ON CHARACTERISTIC HYPERSURFACES OF SUBMANIFOLDS IN EUCLIDEAN SPACE

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The main purpose of this paper is to prove that \( M^n \subset E^N \), where \( N = n(n + 1)/2 \), the characteristic \((n - 1)\)-dimensional submanifolds of \( M^n \) are the asymptotic hypersurfaces.

1. Introduction. The concept of a characteristic submanifold of a given solution for a differential system, was introduced by E. Cartan in his theory of partial differential equations ([2], p. 79). Its importance appears in the treatment of the Cauchy problem.

Given an \( n \)-dimensional submanifold \( M^n \) of the Euclidean space \( E^N \), we can define geometrically the notion of asymptotic submanifolds of \( M^n \). The asymptotic lines have been used extensively for the study of the geometry of a surface in \( E^3 \). For higher dimension and codimension some results have been obtained, using the generalized concept [3], [4], [9], [10]. It is well known, that the characteristic curves of a surface in \( E^3 \) are the asymptotic lines ([2], p. 143).

In §2 we start with a brief introduction to the Cartan-Kähler theory of differential equations. Then given a Riemannian manifold \( M^n \), we consider the differential ideal, whose integral submanifolds determine local isometries of \( M^n \) into \( E^N \), \( N = n(n + 1)/2 \). Next assuming \( M^n \subset E^N \), we characterize the \((n - 1)\)-dimensional characteristic submanifolds of \( M^n \).

In §3, we define the concept of asymptotic submanifolds of \( M^n \subset E^N \), prove the main result and obtain a first order partial differential equation whose solutions are the characteristic hypersurfaces of \( M \).

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2. Characteristic submanifold. Let \( M \) be an \( n \)-dimensional differentiable manifold. We denote by \( \mathcal{A}_k(M) \) the vector space of differential \( k \)-forms on \( M \) and \( \mathcal{A}(M) = \bigoplus_{k=0}^{n} \mathcal{A}_k(M) \). A differential ideal is an ideal \( U \) in \( \mathcal{A}(M) \) which is finitely generated, homogeneous (i.e., \( U = \bigoplus_{k=0}^{n} U_k \) where \( U_k = U \cap \mathcal{A}_k(M) \) are closed under exterior differentiation. We assume that \( U \) is a differential ideal which does not contain functions i.e., \( U_0 = 0 \). A \( p \)-dimensional submanifold \( S \) of \( M \) is said to be an \((p\)-dimensional\) integral submanifold for \( U \), if \( i^*(U) = 0 \) i.e., \( i^*(U_p) = 0 \) where \( i: S \to M \) is the inclusion map.

We denote by \( T_xM \) the tangent space to \( M \) at \( x \in M \); \( G^p_x(M) \) denotes the Grassman manifold of \( p \)-dimensional subspaces of \( T_xM \).
and $G^p(M) = \bigcup_{x \in M} G^p_x(M)$ is given the usual manifold structure. An element $E^p_x \in G^p_x(M)$ is said to be an integral element for $U$, if all the differential forms of $U$ vanish when restricted to the elements of $E^p_x$.

Let $I^p_x(U)$ denote the set of $p$-dimensional integral elements for $U$ at $x$, and let $I^p(U) = \bigcup_{x \in M} I^p_x(M)$ be given the topology as a subspace of $G^p(M)$. If $E^p_x$ is an integral element for $U$ generated by \{\(v_1, \ldots, v_p\)\}, we define the polar space $H(E^p_x)$ by

\[ H(E^p_x) = \{v \in T_xM; \phi(v, v_1, v_2, \ldots, v_p) = 0, \forall \phi \in U_{p+1}\}. \]

An integral element $E^p_x$, $p \geq 1$ is said to be ordinary if there exist integral elements $E^0_x, E^1_x, \ldots, E^{p-1}_x$ with $E^0_x \subset E^1_x \subset \cdots \subset E^{p-1}_x \subset E^p_x$ such that $\dim H(E^i_x)$ is constant on a neighborhood of $E^i_x$ in $I^i(U)$ for $i = 0, 1, \ldots, p - 1$. A zero-dimensional integral element $E^0_x$ is said to be regular if $\dim H(E^0_x)$ is constant on a neighborhood of $E^0_x$ in $I^0(U)$. A $p$-dimensional integral element $E^p_x$, $p \geq 1$ is said to be regular if it is ordinary and $\dim H(E^p_x)$ is constant on a neighborhood of $E^p_x$ in $I^p(U)$. We remark that when $M$ is connected, this definition of regularity is equivalent to Cartan's ([2], pp. 61-67) according to which, an integral element $E^p_x$ is regular if it is ordinary and $\dim H(E^p_x)$ is equal to the dimension of a generic $p$-dimensional ordinary integral element.

It follows from Cartan-Kähler theorem ([2, pp. 68-74], [7, p. 26]) under the assumption that the manifold $M$ and the differential forms are analytic, that given a $q$-dimensional ordinary integral element $E^q_x$, then there exists a $q$-dimensional integral submanifold $S$, which contains $x$ and satisfies the requirement $T_xS = E^q_x$.

An integral submanifold $S$ for $U$ is said to be singular if $\forall x \in S$, the integral element $T_xS$ is not ordinary. We remark, that an integral submanifold $S$ may be singular because none of its points is regular, or none of its tangential subspaces of dimension one, or two, $\ldots$, etc., or $p - 1$ is regular, where $p$ is the dimension of $S$. Hence one may have different classes of singular integral submanifolds, whose degree of singularity decreases in a certain sense when one goes from one class to the next one.

Let $S$ be a $p$-dimensional nonsingular integral submanifold for $U$, a submanifold $\tilde{S} \subset S$ of dimension $q < p$ is called characteristic if $\forall x \in \tilde{S}$, the integral element $T_xS$ is not regular.

The concepts introduced above, can be found with more details in [2] and [7]. The Cartan-Janet theorem [1], [6] asserts that any real analytic, $n$-dimensional, Riemannian manifold can be locally mapped by a real analytic isometric embedding, into a Euclidean space $E^n$ of dimension $N = n(n + 1)/2$. In what follows we consider the differential ideal, whose integral submanifolds give local isome-
tries of $M$ into $E^N$. Next assuming $M \subset E^N$, we characterize the $(n-1)$-dimensional characteristic submanifolds of $M$. We adopt the following indices convention

\[
1 \leq i, j, k, l \leq n; \quad n + 1 \leq \lambda, \mu, \alpha \leq N;
\]

\[
1 \leq I, J, K \leq N; \quad N = n(n + 1)/2
\]

and the summation convention with regard to repeated indices.

Let $M$ be an $n$-dimensional Riemannian manifold with metric $g$. Let $F(M)$ denote the bundle of orthonormal frames over $M$, with the usual manifold structure. Under the action of the orthogonal group $O(n)$, $F(M)$ is a principal fiber bundle over $M$, with structural group $O(n)$. Let $\pi: F(M) \to M$ be the usual projection. We define the canonical forms $\omega^i$, $\cdots$, $\omega^n$ on $F(M)$ by

\[
\pi_z^*z(y) = \omega^i(x, e_i, \cdots, e_n) \in F(M) \text{ and } v \in T_z(F(M)),
\]

hence $\pi_z^*g = \sum_i \omega^i \otimes \omega^i$. The connection forms $\omega^i_1$ on $F(M)$ are uniquely defined by

\[
d\omega^i = \omega^i \wedge \omega^i; \quad \omega^i_1 + \omega^i_2 = 0.
\]

Finally, if we consider

\[
\Omega^i = d\omega^i - \omega^i_1 \wedge \omega^i_1
\]

then there exist functions $R_{ijkl}$, the components of the Riemann curvature tensor, defined on $F(M)$ such that

\[
\Omega^i = -\frac{1}{2} R_{ijkl} \omega^k \wedge \omega^l, \quad R_{ijkl} = -R_{lijk}.
\]

Similarly for $E^N$, we denote by $F(E^N)$ the bundle of orthonormal frames over $E^N$, $\bar{\pi}: F(E^N) \to E^N$ the projection, $\bar{\omega}^i$ the canonical forms on $F(E^N)$, $\bar{\omega}^i_1$ the connection forms on $F(E^N)$.

We consider the product manifold $B = F(M) \times F(E^N)$, and define the differential ideal on $B$. Let $\rho: B \to F(M)$ and $\bar{\rho}: B \to F(E^N)$ be the usual projections. Using $\rho$ and $\bar{\rho}$ we can pull the differential forms $\omega^i$, $\omega^i_1$, $\bar{\omega}^i$, $\bar{\omega}^i_1$ back to $B$, we will denote the pulled-back forms by the same symbols. Let $U$ be the differential ideal on $B$ generated by

\[
\bar{\omega}^i - \omega^i
\]

\[
\bar{\omega}^i_1
\]

\[
\bar{\omega}^i_1 - \omega^i_1
\]

\[
\omega^i \wedge \bar{\omega}^i_1
\]

\[
\bar{\omega}^i_1 \wedge \bar{\omega}^i_1 + \frac{1}{2} R_{ijkl} \omega^i \wedge \omega^k
\].
We remark that there is a left action of $O(n)$ on $B$ which preserves the differential ideal $U$. Namely if $A = (a_{ij}) \in O(n)$ we consider $L_A: B \to B$, which associates to
\[
(z = ((x, e_1, \ldots, e_n), (\bar{x}, \bar{e}_1, \ldots, \bar{e}_N)) \in B)
\]
the point
\[
L_A(z) = \left((x, \sum_i a_{i1}e_i, \ldots, \sum_i a_{in}e_i),
(\bar{x}, \sum_i a_{i1}\bar{e}_i, \ldots, \sum_i a_{in}\bar{e}_i, \bar{e}_{n+1}, \ldots, \bar{e}_N)\right).
\]
It is not difficult to verify that $L^*_A(U \cap A_1(B)) \subset U \cap A_1(B)$ and hence $L^*_A(U) = U$.

Since we want to determine the $(n - \frac{1}{2})$-dimentional characteristic submanifolds of $M^* \subset E^N$, we start characterizing the nonregular $(n - 1)$-dimentional integral elements $E^*_{x_{1}}$ for $U$ in $B$, whose projections $\pi_* \circ \rho_*(E^*_{x_{1}})$ are $(n - 1)$-dimentional. This characterization is obtained in Lemma 1(c).

Let $p$ be an integer $0 \leq p < n$, we adopt the additional index conventions
\[
1 \leq a, b, c \leq p ; \quad p + 1 \leq r, s, t \leq n .
\]
Suppose that $E^*_{x_{1}}$ is a $p$-dimentional integral element for $U$, generated by vectors $e_1, \ldots, e_p$ such that
\[
\omega^a(e_b) = \delta^a_b , \quad \omega^r(e_b) = 0 .
\]
If we denote, $h^i_{ba} = \omega^i(e_a)$ then it follows, from the fact that the generators of $U$ vanish when restricted to $E^*_{x_{1}}$, that
\[
(1) \quad h^i_{ba} = h^i_{ba}
\]
\[
(2) \quad \sum_i (h^i_{ia}h^i_{ja} - h^i_{i}h^i_{i}) - R_{iab} = 0 .
\]
Denote by
\[
H_{ia} = (h_{ia}^{p+1}, \ldots, h_{ia}^N)
\]
the vector in the $(N - n)$-dimentional Euclidean space.

Let $J^p$ denote the set of $p$-dimentional integral elements $E^*_{x_{1}}$, which satisfy the following conditions:
1. $\omega^1 \wedge \cdots \wedge \omega^p \neq 0$ and $\omega^{p+1} = \cdots = \omega^* = 0$ when restricted to $E^*_{x_{1}}$.
2. the vectors $\{H_{ma} : 1 \leq a \leq p, a \leq m \leq n - 1\}$ are linearly independent. Let $V^p = \{E^*_{x_{1}} \in I^p(U) : L_A(E^*_{x_{1}}) \in J^p$ for some $A \in O(n)\}$.
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Then $V^p$ is an open subset of $P(U)$. Part of the next lemma is proved following ([5], with the obvious modifications).

**Lemma 1.**

(a) If $0 \leq p < n$, then $\dim H(E^p)$ is constant on $V^p$;
(b) For $0 \leq p < n$, if $E^p \in V^p$, then it is a regular element;
(c) If $p = n - 1$, and $E^{n-1}$ is an integral element such that $\pi_\ast \circ \rho_\ast(E^{n-1})$ is $(n - 1)$-dimensional, then $E^{n-1}$ is regular if and only if $E^{n-1} \in V^{n-1}$.

**Proof.** (a) Since $L^\ast(U) = U$ it suffices to show that $\dim H(E^p)$ is constant on $J^p$. Assume that $E^p$ is generated by $e_1, \ldots, e_p$ such that $\omega^i(e_a) = \delta^i_a$ and $\omega^i(e_a) = 0$. We consider the polar space

$$H(E^p) = \{v \in T_zB; \phi(v, e_1, \ldots, e_p) = 0 \forall \phi \in U_n\}$$

$$= \{v \in T_zB; \phi_i(v) = 0 \text{ and } \phi_a(v, e_a) = 0 \forall \phi_i \in U_1, \phi_a \in U_2\}$$

where last equality follows from the fact that $U$ is generated by (*)

Hence $H(E^p)$ consists of vectors $v \in T_zB$ which satisfy the following system of equations:

1. $\tilde{\omega}^i(v) - \omega^i(v) = 0$
2. $\tilde{\omega}^i(v) = 0$
3. $\tilde{\omega}^i(v) - \omega^i(v) = 0$
4. $h^i_\alpha \omega^i(v) - \tilde{\omega}^i(v) = 0$
5. $\sum h^i_\alpha \tilde{\omega}^i(v) + \sum h^i_\alpha \omega^i(v) - R_{t\alpha} \omega^i(v) = 0$, $i < j$.

If we specify $\omega^i(v), \tilde{\omega}^i(v)$ then equations (3)–(6) will uniquely determine $\tilde{\omega}^i(v), \omega^i(v)$ and $\tilde{\omega}^i(v)$. Moreover we remark that for $1 \leq i, j \leq p$, equation (7) is an immediate consequence of (1), (2) and (6). So we need only to consider (7) where $1 \leq i \leq p, p + 1 \leq j \leq n$ and $p + 1 \leq i < j \leq n$, i.e.,

$$\sum h^i_\alpha \tilde{\omega}^i(v) + \sum h^i_\alpha \omega^i(v) - R_{b\alpha} \omega^i(v) = 0$$

Since in (8), for $a \neq b$, interchanging $a$ and $b$ does not modify the equation, we need only to consider

$$\sum h^i_\alpha \tilde{\omega}^i(v) = (\sum h^i_\alpha h^i_\beta - R_{b\alpha}) \omega^i(v), \quad a \leq b$$

$$\sum h^i_\alpha \tilde{\omega}^i(v) - \sum h^i_\alpha \omega^i(v) = R_{t\alpha} \omega^i(v), \quad s < t.$$
Denote the vectors

\[ H_i(v) = (\tilde{\omega}_{i}^{a}(v), \cdots, \tilde{\omega}_{i}^{m}(v)) . \]

We determine the vectors \( H_{p+1}(v), \cdots, H_{n}(v) \) so that they satisfy (9) and (10). The system (9) determines the dot product of \( H_{p+1}(v) \) with the \( p(p + 1)/2 \) linearly independent vectors \( H_{ba}, a \leq b \). Once we have chosen a particular \( H_{p+1}(v) \) which satisfies this linear system of rank \( p(p + 1)/2 \), the dot product of \( H_{p+2}(v) \) with each of the \( p(p + 1)/2 + p \) linearly independent vectors \( \{ H_{ma} : 1 \leq a \leq p, a \leq m \leq p + 1 \} \) is completely determined by (9) and (10). We continue in this fashion. Finally we find that the dot product of \( H_{n}(v) \) with each of the \( p(p + 1)/2 + p(n - p - 1) \) linearly independent vectors \( \{ H_{ma} : 1 \leq a \leq p, a \leq m \leq n - 1 \} \) is completely determined. Hence we find that \( \tilde{\omega}_{i}^{a}(v) \) must satisfy a consistent system of linear equations which has rank \( np(n - p)/2 \). The polar system of \( E^p \) consists of these equations together with (3)-(6). Hence dim \( H(E^p) \) depends only on \( n \) and \( p \) whenever \( E^p \in J^p \).

(b) Suppose that \( E^p \in J^p \) is generated by \( e_1, \cdots, e_p \), such that \( \omega^{a}(e_b) = \delta_b^a \) and \( \omega^{a}(e_b) = 0 \). If \( 0 \leq q \leq p \), we let \( E^q \) be the \( q \)-dimensional integral element generated by \( e_1, \cdots, e_q \). Then \( E^q \in J^q \) and hence dim \( H(E^q) \) is constant in a neighborhood of \( E^q \in I^q(U) \). It follows that \( E^q \) is regular. Consequently if \( E^q \in V^p \), then it is a regular integral element.

(c) From (b) we only need to prove that if \( E^{n-1} \) is a regular integral element then \( E^{n-1} \in V^{n-1} \). Since \( \pi_{*} \circ \rho_{*}(E^{n-1}) \) is \( (n - 1) \)-dimensional, we can find an element \( A \in O(n) \) such that \( \omega^a = 0 \) on \( L_A(E^{n-1}) \). Hence, we can assume that \( E^{n-1} \) is generated by \( e_1, \cdots, e_{n-1} \), such that \( \omega^{a}(e_b) = \delta_b^a \) and \( \omega^{a}(e_b) = 0 \), where \( 1 \leq a, b \leq n - 1 \). Since \( E^{n-1} \) is regular, it follows that dim \( H(E^{n-1}) \) is constant in a neighborhood of \( E^{n-1} \in I^{n-1}(U) \). The polar system of \( E^{n-1} \) is given by (3)-(6) and (7) reduces to

\[ \sum_{a} h_{ba}^{i} \omega_{i}^{b}(v) = \left( \sum_{a} h_{ma}^{i} h_{ib}^{j} - R_{bma}^{i} \right) \omega^{i}(v), \quad a \leq b. \]

As in (a) if we specify \( \omega^{i}(v), \omega_{i}^{a}(v) \) then \( \tilde{\omega}_{i}^{a}(v), \tilde{\omega}_{i}^{b}(v) \) and \( \tilde{\omega}_{i}^{m}(v) \) will be uniquely determined by (3)-(6). Moreover the \( n(n - 1)/2 \) components \( \tilde{\omega}_{i}^{a}(v) \) must satisfy the linear system (11) which has exactly \( n(n - 1)/2 \) equations. Hence, if dim \( H(E^{n-1}) \) is constant in a neighborhood of \( E^{n-1} \), then the determinant of the coefficient matrix in (11) is nonzero, i.e., the vectors \( \{ H_{ma} : 1 \leq a \leq b \leq n - 1 \} \) are linearly independent, which implies \( E^{n-1} \in J^{n-1} \).

Let \( M \) be an \( n \)-dimensional Riemannian manifold and \( f : M \to E^{n} \) an isometric imbedding. If \( x_0 \in M \), there exists a neighborhood \( V \) of
$x_0$ in $M$ and a section $\tilde{\sigma}: V \to F(E^N)$ such that if $\tilde{\sigma}(x) = (f(x), \tilde{e}_i(x), \cdots, \tilde{e}_N(x))$, then $\tilde{e}_i(x), \cdots, \tilde{e}_n(x)$ are tangent to $f(M)$. We consider the section $\sigma: V \to F(M)$, defined by $\sigma(x) = (x, e_i(x), \cdots, e_n(x))$ where $f_*(e_i(x)) = \tilde{e}_i(x)$. For simplicity, we denote by $\omega_i$, $\omega'_i$ the differential forms $\sigma^*\omega$, $\sigma^*\omega'_i$ induced on $V$ and similarly $\tilde{\omega}_i$, $\tilde{\omega}'_i$ will denote the pulled-back forms $\tilde{\sigma}^*\omega$, $\tilde{\sigma}^*\omega'_i$ on $V$. Consider the map $\Gamma: V \to B$ defined by $\Gamma(x) = (\alpha_i(x), \sigma(x))$. Since $f$ is an isometry, $\Gamma(V)$ is an integral submanifold for $U$ in $B$. We say that a $g$-dimensional vector space $L \subset T_{x_0}M$, $0 \leq g < n$ is regular if $\Gamma(L)$ is a regular integral element for $U$. Similarly, a $q$-dimensional submanifold $S$ of $V$ is said to be characteristic, if $\Gamma(S)$ is a characteristic submanifold of $\Gamma(V)$. The characteristic hypersurfaces of $M$ have at each point a nonregular tangent space. Our next lemma characterizes the non-regular $(n - 1)$-dimensional spaces tangent to $M$.

We denote the matrix $H^i = (h^i_j)$ where $h^i_j = \omega_i e_j$. Moreover, given a matrix $A$, we denote by $A_b$ the $b$th row of $A$ and $A^t$ denotes the transpose of $A$. Assume $\Gamma(V)$ is not a singular integral submanifold for $U$, then as an immediate consequence of Lemma 1(c), we obtain

**Lemma 2.** Let $u_i\omega^i = 0$ be an $(n - 1)$-dimensional subspace of $T_{x_0}M$. We may assume that $\sum_{i=1}^{n} u_i^2 = 1$. Choose $A = (a_{is}) \in O(n)$ such that $a_{is} = u_i$. Then $u_i\omega^i = 0$ is nonregular if and only if the vectors

$$(A_a H^{a+1} A_b, \cdots, A_a H^N A_b), \quad 1 \leq a \leq b \leq n - 1$$

are linearly dependent, as vectors in $E^{N-n}$.

We remark that this condition determines a first order partial differential equation, and the characteristic hypersurfaces of $M$ are the solutions of this equation. In the next section as a consequence of Lemma 3, the partial differential equation will be given in another form, which will not involve the choice of matrix $A$.

3. Asymptotic submanifolds; proof of main result. Let $M$ be an $n$-dimensional $C^\infty$ submanifold of $E^N$, $N = n(n + 1)/2$ with the induced metric and such that the inclusion $i: M \to E^N$ is nondegenerate. Let $x \in M$ and denote by $s$ the second fundamental form. A $q$-dimensional $0 < q < n$ linear subspace $L$ of the tangent space $T_xM$ is called asymptotic if there exists a vector $\xi$ normal to $T_xM$ such that $\langle s(X, Y), \xi \rangle = 0$, $\forall X, Y \in L$ where $\langle , \rangle$ denotes the Euclidean metric. If $L$ is of codimension one, we have an asymptotic hyperplane at $x$. A $q$-dimensional submanifold $V$ of $M$, $q < n$ is called asymptotic at $x \in V$ if $T_xV$ is asymptotic and asymptotic if this is
true for each $x \in V$. It is not difficult to see that $V$ is an asymptotic hypersurface of $M$ if and only if there exists a normal to the osculating space of $V$, which is also normal $M$. The notation of asymptotic submanifold in a more general context can be found in [4].

Let $e_1, \ldots, e_N$ be an orthonormal frame defined on a neighborhood of $x \in M$, such that $e_1, \ldots, e_n$ are tangent to $M$ and $e_{n+1}, \ldots, e_N$ are normal to $M$. Let $\omega^1, \ldots, \omega^N$ be the dual frame. With the same indices convention as in §2, we denote by $h^i_j = \omega^i(e_j)$ where $\omega^i$ are the connection forms. It follows from the definition that a hyperplane $u_i \omega^i = 0$ is asymptotic if and only if the second fundamental forms $h^i_j \omega^j \otimes \omega^i$ are linearly dependent when restricted to $u_i \omega^i = 0$.

The following algebraic lemma shows that the condition obtained in Lemma 2 is equivalent to saying that $u_i \omega^i = 0$ is asymptotic. As in §2 given a matrix $A$ we denote by $A_b$ the $b$th row of $A$ and $A^t_b$ denotes the transpose of $A_b$.

**Lemma 3.** Let $\varphi^i$ be $n \times n$ symmetric matrices $\lambda = n + 1, \ldots, N$, $N = n(n + 1)/2$ and let $A = (a_{ij}) \in O(n)$. Then the vectors

\[(A_b \varphi^{n+1} A_t^i, \ldots, A_b \varphi^N A_t^i), \quad 1 \leq b \leq c \leq n - 1\]

are linearly dependent, as vectors in $E^{N-n}$, if and only if the quadratic forms $\varphi^i \omega^j \otimes \omega^i$ are linearly dependent when restricted to $a_{n} \omega^i = 0$, where $\omega^i$ are $n$ independent 1-forms.

**Proof.** The vectors $(A_b \varphi^{n+1} A_t^i, \ldots, A_b \varphi^N A_t^i)$ are linearly dependent iff $\exists a_i \in R$ not all zero, such that

\[A_b \left( \sum_{i=n+1}^{N} a_i \varphi^i \right) A_t^i = 0, \quad 1 \leq b \leq c \leq n - 1.\]

We denote by $D$ the matrix $D = \sum_i a_i \varphi^i$ and $W = (\omega^1, \ldots, \omega^n)$. We will prove that $A_b DA_t^i = 0 \forall 1 \leq b \leq c \leq n - 1$ if and only if $WDW^t = 0$ whenever $A_n W^t = 0$.

Consider

\[(12) \quad WDW^t = WA^t(ADA^t)AW^t.\]

Suppose $A_b DA_t^i = 0, \forall 1 \leq b \leq c \leq n - 1$, then since $D$ is symmetric

\[WDW^t = \begin{bmatrix} WA_1, \ldots, WA_{n-1} \end{bmatrix} \begin{bmatrix} A_1 DA_n^t & \vdots & \vdots \\ \vdots & A_{n-1} DA_n^t & \vdots \\ A_n DA_1^t & \ldots & A_n DA_n^t \end{bmatrix} \begin{bmatrix} A_1 W^t \\ \vdots \\ A_{n-1} W^t \\ A_n W^t \end{bmatrix}.\]
Hence if $A_n W^i = 0$ then $W D W^i = 0$, i.e., the quadratic forms $W \varphi^i W^i$ are linearly dependent whenever $A_n W^i = 0$.

Conversely, suppose $W D W^i = 0$ when $A_n W^i = 0$, then it follows from (12) that

\begin{equation}
0 = \sum_{b=1}^{n-1} A_s D A^i_i (\sum_{k=1}^{n} a_{bk} \omega^k)^2 + 2 \sum_{b,c=1}^{n-1} A_s D A^i_i (\sum_{k,l=1}^{n} a_{bk} a_{cl} \omega^k \otimes \omega^l).
\end{equation}

Let $e_i$ be the dual basis of $\omega^i$, i.e., $\omega^i(e_j) = \delta^i_j$. If we evaluate (13) at the pair $(e_k, e_k)$ we get

$$
\sum_{b=1}^{n-1} A_s D A^i_i a_{bk}^2 + 2 \sum_{b,c=1}^{n-1} A_s D A^i_i a_{bk} a_{ck} = 0, \quad \forall k = 1, \ldots, n.
$$

Adding over $k$, since $A \in O(n)$ we get

\begin{equation}
\sum_{b=1}^{n-1} A_s D A^i_i = 0.
\end{equation}

If we apply (13) to the pairs $(e_k, e_l), (e_l, e_k)$, $l \neq k$ and subtract we get

\begin{equation}
\sum_{b,c=1}^{n-1} A_s D A^i_i (a_{bk} a_{cl} - a_{bk} a_{ck}) = 0, \quad \forall 1 \leq k \leq l \leq n.
\end{equation}

This is an homogeneous linear system of $n(n-1)/2$ equations with $(n-1)(n-2)/2$ unknowns $A_s D A^i_i$, $1 \leq b < c \leq n - 1$. We claim that the rank of this system is $(n-1)(n-2)/2$. In fact, otherwise it follows from Sylvester-Franke theorem on determinants ([8], p. 94, take $m = 2$), that the cofactor of $a_{ni}$ in $A$ is zero, $\forall i = 1, \ldots, n$, which contradicts the fact that $\det A \neq 0$. Hence from (15) we have that

\begin{equation}
A_s D A^i_i = 0, \quad 1 \leq b < c \leq n - 1.
\end{equation}

Now (13) reduces to

\begin{equation}
\sum_{b=1}^{n-1} A_s D A^i_i (\sum_{k=1}^{n} a_{bk} \omega^k)^2 = 0
\end{equation}

and from (14) we have

\begin{equation}
A_{n-1} D A^i_{n-1} = - \sum_{b=1}^{n-2} A_s D A^i_i.
\end{equation}

If we substitute (18) in (17) we get

$$
\sum_{b=1}^{n-2} A_s D A^i_i ((a_{bk} - a_{n-1k}) \omega^k)(\sum_{k=1}^{n} (a_{bk} + a_{n-1k}) \omega^k) = 0.
$$

Applying this equation to the pairs of vectors $(e_k, e_l), (e_l, e_k)$, $l \neq k$ and subtracting we get
This is a linear system of \( n(n - 1)/2 \) equations with \( n - 2 \) unknowns \( A_b D A_i^j (a_{b \theta} a_{\mu - 1} - a_{\mu - 1 b} a_{b \theta}) = 0 \), \( 1 \leq k < l \leq n \).

Let \( f: M \to E^N \) be an isometric embedding, with the same notation as in 2, we say that \( f \) is singular if \( \forall x \in M, \Gamma_x^* (T_x M) \) is not an ordinary integral element for \( U \) in \( B \). Then our main result follows immediately from Lemmas 2 and 3:

**THEOREM.** Let \( f: M \to E^N \) be a nonsingular isometric imbedding. An \( (n - 1) \)-dimensional submanifold of \( M \) is characteristic if and only if it is asymptotic.

We remark that \( f \) being nonsingular implies that \( f \) is non-degenerate, but for \( n > 2 \) it may exist a nondegenerate isometric imbedding which is singular; in this case all hypersurfaces would be asymptotic.

We observe that it is not difficult to prove that \( u_i \omega^i = 0 \) is asymptotic if and only if there exist real numbers \( a_2, b_i \) not all zero, such that

\[
a_2 h_i^j \omega^i \otimes \omega^j = u_i \omega^i \otimes b_j \omega^j.
\]

This reduces to a homogeneous equation in \( u_i \) of degree \( n, P(u_1, u_2, \ldots, u_n) = 0 \). In order to describe the polynomial \( P \) we consider the matrices

\[
U_0 = \begin{bmatrix}
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0
\end{bmatrix}, \quad U_p = \begin{bmatrix}
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where \( U_p \) has the first \( (p - 1) \) rows equal to zero, \( 1 \leq p \leq n - 1 \).
Then
\[
A_0 = \begin{bmatrix}
  h_{11}^{n+1} & h_{22}^{n+1} & \cdots & h_{nn}^{n+1} \\
  \vdots & \vdots & & \vdots \\
  h_N & h_N & \cdots & h_N \\
\end{bmatrix}
\quad A_p = 2 \begin{bmatrix}
  h_{pp+1}^{n+1} & h_{pp+2}^{n+1} & \cdots & h_{pn}^{n+1} \\
  \vdots & \vdots & & \vdots \\
  h_{pp+1}^N & h_{pp+2}^N & \cdots & h_{pn}^N \\
\end{bmatrix},
\]
\[
1 \leq p \leq n - 1.
\]

Then
\[
P(u_1, u_2, \cdots, u_n) = \det \begin{bmatrix}
  U_2 & U_1 & \cdots & U_{n-1} \\
  A_0 & A_1 & \cdots & A_{n-1} \\
\end{bmatrix} = 0.
\]

Hence the characteristic hypersurfaces of \( M \) are the solutions of the first order partial differential equation defined by \( P(u_1, \cdots, u_n) = 0 \). For \( n = 3 \) this equation was obtained by Cartan ([2], p. 208).

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