SETS IN $R^d$ HAVING $(d - 2)$-DIMENSIONAL KERNELS

MARILYN BREEN
SETS IN $R^d$ HAVING $(d - 2)$-DIMENSIONAL KERNELS

Marilyn Breen

Let $S$ be a $d$-dimensional set, $d \geq 2$, and assume that for every $(d + 1)$-member subset $T$ of $S$, there corresponds a $(d - 2)$-dimensional convex set $K_T \subseteq S$ such that every point of $T$ sees $K_T$ via $S$ and $(\text{aff } K_T) \cap S = K_T$. Furthermore, assume that when $T$ is affinely independent, then $K_T$ is the kernel of $T$ relative to $S$. Then $S$ is starshaped and the kernel of $S$ is $(d - 2)$-dimensional.

1. Introduction. Let $S$ be a subset of $R^d$, $d \geq 2$. For points $x, y$ in $S$, we say $x$ sees $y$ via $S$ if and only if the corresponding segment $[x, y]$ lies in $S$. Similarly, for $T \subseteq S$, we say $x$ sees $T$ (and $T$ sees $x$) via $S$ if and only if $x$ sees each point of $T$ via $S$. The set of points in $S$ seen by $T$ is called the kernel of $T$ relative to $S$ and is denoted $\ker_S T$. Finally, if $\ker_S S = \ker S$ is not empty, then $S$ is said to be starshaped.

An interesting problem is that of determining necessary and sufficient conditions for $S$ to be a starshaped set whose kernel is $k$-dimensional, $0 \leq k \leq d$. Several papers have considered this question (Hare and Kenelly [2], Kenelly, Hare, et al. [3], Toranzos [4]), and Foland and Marr [1] have proved that a set $S$ will have a zero-dimensional kernel provided $S$ contains a noncollinear triple and every three noncollinear members of $S$ see via $S$ a unique common point. Hence the purpose of this paper is to obtain an analogue of these results for subsets of $R^d$ whose kernel is $(d - 2)$-dimensional.

The following familiar terminology will be used. Throughout the paper, conv $S$, aff $S$, cl $S$, bdry $S$, rel int $S$, and ker $S$ will denote the convex hull, affine hull, closure, boundary, relative interior, and kernel, respectively, of the set $S$. The cone of $x$ over $S$, defined to be the union of all rays emanating from $x$ through points of $S$, will be denoted cone($x$, $S$). Also, if $S$ is convex, dim $S$ will represent the dimension of $S$.

2. Proof of the theorem.

THEOREM. Let $S$ be a $d$-dimensional set, $d \geq 2$, and assume that for every $(d + 1)$-member subset $T$ of $S$, there corresponds a $(d - 2)$-dimensional convex set $K_T \subseteq S$ such that every point of $T$ sees $K_T$ via $S$ and $(\text{aff } K_T) \cap S = K_T$. Furthermore, assume that when $T$ is affinely independent, then $K_T$ is the kernel of $T$ relative to $S$. Then $S$ is starshaped and the kernel of $S$ is $(d - 2)$-dimensional.
Proof. The proof of the theorem is lengthy and will be accomplished by a sequence of lemmas. The first lemma and its corollary are immediate consequences of our hypothesis, while the second, third and fourth lemmas present the main arguments in the proof.

**Lemma 1.** If \( T = \{t_1, \cdots, t_{d+1}\} \) is an affinely independent subset of \( S \), then \( \cup \{[t_i, t_j]: 1 \leq i < j \leq d+1\} \subseteq S \).

**Proof of Lemma 1.** Otherwise, \( T \subseteq \ker_s T = K_T \), contradicting the fact that \( K_T \) is a convex set of dimension \( d-2 \).

**Corollary 1.** Let \( T = \{t_1, \cdots, t_d\} \) be a subset of \( S \), with \( t_1, \cdots, t_d \) affinely independent and \( \conv \{t_1, \cdots, t_d\} \subseteq S \). Then \( K_T \subseteq \aff \{t_1, \cdots, t_d\} \).

**Lemma 2.** Assume that \( \conv (K \cup \{p\}) \cup \conv (K \cup \{q\}) \subseteq S \), where \( K \) is a convex set of dimension \( d-2 \), \( p \not\in \aff K \), and \( q \notin \aff (K \cup \{p\}) \). Then for \( x \in S \) and \( x \in \pi = \aff (K \cup \{p\}) \), \( x \) sees each point of \( K \) via \( S \).

**Proof of Lemma 2.** To begin, note that for \( k_1, \cdots, k_{d-1} \) any \( d-1 \) affinely independent points in \( K \), the set \( \{k_1, \cdots, k_{d-1}, p, q\} = T \) is affinely independent. Hence the set \( K_T \) described in the theorem is exactly \( \ker_s T \), and so \( K \subseteq K_T \). Thus without loss of generality we may assume \( K = K_T \) and therefore \( (\aff K) \cap S = K \). Also, we assume that \( x \not\in \aff K \), for otherwise \( x \in K \), finishing the proof.

Now by the hypothesis of the theorem, the points \( k_1, \cdots, k_{d-1}, p, x \) see via \( S \) a convex set \( D \) of dimension \( d-2 \) such that \( (\aff D) \cap S = D \), and since \( \conv \{k_1, \cdots, k_{d-1}, p\} \subseteq S \cap \pi, D \subseteq \pi \) also (by the corollary to Lemma 1). Similarly, \( k_1, \cdots, k_{d-1}, q, x \) see a \((d-2)\)-dimensional convex set \( D' \) with \((\aff D') \cap S = D' \), and \( D' \subseteq \pi' = \aff (K \cup \{q\}) \).

If either \( D = K \) or \( D' = K \), the argument is complete. Hence we assume \( D \neq K \) and \( D' \neq K \) to reach a contradiction. The set \( D' \cup D \) cannot contain a set \( P \) of \( d+1 \) affinely independent points, for these points would see \( k_1, \cdots, k_{d-1}, x \) via \( S \), contradicting the fact that \( \ker_s P \) is a convex set of dimension \( d-2 \). A similar argument implies that all points seen by \( k_1, \cdots, k_{d-1}, x \) necessarily lie in the \((d-1)\)-dimensional flat \( \aff(D \cup D') \). Then since \([\aff(D \cup D')] \cap \pi \cap S = D \), the subset of \( \pi \) seen by \( k_1, \cdots, k_{d-1}, x \) is exactly \( D \).

We assert that \( x \in (\aff D) \cap S = D \): Consider the \((d-1)\)-dimensional flat \( \pi \), and let \( D_1, D_2 \) denote distinct open halfspaces of \( \pi \) determined by \( D \). Since \( K \not\subseteq \aff D \), without loss of generality assume \( k_1 \in D_1 \). There are two cases to consider, depending on the location of the remaining \( k_i \) points.
Case 1. If for every $k_i \not\in D$, we have $k_i \in D_1$, then the sets $\text{conv}(D \cup \{k_i\})$, $k_i \not\in D$, intersect in a $(d - 1)$-dimensional convex set $C$ in $\text{cl}(D)$, $D \subseteq \text{bdry} C$, and each point $k_1, \cdots, k_{d-1}$ sees $C$ via $S$. In case $x \not\in \text{aff} D$, $\text{cone}(x, D)$ would intersect $C \sim D$ at some point $a$, and clearly $[x, a] \subseteq S$. Therefore each of $k_1, \cdots, k_{d-1}, x$ would see $a$ via $S$, contradicting the fact that the subset of $\pi$ seen by $k_1, \cdots, k_{d-1}, x$ is exactly $D$. We conclude that if Case 1 occurs then $x \in \text{aff} D$.

Case 2. If for some $k_i \not\in D$, we have $k_i \in D_2$, then the sets $\text{cone}(k_i, D)$, $k_i \not\in D$, intersect in a $(d - 1)$-dimensional convex set $C$, $C \subseteq D$, and $k_1, \cdots, k_{d-1}$ see $C$ via $S$. Again, if $x \not\in \text{aff} D$, $\text{cone}(x, D)$ would intersect $C \sim D$ at some point, impossible by the argument in Case 1. We conclude that $x \in \text{aff} D$ if Case 2 occurs, and our assertion is proved.

Thus we have $x \in (\text{aff} D) \cap S = D$, so $x$ sees $k_1, \cdots, k_{d-1}$ via $S$. However, this is impossible since the subset of $\pi$ seen by $x, k_1, \cdots, k_{d-1}$ is exactly $D$ and $k_i \not\in D$. Our original assumption is false and either $D = K$ or $D' = K$. In either case, $x$ sees $K$ via $S$, completing the proof of Lemma 2.

**Corollary 2.** Assume that $\text{conv}(K \cup \{p\}) \cup \text{conv}(K \cup \{q\}) \subseteq S$, where $K$ is a convex set of dimension $d - 2$, $p \not\in \text{aff} K$, and $q \not\in \text{aff} (K \cup \{p\})$. If $x \in (S \cap \text{aff}(K \cup \{p\})) \sim \text{aff} K$ and $y \in (S \cap \text{aff}(K \cup \{q\})) \sim \text{aff} K$, then $[x, y] \not\subseteq S$.

**Proof.** Otherwise the set $K \cup \{x, y\}$ would contain $d + 1$ affinely independent points with each corresponding segment in $S$, violating Lemma 1.

**Lemma 3.** Assume that $\text{conv}(K \cup \{p\}) \cup \text{conv}(K \cup \{q\}) \subseteq S$, where $K$ is a convex set of dimension $d - 2$, $p \not\in \text{aff} K$, and $q \not\in \text{aff} (K \cup \{p\})$. Let $\pi = \text{aff}(K \cup \{p\})$, $\pi' = \text{aff}(K \cup \{q\})$. Select $r \not\in \pi \cup \pi'$, and let $\pi_1$ and $\pi_1'$ denote the open halfspaces determined by $\pi$ and $\pi'$, respectively, and containing $r$. If $u \in \pi \cup \pi'$ and if $[r, u] \subseteq S$, then $[r, u] \subseteq (\text{cl} \pi_1) \cap (\text{cl} \pi_1')$.

**Proof of Lemma 3.** If $u \in \text{aff} K = \pi \cap \pi'$, the result is trivial. Hence without loss of generality we assume that $u \in \pi' \sim \text{aff} K$. Then clearly $[r, u] \subseteq \text{cl} \pi_1'$, and we need only show that $[r, u] \subseteq \text{cl} \pi_1$.

It suffices to prove that $(r, u) \cap \pi = \emptyset$: Suppose on the contrary that $v \in (r, u) \cap \pi$. Now $v \not\in \pi'$, for otherwise the line determined by $u$ and $v$ would lie in $\pi'$ and $r \in \pi'$, contradicting our hypothesis. Hence
LEMMA 4. Assume that \(\text{conv}(K \cup \{p\}) \cup \text{conv}(K \cup \{q\}) \subseteq S\), where \(K\) is a convex set of dimension \(d - 2\), \(p \notin \text{aff} K\), and \(q \notin \text{aff} (K \cup \{p\})\). If \(z \in S\), then \(z\) sees \(K\) via \(S\).

**Proof of Lemma 4.** As in the proof of Lemma 2, let \(\pi = \text{aff}(K \cup \{p\})\), \(\pi' = \text{aff}(K \cup \{q\})\), and assume that \(K = (\text{aff} K) \cap S\). Furthermore, we may suppose that \(z \notin \pi \cup \pi'\), for otherwise the result is an immediate consequence of that lemma. Then for \(k_1, \cdots, k_{d-1}\) affinely independent in \(K\), the points \(k_1, \cdots, k_{d-1}, p, z\) are affinely independent and see via \(S\) a unique \((d - 2)\)-dimensional convex subset \(A\). By the corollary to Lemma 1, since \(\text{conv}\{k_1, \cdots, k_{d-1}, p\} \subseteq S \cap \pi\), we have \(A \subseteq \pi\), and by Lemma 2, \(A\) sees \(K\) via \(S\). Similarly, \(k_1, \cdots, k_{d-1}, q, z\) see a \((d - 2)\)-dimensional convex set \(A', A' \subseteq \pi'\), and \(A'\) sees \(K\) via \(S\).

As in Lemma 3, let \(\pi_1\) and \(\pi'_1\) denote the open halfspaces determined by \(\pi\) and \(\pi'\), respectively, and containing \(z\). Since \(A \cup A' \subseteq \pi \cup \pi'\), it follows directly from the lemma that \(\text{conv}(A \cup \{z\}) \cup \text{conv}(A' \cup \{z\}) \subseteq \text{cl} \pi_1 \cap \text{cl} \pi'_1\).

If \(A = K\) or \(A' = K\), the argument is complete. Hence we assume \(A \neq K, A' \neq K\), to reach a contradiction. The argument is given in two steps.

**Step 1.** We show that for an appropriate choice of point \(t\) in \(\pi'\) and convex set \(D\) in \(\alpha = \text{aff}(A \cup \{z\}), [t, d] \cup K \cup A\) lies in \(S\) and in the boundary of its convex hull for every \(d\) in \(D \sim A\). To begin, select \(t \in (\text{rel int} \text{conv}(K \cup A')) \sim \alpha\) and let \(\alpha_1\) denote the open halfspace determined by \(\alpha\) and containing \(t\). Then \(t \in \pi_1 \cap \alpha_1\) and, by Lemma 3, \(\text{conv}(K \cup \{t\}) \subseteq \text{cl} \pi_1 \cap \text{cl} \alpha_1\). By the corollary to Lemma 1, for \(a_1, \cdots, a_{d-1}\) affinely independent in \(A\), the points \(a_1, \cdots, a_{d-1}, z, t\) see some \((d - 2)\)-dimensional convex set \(D\) in \(\alpha\), and \((\text{aff} D) \cap S = D\). Furthermore, by Lemma 3 applied to \(\pi\) and \(\alpha\), \(\text{conv}(D \cup \{t\}) \subseteq \text{cl} \pi_1 \cap \text{cl} \alpha_1\). Similarly, by the results in Lemmas 2 and 3, \(D\) sees \(A\) via \(S\) and \(D\) and \(A\) are in \(\text{cl} \pi'_1\). We conclude that \(\text{conv}(D \cup \{t\})\) lies in \(\text{cl} \pi_1 \cap \text{cl} \pi'_1 \cap \text{cl} \alpha_1\).

Note that \(K\) and \(D\) lie in some common hyperplane: Otherwise, for \(T\) a subset of \(K \cup D\) consisting of \(d + 1\) affinely independent points, the corresponding set \(K_T\) would contain \(A \cup \{t\}\), contradicting the fact that \(K_T = \ker_s T\) is a convex set of dimension \(d - 2\). Since \(A\) and \(D\) also lie
in a common hyperplane, we have $K \cap A \subseteq (\text{aff } K) \cap (\text{aff } A) \subseteq \text{aff}(K \cup D) \cap \text{aff}(A \cup D) = \text{aff } D$, $K \cap A \subseteq (\text{aff } D) \cap S = D$, and $D$ contains $K \cap A$. Similarly, $D \cap A \subseteq K \cap A$, and $D \cap \pi = D \cap A = K \cap A$. (Of course, $K \cap A$ may be empty.) By our choice of $t$, $t$ and $K \sim A$ lie in the same open halfspace determined by $\alpha$. Moreover, for any point $d$ in $D \sim A$, $K \sim A$ and $[t, d)$ lie in $\alpha_i$, and it is easy to see that $[t, d) \cup K \cup A$ lies in the boundary of its convex hull, the desired result for Step 1.

**Step 2.** Next we show that $\text{conv}([t, d] \cup K \cup A) \subseteq S$. This will violate Lemma 1, since $\text{conv}([t, d] \cup K \cup A)$ is a convex set of dimension $d$. Recall that $d \notin \pi$ so $[t, d) \cap \pi = \emptyset$. By previous comments, for $z_0$ on $[t, d)$, there corresponds a $(d - 2)$-dimensional convex set $E_0 = E_0$ in $\pi$ such that $z_0$ sees $E_0$ via $S$ and $E_0 = (\text{aff } E_0) \cap S$. Lemma 3 implies that $\text{conv}(E_0 \cup \{z_0\}) \subseteq \text{cl } \pi, \cap \text{cl } \pi_i \cap \text{cl } \alpha_i$. Also, using Lemma 2, $z_0$ and $A$ see all points of $E_0 \cup D$ via $S$, so $E_0 \cup D$ cannot contain $d + 1$ affinely independent points. Thus $E_0$ and $D$ lie in a common hyperplane. Hence for $z_1$ and $z_2$ on $[t, d)$, the corresponding sets $E_1$ and $E_2$ are in hyperplanes containing $D$, $E_i = K$ and $E_d = A$. If $K$ and $A$ are parallel, then since each $E_0$ set is in a hyperplane containing $D$, the sets $E_1$ and $E_2$ must be parallel. In case $K$ intersects $A$, then $\text{aff } K \cap \text{aff } A \subseteq \text{aff}(K \cup D) \cap \text{aff}(A \cup D) = \text{aff } D$, and $\text{aff } K \cap \text{aff } A \cap \text{aff } D = \text{aff } K \cap \text{aff } A \neq \emptyset$. Also, for every $E_0$ set, $\text{aff } D \cap \text{aff } A \subseteq \text{aff}(D \cup E_0) \cap \text{aff}(A \cup E_0) = \text{aff } E_0$, and $\text{aff } E_0$ contains the $(d - 3)$-dimensional set $\text{aff } D \cap \text{aff } A \cap \text{aff } K = \text{aff } K \cap \text{aff } A$. Therefore each pair of distinct $\text{aff } E_0$ sets will intersect in exactly $\text{aff } K \cap \text{aff } A$. Furthermore, it is not hard to show that for $z_1 \neq z_2$, the sets $\text{conv}(E_1 \cup \{z_1\}) \sim \pi$ and $\text{conv}(E_2 \cup \{z_2\}) \sim \pi$ are disjoint: If $\text{conv}(E_1 \cup \{z_1\})$ intersected $\text{conv}(E_2 \cup \{z_2\})$ at point $b \notin \pi$, then $E_1 \cup E_2 \cup \{b\}$ would contain $d + 1$ affinely independent points with corresponding segments in $S$, violating Lemma 1. Hence the sets must be disjoint.

Now we select $c \in \text{conv}([t, d] \cup K \cup A)$ to show that $c \in S$, and without loss of generality, we assume that $c \in \text{conv}((t, d) \cup K \cup A)$. Our argument is motivated by a planar construction employed in [1, Lemma 2]. For $z_0 \in [t, d]$, $c \in \text{conv}(K \cup A)$, and for $z_0 \in [t, d]$, we have $\text{conv}(K \cup A) \cap (\text{aff } E_0) \subseteq E_0$, so either $c \in \text{conv}([t, z_0] \cup K \cup E_0)$ or $c \in \text{conv}([z_0, d] \cup K \cup E_0)$. Thus we may define sets $F, G$ in the following manner: Let $F = \{z_0: z_0 \in [t, d] \text{ and } c \in \text{conv}([t, z_0] \cup K \cup E_0)\}$ and $G = \{z_0: z_0 \in [t, d] \text{ and } c \in \text{conv}([z_0, d] \cup K \cup E_0)\}$. By previous comments, if $t < z_1 < z_2 < d$, then $\text{conv}(E_1 \cup \{z_1\}) \cap \text{conv}(E_2 \cup \{z_2\}) \subseteq K \cap A$. Hence, if $z_1 \in F$, we have $z_2 \in F$, and similarly if $z_2 \in G$, then $z_1 \in G$. Therefore, $F$ and $G$ are connected intervals whose union is $[t, d]$, and each of $F$ and $G$ is nonempty since $t \in G$ and $d \in F$. Clearly we may select a point
$u \in (t, d)$ such that $(t, u) \subseteq G$ and $(u, d] \subseteq F$, and without loss of generality, we assume that $u \in F$. We will show that for this $u$ and for $E_u$ the corresponding subset of $\pi$ seen by $u$, $c \in \text{conv}(E_u \cup \{u\})$.

Select a sequence $\{u_n\}$ in $(t, u)$ converging to $u$, and let $\{E_n\}$ represent the corresponding sequence of $(d - 2)$-dimensional convex sets in $\pi$. Since the sets $\text{aff} E_n$ either are parallel to both $A$ and $K$ or intersect in the $(d - 3)$-flat $\text{aff} A \cap \text{aff} K$, and since each $E_n$ lies in the subset of $\pi$ bounded by $\text{aff} A$ and $\text{aff} K$, clearly the sequence $\{E_n \cap \text{conv}(A \cup K)\}$ converges to a $(d - 2)$-dimensional convex set $E'$ in $\text{conv}(K \cup A) \subseteq S$.

We assert that $E' \subseteq E_u$. First we remark that for $w$ in $(\pi \cap S) \sim A$, $w$ cannot lie in $\text{aff}(D \cup \{t\})$. Otherwise, by Lemma 2, $w$ would see both $A$ and $D$ via $S$, and the set $A \cup D \cup \{w\}$ would contain $d + 1$ affinely independent points with corresponding segments in $S$, violating Lemma 1. Therefore, by previous arguments, each point $w$ of $(\pi \cap S) \sim A$ sees via $S$ a unique $(d - 2)$-dimensional subset $J_w$ of $\text{aff}(D \cup \{t\})$, $J_w = (\text{aff} J_w) \cap S$, and either $\text{aff} J_w$ is parallel to sets $A$, $K$, and $D$ or $\text{aff} J_w$ contains the $(d - 3)$-dimensional flat $\text{aff} A \cap \text{aff} K$. Furthermore, for $w$ in $E_n$, the set $J_w$ necessarily contains $u_n$, so $J_w$ is uniquely determined by $n$, and therefore each $w$ in $E_n$ is associated with the same $(d - 2)$-dimensional subset of $\text{aff}(D \cup \{t\})$, call it $J_n$. Similarly, let $J_u$ and $J'$ denote the $(d - 2)$-dimensional subsets of $\text{aff}(D \cup \{t\})$ seen by $E_u$ and $E'$, respectively, and note that $u \in J_u$. By an earlier argument involving Lemma 1, distinct sets $\text{conv}(E_u \cup J_u) \sim \pi$ are disjoint, and each of these is disjoint from $\text{conv}(E_u \cup J_u) \sim \pi$ and from $\text{conv}(E' \cup J') \sim \pi$. Also, $\text{conv}(E_u \cup J_u) \sim \pi$ and $\text{conv}(E' \cup J') \sim \pi$ are either disjoint or one is a subset of the other. In any event, $J'$ is necessarily bounded in $\text{aff}(D \cup \{t\})$ by $\text{aff} J_u$ and $\text{aff} J_n$ for every $n$, and since $\{u_n\}$ converges to $u$, this implies that $u \in J'$. Therefore, $J'$ and $J_n$ both contain $u_n$ and $J' = J_u$. Then by Lemma 1, for $w'$ in $E'$, $w'$ must belong to $E_u, E' \subseteq E_u$, and the assertion is proved.

Now since $u_n \in G$, we have $c \in \text{conv}([u_n, d] \cup A \cup E_n)$ for each $n$, and so $c \in \text{conv}([u, d] \cup A \cup E_u)$. Since $c \in \text{conv}([t, u] \cup K \cup E_u)$, we have $c \in \text{conv}(E_u \cup \{u\}) \subseteq S$, the desired result. We conclude that $\text{conv}([t, d] \cup K \cup A) \subseteq S$, finishing Step 2.

To complete the proof of Lemma 3, notice that $\text{conv}([t, d] \cup K \cup A)$ is a $d$-dimensional subset of $S$. Clearly we have a violation of Lemma 1, our preliminary assumption must be false, and one of the sets $A, A'$ must be $K$. (In fact, by Lemma 1, it is easy to see that $A = A' = K$.) Therefore $z$ sees $K$ via $S$, and Lemma 4 is proved.

At last, using the lemmas above, the proof of the theorem is immediate. Select a set $T$ consisting of $d + 1$ affinely independent points of $S$, and let $K = \text{ker}_S T$. Since $\dim K = d - 2$, clearly we may
select points \( p, q \) in \( T \) such that \( p \not\in \text{aff} K \) and \( q \not\in \text{aff}(K \cup \{p\}) \). Hence \( K, p, \) and \( q \) satisfy the hypotheses of Lemmas 2, 3 and 4, and so for \( z \in S, z \) sees \( K \) via \( S \). Thus \( K \subseteq \ker S \), and since \( \ker S \subseteq \ker_r T \), we have \( K = \ker S \). Therefore \( S \) is indeed starshaped, and \( \ker S \) has dimension \( d - 2 \), the desired result.

The author would like to thank the referee for his conjecture of the following corollary.

**Corollary 3.** The hypothesis of the theorem above provides a characterization of \( d \)-dimensional sets \( S, d \geq 2 \), for which \( K = \ker S \) has dimension \( d - 2 \), \( (\text{aff} K) \cap S = K \), and the maximal convex subsets of \( S \) have dimension \( d - 1 \).

**Proof.** If set \( S \) satisfies the properties above, then clearly to every \((d + 1)\)-member subset \( T \) of \( S \), the set \( K \) serves as an appropriate \( K_T \) set. Furthermore, if \( T \) is affinely independent, we assert that \( K = \ker_s T \): Clearly we may select points \( t_1 \) and \( t_2 \) in \( T \) with \( t_i \not\in K \) and \( t_i \not\in \text{aff}(K \cup \{t_i\}) \). For \( y \) any point which sees \( T \) via \( S \), if \( y \not\in \text{aff}(K \cup \{t_i\}) \), then \( \text{conv}(K \cup \{y, t_i\}) \) would be a full \( d \)-dimensional subset of \( S \), contradicting the fact that maximal convex subsets of \( S \) have dimension \( d - 1 \). Hence \( y \in \text{aff}(K \cup \{t_i\}) \). Similarly \( y \in \text{aff}(K \cup \{t_2\}) \), and \( y \in (\text{aff} K) \cap S = K \). We conclude that \( K = \ker_s T \), and \( S \) indeed satisfies the hypothesis of the theorem.

Conversely, if \( S \) satisfies the hypothesis of the theorem, then the dimension of \( K = \ker S \) is \( d - 2 \) and \((\text{aff} K) \cap S = K \). We need only show that for \( M \) a maximal convex subset of \( S \), \( \dim M = d - 1 \). Clearly \( \dim M \leq d - 1 \), and since \( K \subseteq M \), \( \dim M \geq d - 2 \). If \( \dim M = d - 2 \), then \( M \subseteq (\text{aff} K) \cap S = K \), and \( M = K \). However, since \( M \) is maximal, this implies that there are no points of \( S \) not in \( K \), impossible since \( S \) is a full \( d \)-dimensional. Thus \( \dim M = d - 1 \), finishing the proof of the corollary.

In conclusion, note that for \( d \geq 3 \), the result fails without the requirement that \((\text{aff} K_T) \cap S = K_T \), as the following example illustrates.

**Example 1.** Let \( \{S_n\} \) be a sequence of \((d - 1)\)-dimensional simplices in \( R^d \), \( \{E_n\} \) a corresponding sequence of \((d - 2)\)-dimensional simplices, so that \( E_n \) is a facet of \( S_n \), \( E_{n+1} \subseteq E_n \), \( \cap \{S_i : 1 \leq i \leq n + 1\} = E_{n+1} \), and \( K = \cap \{E_n : 1 \leq n\} \) is a singleton set. Define \( S = \text{cl}(\cup \{S_n : 1 \leq n\}) \). Then for \( T \) any finite subset of \( S \) and \( k \) the largest integer such that \( S_k \cap T \neq \emptyset \), \( E_k \subseteq \ker_s T \). Moreover, if \( T \) contains \( d + 1 \) affinely independent points, then \( E_k = \ker_s T \). However, \( \dim(\ker S) = \dim K = 0 \). Hence for \( d \geq 3 \), the theorem fails without the requirement that \((\text{aff} K_T) \cap S = K_T \). Of course, if \( d = 2 \), each set \( K_T \) is a singleton set so that \((\text{aff} K_T) \cap S = K_T \) will be satisfied automatically.
An easy adaptation of Example 1 shows that $S$ need not even be starshaped unless $(\text{aff } K_T) \cap S = K_T$.

REFERENCES


Received January 10, 1977 and in revised form June 10, 1977.

UNIVERSITY OF OKLAHOMA
NORMAN, OK 73019
PACIFIC JOURNAL OF MATHEMATICS
EDITORS
ICHARD ARENS (Managing Editor)
University of California
Los Angeles, CA 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90007

. A. BEAUMONT
University of Washington
Seattle, WA 98105

R. FINN AND J. MILGRAM
Stanford University
Stanford, CA 94305

. C. MOORE
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN AND J. MILGRAM
Stanford University
Stanford, CA 94305

. C. MOORE
University of California
Berkeley, CA 94720

ASSOCIATE EDITORS
F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA UNIVERSITY OF SOUTHERN CALIFORNIA
ALIFORNIA INSTITUTE OF TECHNOLOGY STANFORD UNIVERSITY
ONTANA STATE UNIVERSITY UNIVERSITY OF HAWAII
IVERSITY OF CALIFORNIA UNIVERSITY OF TOKYO
IVERSITY OF NEVADA UNIVERSITY OF UTAH
EW MEXICO STATE UNIVERSITY WASHINGTON STATE UNIVERSITY
ERGON STATE UNIVERSITY UNIVERSITY OF WASHINGTON
IVERSITY OF OREGON *
AKA UNIVERSITY AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typewritten form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in reference to text, and script in blue. The first paragraph or two must be capable of being used separately as an abstract of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, if applicable, may be sent to any one of the four editors. Please classify according to the scheme of Math. Review index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Bart University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $72. per year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, 102 Highland Boulevard, Berkeley, California, 94708.

UBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1978 Pacific Journal of Mathematics
All Rights Reserved
Mieczyslaw Altman, *General solvability theorems* ........................................... 1
Denise Amar and Eric Amar, *Sur les suites d’interpolation en plusieurs variables* ................................................................. 15
Herbert Stanley Bear, Jr. and Gerald Norman Hile, *Algebras which satisfy a second order linear partial differential equation* ............................................................... 21
Marilyn Breen, *Sets in $R^d$ having $(d - 2)$-dimensional kernels* .................... 37
Gavin Brown and William Moran, *Analytic discs in the maximal ideal space of $M(G)$* .............................................................. 45
Ronald P. Brown, *Quadratic forms with prescribed Stiefel-Whitney invariants* .............................................................. 59
Gulbank D. Chakerian and H. Groemer, *On coverings of Euclidean space by convex sets* .............................................................. 77
S. Feigelstock and Z. Schlussel, *Principal ideal and Noetherian groups* .......... 87
Ralph S. Freese and James Bryant Nation, *Projective lattices* ......................... 93
Harry Gingold, *Uniqueness of linear boundary value problems for differential systems* .............................................................. 107
John R. Hedstrom and Evan Green Houston, Jr., *Pseudo-valuation domains* .............................................................. 137
William Josephson, *Coallocation between lattices with applications to measure extensions* .............................................................. 149
M. Koskela, *A characterization of non-negative matrix operators on $l^p$ to $l^q$ with $\infty > p \geq q > 1$* .............................................................. 165
Kurt Kreith and Charles Andrew Swanson, *Conjugate points for nonlinear differential equations* .............................................. 171
Shoji Kyuno, *On prime gamma rings* .............................................................. 185
Alois Andreas Lechicki, *On bounded and subcontinuous multifunctions* ......... 191
Roberto Longo, *A simple proof of the existence of modular automorphisms in approximately finite-dimensional von Neumann algebras* .............................................................. 199
Kenneth Millett, *Obstructions to pseudoisotopy implying isotopy for embeddings* .............................................................. 207
William F. Moss and John Piepenbrink, *Positive solutions of elliptic equations* .............................................................. 219
Mitsuru Nakai and Leo Sario, *Duffin’s function and Hadamard’s conjecture* .............................................................. 227
Mohan S. Putcha, *Word equations in some geometric semigroups* .................... 243
Walter Rudin, *Peak-interpolation sets of class $C^1$* ..................................................... 267
Elias Saab, *On the Radon-Nikodým property in a class of locally convex spaces* .............................................................. 281
Stuart Sui Sheng Wang, *Splitting ring of a monic separable polynomial* ............ 293