characteristic,

$$\begin{aligned} g_{k_1} \cdots g_{k_s} &= 2^{-s}(\chi_{\mathfrak{p}^{-k_1}} + \chi_{-\mathfrak{p}^{-k_1}}) \cdots (\chi_{\mathfrak{p}^{-k_s}} + \chi_{-\mathfrak{p}^{-k_s}})\Phi_0 \\ &= 2^{-s} \sum \chi_{\pm\mathfrak{p}^{-k_1}} \cdots \chi_{\pm\mathfrak{p}^{-k_s}}\Phi_0 = 2^{-s} \sum \chi_{(\pm\mathfrak{p}^{-k_1} \pm \cdots \pm \mathfrak{p}^{-k_s})}\Phi_0\,. \end{aligned}$$

Thus,

$$(g_{k_1} \cdots g_{k_s})^\wedge = \sum \tau_{(\pm\mathfrak{p}^{-k_1} \pm \cdots \pm \mathfrak{p}^{-k_s})}\Phi_0\,.$$

Each term is the characteristic function of a coset of $\mathfrak{D}$ in one or the other of $\mathfrak{p}^{-k_s} + \mathfrak{P}^{-k_s+1}$ or $-\mathfrak{p}^{-k_s} + \mathfrak{P}^{-k_s+1}$. For K of finite characteristic we proceed more carefully.

$$g_{k_1} \cdots g_{k_s} = 2^{-s}(h_{k_1} + \overline{h_{k_1}}) \cdots (h_{k_s} + \overline{h_{k_s}}) .$$

$$\hat{h}_k = [q^{-k+1} \textstyle\sum_{l=1}^{q(k-1)} \overline{\chi}_{c_l^{k-1}}] \tau_{\mathfrak{p}^{-k}} \Phi_{-k+1} .$$

Since $c_l^{k-1} \in \mathfrak{D}$ it follows that the term in the "square" brackets is constant on cosets of $\mathfrak{D}$. $\tau_{\mathfrak{p}^{-k}} \Phi_{-k+1}$ is the characteristic function of $\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}$ so $\hat{h}_k$ is a finite linear combination of characteristic functions of cosets of $\mathfrak{D}$ contained in $\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}$. Thus h_k is a finite linear combination of terms of the form $\chi_u \Phi_0$, $u \in \mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}$, $k > 0$. Similarly, $\bar{h}_k$ is a finite linear combination of such terms with $u \in -\mathfrak{p}^{-k} + \mathfrak{P}^{-k+1}$. The proof now proceeds as in the finite characteristic case.

This completes the proof of Lemma 2.

Note. μ is defined as a local field version of a Riesz product. See [5; V § 7]. It should then come as no surprise that μ is a continuous singular measure when $q \geqq 3$. We also note that if $q = 3$, then μ (except for a trivial factor) is the Cantor-Lebesgue measure supported on the Cantor set, if one identifies $\mathfrak{D}$ with $[0, 1]$ in the usual way.

COROLLARY. *Let π be a multiplicative character on K that is ramified of degree 1, homogeneous of degree zero, and is even. Let μ be the real-valued, singular measure defined in Lemma 2. Then $\pi\hat{\mu} = \hat{\mu}$.*

Proof. We show that $\pi(x) \equiv 1$ on C. π is ramified of degree 1 so π is constant on each coset $\pm\mathfrak{p}^k + \mathfrak{P}^{k+1}$ so we only need to determine $\pi(\mathfrak{p}^k)$ and $\pi(-\mathfrak{p}^k)$. π is homogeneous of degree zero so we only need to determine $\pi(1)$ and $\pi(-1)$. π is even so $\pi(-1) = \pi(1)$. π is a multiplicative character so $\pi(1) = 1$. This completes the proof.

THEOREM. *Let μ be as above, and let $\{c_k\}$ be a collection of distinct coset representatives of $\mathfrak{D}$ in K. Then there is a sequence $\{a_k\}$ of real numbers such that if $f(x) = \sum_{k=1}^{\infty} a_k \tau_{c_k} \mu(x, -k)$, then $f \in L^1$, but $f \notin H^1$. Furthermore, $\hat{f}$ is supported on C.*

Proof. Let $f_k = \tau_{c_k} \mu(\cdot, -k)$. f_k is supported on $c_k + \mathfrak{D}$.

$$f_k(x, l) = \begin{cases} \mu(x - c_k, l) & , \quad l > -k \\ \mu(x - c_k, -k) , & l \leqq -k . \end{cases}$$

Thus $f_k(\cdot, l)$ is supported on $(c_k + \mathfrak{D}) \times \boldsymbol{Z}$. Consequently,

$$\int |f| = \sum |a_k| \int |f_k| = \int |g_1| \sum |a_k| , \quad \text{and} \quad \int f^* = \sum |a_k| \int (f_k)^* .$$

We claim that $\left\{\int (f_k)^*\right\}$ is unbounded. If this claim is valid we simply choose $\{a_k\}$ so $\sum |a_k| < \infty$ and $\sum a_k \int (f_k)^* = \infty$. To prove the claim suppose $\left\{\int (f_k)^*\right\}$ is bounded. We note that $(f_k)^*(x) = \sup_{l \geqq -k} |\mu(x-c_k, l)|$, so $\{(f_k)^*(x+c_k)\}$ is a nondecreasing sequence with limit μ^*. By the Lebesgue monotone convergence theorem $\mu^* \in L^1$. But $\mu(x, k)$ converges a.e. so by the Lebesgue dominated convergence theorem $\{\mu(\cdot, k)\}$ converges in L^1 and hence is Cauchy in L^1. But $\{\mu(\cdot, k)\}$ is not Cauchy in L^1, a contradiction.

We need to show that $\hat{f}$ is supported on C. But $\hat{f} = \sum a_k \chi_{c_k} \hat{\mu}(\cdot, k)$, and $\hat{\mu}(\cdot, k)$ is supported on C for all k, so $\hat{f}$ is also supported on C. This completes the proof of the theorem.

Acknowledgment. Ms. Anna Gandulfo provided some heroic calculations for a variety of special cases of Lemma 2. These examples established the background for the more general results that appear in this paper.

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Received August 10, 1976. Research supported in part by the National Science Foundation under Grant No. MPS75-02411.

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Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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