COALLOCATION BETWEEN LATTICES WITH APPLICATIONS TO MEASURE EXTENSIONS

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It is well known that in a locally compact Hausdorff space every countably additive measure on $R_\sigma(K_\delta)$, the $\sigma$-ring generated by the compact $G_\delta$ sets, can be extended to a countably additive measure on $\sigma(F)$, the $\sigma$-algebra generated by the closed sets. In a locally compact Hausdorff space $F$, the lattice of closed sets, countably coallocates (Definition 4.7) the lattice of compact $G_\delta$ sets. Our purpose is to show that coallocation and countable coallocation are properties basic to many extension theorems.

Dubins [5] considered the following situation. $K \subseteq L$ are two lattices containing the null set (a lattice is a collection of subsets of some set closed under finite unions and intersections). $u$ is a bounded measure defined on $K$. Dubins asked when $u_*$, defined by $u_*(b) = \sup\{u(k)/k \subseteq b, k \in K\}$, is a measure on $L$. A necessary and sufficient condition is for $L$ to allocate $K$. $L$ allocates $K$ if the following is true. For any $k \in K$ contained in the union of two sets $l$ and $h$ from $L$ there exist sets $p$ and $q$ from the lattice $K$ such that $k = p \cup q$ and $p \subseteq l$, $q \subseteq h$.

With two lattices $K \subseteq L$ and $u$ a measure on $K$, we show that a sufficient condition for $u^{**}$, defined by $u^{**}(b) = \inf\{u_*(l')/b \subseteq l', l' \in L:\}$, to be a measure on the algebra generated by $L$ is for $u_*$ to be modular on $L'$. $l'$ is the complement of the set $l$ and $L' = \{l'/l \in L\}$. It follows that if $u$ is a $K$ inner regular measure on $R(K)$, the ring generated by $K$, then $u^{**}$ is a $L$ inner regular extension of $u$ to $A(L)$, the algebra generated by $L$.

Thus when $L$ coallocates $K$ (i.e. $L'$ allocates $K$) Dubin's result shows that for every $K$ regular bounded measure $u$ on $R(K)$, $u^{**}$ is a $L$ regular extension of $u$ to $A(L)$. If $L$ countably coallocates $K$ then $u^{**}$ is countably additive when $u$ is countably additive. From this we obtain the stated result on locally compact Hausdorff spaces [Halmos 7] as well as a related result by Levin and Stiles [8]. Countable coallocation also yields an extension theorem by Marik [9] on countably paracompact normal spaces and a theorem by Berberian [2]. In most instances we can and do prove our results for measures that are not bounded.

We also look at measures that are $\tau$-smooth. A measure $u$ on $K$ is $\tau$-smooth if for any net $\{k_a\}$ decreasing to $\emptyset$, $k_a \in K$, $\lim_{a} u(k_a) = 0$. We
show that any bounded $K$ regular measure $u$ on $R(K)$ that is $\tau$-smooth on $K$ can be extended to a bounded measure on $A(\tau(K))$ that is $\tau$-smooth on $\tau(K)$. $\tau(K)$ is the smallest lattice containing $K$ that is closed under arbitrary intersections. We prove $u_*$ is modular on $\tau(K)'$ and obtain $u^{**}$, defined with respect to $\tau(K)$, as the desired extension.

2. Definitions and notation. All lattices are collections of subsets of an abstract set $X$ that are closed under finite unions and intersections. The fact that $X$ contains points has no importance in this paper — the boolean algebra of all subsets of $X$ can be replaced by any complete boolean algebra. Subsets of $X$ will be denoted by lower case letters. If we are considering a lattice $L$ and a set $l$, it will usually be assumed that $l$ belongs to $L$.

$l'$ denotes the complement of the set $l$ in $X$ and $L' = \{l'/l \in L\}$. $R(L)$ is the ring generated by $L$; $A(L)$ the algebra generated by $L$. $\sigma_r(L)$ is the $\sigma$-ring generated by $L$ and $\sigma(L)$ is the $\sigma$-algebra generated by $L$.

A measure $u$ on a lattice $A$ is an extended real valued set function such that for $a, b \in A$

(i) $u(a) + u(b) = u(a \cup b) + u(a \cap b)$.

(ii) $u(a) + u(b) = u(a \cup b)$ whenever $a \cap b = \emptyset$.

(iii) $a \subseteq b$ implies $u(a) \leq u(b)$.

Let $K$ be a lattice contained in $A$. A measure $u$ on $A$ is $K$ inner regular if for every $a \in A$, $u(a) = \sup\{u(k)/k \subseteq a, k \in K\}$.

A measure $u$ on a lattice $A$ is $\sigma$-smooth if for any sequence $\{a_n\}$ decreasing to $\emptyset$, $\lim_n u(a_n) = 0$. $u$ is countably additive on $A$ if $\Sigma^*_i u(a_n) = u(\bigcup^*_i a_n)$ whenever $\{a_n\}$ is a disjoint sequence of sets from $A$ such that $\bigcup^*_i a_n \in A$. For a ring $A$ any finite valued measure $u$ which is $\sigma$-smooth on $K \subseteq A$ and $K$ inner regular is countably additive on $A$.

A measure $u$ on a lattice $A$ is $\sigma$-finite if for every $a \in A$, $a$ is contained in $\bigcup^*_i a_n$ where $a_n \in A$ and $u(a_n)$ is finite for all $n$. If $A$ is a ring then by the Caratheodory extension theorem any countably additive, $\sigma$-finite measure $u$ on $A$ can be uniquely extended to a countably additive measure on $R_\sigma(A)$. The extension is the outer measure defined by $\hat{u}(b) = \inf\{\Sigma^*_i u(a_n)/b \subseteq \bigcup^*_i a_n, a_n \in A\}$.

The bounded measures on the algebra $A(L)$ are denoted by $M(L)$. It is easy to verify that if $u$ is bounded and $L$ inner regular then $u(a) = \inf\{u(l')/a \subseteq l', l \in L\}$ for $a \in A(L)$. A measure satisfying the last equality is called $L'$ outer regular. If a measure is both $L$ inner regular and $L'$ outer regular then it is $L$ regular. The $L$ regular, bounded measures on $A(L)$ are denoted by $M_r(L)$. Those measures belonging to $M_r(L)$ which are $\sigma$-smooth are denoted by $M^S_r(L)$. These
measures are countably additive and hence can be uniquely extended to a countably additive measure on $\sigma(L)$.

For a measure $u$ on a lattice $K$ which contains $\emptyset$, $u_*$ is defined as in the introduction. The definition of $u^{**}$ as given in the introduction depends on the lattice $L$ used ($L$ must also contain $\emptyset$).

3. The modularity of $u_*$. Let $K \subseteq L$ be two lattices containing $\emptyset$ and $u$ a measure on $K$. $u_*$ is modular on $L'$ if $u_*(l_1') + u_*(l_2') = u_*(l_1' \cup l_2') + u_*(l_1' \cap l_2')$. We now show that if $u_*$ is modular on $L'$ then $u^{**}$ is an $L'$ outer regular measure on $A(L)$ where $u^{**}$ is defined with respect to $L$. Furthermore, $u^{**}$ is a complete measure on $A(u,L') = \{e/u^{**}(l') = u^{**}(e \cap l') + u^{**}(e' \cap l') \text{ for all } l \in L \}$. The easy proofs of the following lemmas are omitted.

**Lemma 3.1.** Let $u$ be a measure on $K$. If $u_*$ is modular on $L'$ then for $a, b$ subsets of $X$,

$$u^{**}(a \cup b) + u^{**}(a \cap b) \leq u^{**}(a) + u^{**}(b).$$

**Lemma 3.2.** Let $u$ be a measure on $K$ and $u_*$ be modular on $L'$. Suppose $l' \cap a = \emptyset$, where $a$ is any subset of $X$. Then $u^{**}(l') + u^{**}(a) = u^{**}(a \cup l')$.

$u_*$ is $\sigma$-smooth on $L'$ if $\lim_n u_*(l_n') = u_*(\bigcup_1^\infty l_n')$ whenever $\{l_n\}$ is an increasing sequence such that $\bigcup_1^\infty l_n' \in L'$.

**Theorem 3.3.** Let $u$ be a measure on $K$.

(i) The modularity of $u_*$ on $L'$ is equivalent to $u^{**}$ being an $L'$ outer regular measure on $A(L)$.

(ii) If $u_*$ is modular on $L'$ then $E(u,L')$ is an algebra containing $A(L)$ and $u^{**}$ is a complete measure on $E(u,L')$.

(iii) Suppose $L$ is closed under countable intersections. If $u_*$ is modular and $\sigma$-smooth on $L'$ then $E(u,L')$ is a $\sigma$-algebra containing $\sigma(L)$ and $u^{**}$ is countably additive on $E(u,L')$.

**Proof.** (i), (ii). That modularity is necessary is obvious. The sufficiency of (i) and (ii) will be proved. If $u_*$ is modular on $L'$ then $E(u,L')$ is closed under complementation and by Lemma 3.2 it contains $L'$.

Fix $l' \in L'$. It is sufficient to assume $u_*(l')$ is finite. Let $e_1, e_2$ belong to $E(u,L')$. By Lemma 3.1,

$$u_*(l') \leq u^{**}((e_1 \cup e_2) \cap l') + u^{**}((e_1 \cup e_2)' \cap l').$$
For the reverse inequality choose \( l', h' \) from \( L' \) such that \( l' \supseteq e_i \cap l', \ h' \supseteq e_i' \cap l' \) and

\[
(2) \quad u_*(l') \geq u_*(l'_i) + u_*(h'_i) - \epsilon/3 \quad j = 1, 2.
\]

We claim that

\[
(3) \quad u_*(l') \geq u_*(l'_1 \cup l'_2) + u_*(h'_1 \cap h'_2) - \epsilon.
\]

This inequality is implied by

\[
u_*(l'_1 \cup l'_2) + u_*(h'_1 \cap h'_2) \leq u_*(l'_1) + u_*(h'_1) + \frac{3}{2} \epsilon
\]

which is equivalent to

\[
u_*(l'_2) + u_*(h'_2) \leq u_*(l'_1 \cap l'_2) + u_*(h'_1 \cup h'_2) + \frac{3}{2} \epsilon
\]

by the modularity of \( u_* \). The last inequality is true by (2) and the modularity of \( u_* \).

(3) implies the reverse direction of (1) and hence \( e_1 \cup e_2 \) belongs to \( \mathcal{C}(u, L') \). Hence \( \mathcal{C}(u, L') \) is an algebra containing \( L \).

To show \( u^{**} \) is a measure suppose \( l' \) contains \( e_1 \cup e_2 \) and that \( u(l') - u(e_1 \cup e_2) < \epsilon \). Then by (3)

\[
u_*(l'_1) + u_*(l'_2) \leq u_*(l') + u_*(l'_1 \cap l'_2) + \epsilon.
\]

Therefore

\[
u^{**}(e_1) + u^{**}(e_2) \leq u^{**}(e_1 \cup e_2) + u^{**}(e_1 \cap e_2) + 2\epsilon.
\]

By Lemma 3.1 \( u^{**} \) is modular on \( \mathcal{C}(u, L') \) and by Lemma 3.2 \( u^{**}(\emptyset) = 0 \). It is easy to verify that \( \mathcal{C}(u, L') \) contains all \( e \) such that \( u^{**}(e) = 0 \).

(iii) Let \( \{e_n\} \) be a sequence from \( \mathcal{C}(u, L') \). Choose \( l'_1 \supseteq e_n \cap l', \ b'_n \supseteq e_n' \cap l' \) such that

\[
u_*(l') \geq u_*(l'_n) + u_*(h'_n) - \epsilon/2^n.
\]

We can show using (4) and the modularity of \( u_* \) that

\[
u_*( \bigcup_{1}^{n} l'_i ) + u_*( \bigcap_{1}^{n} h'_i ) \leq u_*(l') + \sum_{1}^{n} \frac{\epsilon}{2^i}.
\]

Since \( u_* \) is \( \sigma \)-smooth on \( L' \) there exists an \( n \) large enough such that
It follows that

$$u_*(l') \equiv u_*(\bigcup_1^\infty l_i') + u_*(\bigcap_1^n h_i') - 2\epsilon. \tag{6}$$

Therefore by Lemma 3.1, $\bigcup_l e_i \in \mathcal{E}(u, L')$.

To show $u^{**}$ is countably additive, we can assume $\Sigma_l u^{**}(e_i)$ is finite. Choose $f_i' \supset e_i$, $f_i' \in L'$ such that $u_*(f_i') - u^{**}(e_i) \leq \epsilon/2^i$. Let $l' = \bigcup_l f_i'$. Then since $u_*$ is $\sigma$-smooth and modular on $L'$,

$$u_*(l') \equiv \sum_l u_*(f_i') < +\infty. \tag{7}$$

Inequality (7) holds for $l'$ and since $l' \supset \bigcup_l^\infty e_i$,

$$u^{**}\left(\bigcup_1^\infty e_i\right) \equiv u_*(l') \leq \sum_l u_*(f_i')$$

$$\leq \sum_l u^{**}(e_i) + 2\epsilon.$$ 

Thus $u^{**}$ is countably additive on $\mathcal{E}(u, L')$.

We now give sufficient conditions for $u^{**}$ to extend $u$.

**Theorem 3.4.** Let $u$ be a $K$ inner regular measure on $S(K)$ which represents either $A(K)$ or $R(K)$. If $u_*$ is modular on $L'$ and $u^{**}$ is finite on $K$ then $u = u^{**}$ on $S(K)$.

**Proof.** $u(b) = u^{**}(b)$ when $u(b) = +\infty$. If $u(b)$ is finite then $u^{**}(b)$ is finite. This follows because every $b \in S(K)$ is of the form $\bigcup_{j=1}^n k_j \cap h_i'$ where for all $j, h_i \in K$ and either $k_j \in K$ or $k_j = X$.

Choose $l' \supset b$ such that $u_*(l') - u^{**}(b) \leq \epsilon/3$. Choose $k_0 \subset b$ such that $u(b) - u(k_0) \leq \epsilon/3$ and choose $k_1 \subset l'$ such that $u_*(l') - u(k_1) \leq \epsilon/3$. Let $k = k_0 \cup k_1$. Then since $u$ is $K$ inner regular, $|u(k) - u(b)| \leq \frac{2}{3}\epsilon$. Then $u_*(l') - u(b) < \epsilon$. Hence $u^{**}(b) = u(b)$.

A set function $v$ on a collection of subsets $\mathcal{H}$ is $\sigma$-finite with respect to $\mathcal{L} \subset \mathcal{H}$ if for every $h \in \mathcal{H}$, $h \subset \bigcup_l^\infty s_i$ where $s_i \in \mathcal{L}$ and $v(s_i)$ is finite for
all \( j \). Note that since our measure \( u \) is finite on \( K \), \( u \) is \( \sigma \)-finite with respect to \( R(K) \) when \( u \) is defined on \( R(\sigma)(K) \).

**Theorem 3.5.** Suppose \( L \) is closed under countable intersections. Suppose \( u_* \) is modular and \( \sigma \)-smooth on \( L' \) where \( u \) is a countably additive measure defined on

(i) \( \sigma(K) \), \( \sigma \)-finite with respect to \( A(K) \). If \( u^* \) is finite on \( K \) then it is a countably additive extension of \( u \) to \( \sigma(L) \).

(ii) \( R(\sigma)(K) \). If \( u^* \) is finite valued on \( K \) then \( u^* \) is a countably additive extension of \( u \) to \( \sigma(L) \).

**Proof.** If \( u \) is a countably additive, \( \sigma \)-finite measure on a \( \sigma \)-ring generated by a ring \( S \) then for all \( b \) in the \( \sigma \)-ring,

\[
  u(b) = \inf \left\{ \sum_{i=1}^{\infty} u(s_i)/b \subseteq \bigcup_{i=1}^{\infty} s_i, s_i \in S \right\}.
\]

That \( u = u^* \) follows in both cases from this and Theorem 3.4.

Define \( K \cap L = \{ k \cap l/k \in K, l \in L \} \).

**Corollary 3.6.** Suppose \( u \in M,(K) \) and \( u_* \) modular on \( L' \). Then \( u^* \) is a \( K \cap L \) regular extension of \( u \) to \( A(L) \). If \( u \) is countably additive, \( u_* \) \( \sigma \)-smooth on \( L' \), and \( L \) closed under countable intersections then \( u^* \) is a countably additive measure on \( \sigma(L) \).

### 4. Coallocation and the extension of \( K \) inner regular measures.

We will assume throughout this section that \( K \subseteq L \), and that any measure on \( K \) (or \( A(K), R(K), \sigma(K), R_{\sigma}(K) \)) is finite valued on \( K \). In most examples we consider there should be no confusion as to which lattice is used for \( L \) in the definition of \( u^* \). We specify this lattice only occasionally.

Allocation is defined as in the introduction. A lattice \( L \) coallocates \( K \) if \( L' \) allocates \( K \). Though Dubin’s paper deals with bounded measures on a lattice, we state his theorem for any extended real valued measure. His proof remains valid despite the change.

**Theorem 4.1.** Let \( \emptyset \in H \); and let \( J \) be any other lattice (\( J \) need not contain \( \emptyset \)). The following two statements are equivalent.

(i) For every measure \( u \) on \( H \), \( u_* \) is a measure on \( J \).

(ii) \( J \) allocates \( H \).

**Proof.** Assume \( J \) allocates \( H \). Choose from \( J \) any \( j_1, j_2 \) and choose from \( H \) \( h \subseteq j_1 \cup j_2 \), \( l \subseteq j_1 \cap j_2 \). Then since \( J \) allocates \( H \) there exists
\( p_1, p_2 \in H \) such that

1. \( p_1 \subseteq j_1, p_2 \subseteq j_2 \)
2. \( p_1 \cup p_2 = h \cup l, \, p_1 \cap p_2 \supseteq l \).

Therefore \( u_*(j_1 \cup j_2) + u_*(j_1 \cap j_2) \leq u_*(j_1) + u_*(j_2) \). The reverse inequality is always true. Thus (ii) implies (i).

Assuming \( J \) does not allocate \( H \), Dubins constructed a measure \( u \) on \( H \) for which \( u_\ast \) is not a measure on \( J \). Thus (i) implies (ii).

**Corollary 4.2.** Suppose \( L \) coallocates \( K \) and \( u \in M(K) \). Define \( u^{**} \) with respect to \( L \).

(i) \( u^{**} \) is a complete measure on \( \mathcal{E}(u, L') \supseteq A(L) \) and is the smallest \( L' \) outer regular measure on \( \mathcal{E}(u, L') \) such that \( u^{**} \equiv u \) on \( K \).

(ii) If \( u \in M_r(K) \) then \( u^{**} \in M_r(L) \) and \( u^{**} = u \) on \( A(K) \).

(iii) If \( u \in M^s_r(K) \), \( u_\ast \sigma \)-smooth on \( L' \) and \( L \) closed under countable intersections then \( u^{**} \in M^s_r(L) \) and \( u^{**} = u \) on \( A(K) \).

For any lattice \( K, R(K) \) is an ideal in \( A(K) \), i.e. \( r \cap a \) belongs to \( R(K) \) whenever \( r \in R(K), a \in A(K) \). Thus \( A(K) \) coallocates \( R(K) \). Hence for any \( K \) inner regular measure \( u \) on \( R(K) \), \( u^{**} \) defined with respect to \( A(K) \) is an extension of \( u \) to \( A(K) \). Since \( u^{**} = u_\ast \), the extension is \( K \) inner regular.

In many instances the lattice \( K' \) separates the lattice \( L \). A lattice \( H \) separates \( L \) if whenever \( l_1 \cap l_2 = \emptyset \), there exists disjoint sets \( h_1, h_2 \) such that \( h_1 \supseteq l_1, h_2 \supseteq l_2 \). \( H \) coseparates \( L \) if \( H' \) separates \( L \).

**Theorem 4.3.** Suppose \( K \) coseparates \( L \) and \( K \subseteq L \). Then \( L \) coallocates \( K \).

**Proof.** Suppose \( l'_1 \cup l'_2 \supseteq k \). Then \( l_1 \cap k, l_2 \cap k \) are disjoint members of \( L \). Since \( K \) coseparates \( L \) there exist disjoint sets \( k'_1, k'_2 \) containing \( l_1 \cap k \) and \( l_2 \cap k \) respectively. Since \( k_1 \subseteq k'_1 \cap l'_1, k_1 \cap k \subseteq l'_1 \). Similarly \( k_2 \cap k \subseteq l'_2 \). Now \( (k_2 \cap k) \cup (k_1 \cap k) = k \).

Let \( X \) be a topological space. We give the following notation for some natural lattices occurring in \( X \). \( \mathcal{F} \) is the lattice of closed sets, \( \mathcal{I} \) is the lattice of zero sets, \( \mathcal{K} \) is the lattice of compact sets, and \( \mathcal{K}_\delta \) is the lattice of compact \( G_\delta \) sets. If \( X \) is a normal space then \( \mathcal{I} \) coseparates \( \mathcal{F} \) by Urysohn's lemma. Hence every \( u \in M_r(\mathcal{I}) \) extends to \( u^{**} \in M_r(\mathcal{F}) \).

\( \mathcal{F} \) coseparates itself in a normal space and \( \mathcal{I} \) coseparates itself in an arbitrary topological space. Consequently for any \( u \in M(\mathcal{I}) \), in any space \( X, u^{**} \) is the smallest outer regular measure on \( A(\mathcal{I}) \) such that \( u^{**} \equiv u \). Here \( u^{**} \) is defined with respect to \( \mathcal{I} \).
It will follow from the next theorem that $\mathcal{F}$ coallocates $\mathcal{H}_8$ in any completely regular Hausdorff space.

**Definition 4.4.** A lattice $K$ is an $L$-ideal if $K \cap L \subseteq K$. $K \cap L = \{k \cap l/k \in K, l \in L\}$.

**Theorem 4.5.** Let $K$ be an $H$-ideal where $K \subseteq H \subseteq L$. If $H$ coseparates $K \cap L$ then $L$ coallocates $K$.

**Proof.** Let $l_1' \cup l_2' \supseteq k$. Then $(k \cap l_1) \cap (k \cap l_2) = \emptyset$. There exists $h_1'$ and $h_2'$ which are disjoint and contain $k \cap l_1$, and $k \cap l_2$ respectively. Then $h_1 \cap k \subseteq l_1'$, $h_2 \cap k \subseteq l_2'$ and $(h_1 \cup h_2) \cap k = k$. Since $K$ is an $H$ ideal, $L$ coallocates $K$.

In a completely regular Hausdorff space $\mathcal{H}_8$ is a $\mathcal{F}$-ideal. $\mathcal{F}$ coseparates the compact sets and therefore $\mathcal{F}$ certainly coseparates $\mathcal{H}_8 \cap \mathcal{F}$. Hence $\mathcal{F}$ coallocates $\mathcal{H}_8$. Therefore we have the following.

**Theorem 4.6.** Let $X$ be a completely regular Hausdorff space. Suppose $u \in M^*(\mathcal{H}_8)$. Then $u^{**} \in M^*(\mathcal{F})$ and is a $\mathcal{H}_8 \cap \mathcal{F}$-regular extension of $u$ to $\sigma(\mathcal{F})$.

**Proof.** That $u^{**}$ is $\sigma$-smooth follows from the fact that $\mathcal{H}_8 \cap \mathcal{F}$ is a compact lattice (any collection $\{f_a\}$ from the lattice has a nonempty intersection whenever every finite subcollection has a nonempty intersection). The rest of the theorem follows from Corollary 4.2.

The following definition is useful in determining when $u^{**}$ is countably additive.

**Definition 4.7.** $L$ countably allocates $K$ if whenever $k \subseteq \bigcup_i^* l_i$ then there exist $k_i \in K$ such that each $k_i$ is contained in a finite union of the $l_i$ and $\bigcup_i^* k_i = k$. If $L'$ countably allocates $K$ then $L$ countably coallocates $K$.

**Theorem 4.8.** Suppose $L$ countably coallocates $K$. Consider a countably additive measure $u$ on $\sigma(K)$ (or $R_\sigma(K)$). Then $u^*$ is $\sigma$-smooth on $L'$.

**Proof.** Suppose $l' = \bigcup_i^* l_i'$ and $\bigcup_i^* l_i' \subseteq L'$. Choose $k \subseteq l'$. There exist $k_i \in K$ such that $k_i \subseteq \bigcup_i^* l_i'$ for some $n$ and $\bigcup_i^* k_i = k$. Since $u$ is countably additive, $\lim_i u(k_i) = u(k)$. Thus $u^*(l') \leq \lim_i u^*(\bigcup_i^* l_i')$. The reverse inequality is always true.

In a locally compact Hausdorff space if $k \subseteq \bigcup_i^* o_i$ where the $o_i$ are
open, then \( k = \bigcup_{i} k_{i}, k_{i} \in \mathcal{H}_{o} \) and \( k_{i} \subseteq o_{i} \) for some \( j \). Thus \( \mathcal{F} \) countably coallocates \( \mathcal{H}_{o} \). Also for every \( k \in \mathcal{H}_{o} \) \( k \subseteq z_{1} \subseteq k_{1} \) where \( z_{1} \) is a zero set and \( k \in \mathcal{H}_{o} \). Applying Theorems 4.8, 3.3 and 3.5 we obtain the following.

**Theorem 4.9.** Let \( X \) be a locally compact Hausdorff space. Every countably additive measure \( u \) on \( R_{o}(\mathcal{H}_{o}) \), is \( \mathcal{H}_{o} \)-inner regular. \( u^{**} \) is a countably additive extension of \( u \) to \( \sigma(\mathcal{F}) \).

**Proof.** All that has to be shown is that \( u \) is \( \mathcal{H}_{o} \)-inner regular. This follows from the fact that for each \( b \in R(\mathcal{H}_{o}), b = \bigcup_{i} k_{i}, k_{i} \in \mathcal{H}_{o} \).

Levin and Stiles [8] showed that the conclusions of Theorem 4.9 no longer are true if \( R_{o}(\mathcal{H}_{o}) \) is replaced by \( \sigma(\mathcal{H}_{o}) \) even if \( X \) is locally compact and Hausdorff. Suppose \( X \) is locally compact, paracompact and Hausdorff. Levin and Stiles prove that for any countably additive measure \( u \) on \( \sigma(\mathcal{H}_{o}) \) \( u(b) = \inf\{u(o)/b \subseteq o, o \text{ open and } o \in \sigma(\mathcal{H}_{o})\} \). Thus if \( u \) is also \( \mathcal{H}_{o} \)-inner regular then \( u^{**} \) must be a countably additive extension of \( u \) to \( \sigma(\mathcal{F}) \) according to Theorem 3.3. This result is found in the paper of Levin and Stiles.

In a countably paracompact, normal space the lattice \( \mathcal{F} \) countably coallocates \( \mathcal{I} \). In any topological space, for every zero set \( z, z \subseteq z_{1} \subseteq z_{2} \) where \( z_{1}, z_{2} \) are zero sets. Thus we obtain Marik's [9] result.

**Theorem 4.10.** Every countably additive measure \( u \) on \( \sigma(\mathcal{F}) \) is \( \mathcal{I} \)-inner regular. If \( X \) is countably paracompact and normal then \( u^{**} \) is a countably additive extension of \( u \) to \( \sigma(\mathcal{F}) \).

Let \( X \) be a countable product, \( \prod_{i} X_{i} \), of discrete topological spaces. Define for \( x = (x_{1}, \cdots, y = (y_{1}, \cdots, y = x(m n) \) if \( x_{i} = y_{i}, i = 1, \cdots, n \). For any subset \( A \) of \( X \) define \( t_{A}(x) \) to be the least positive integer \( n \), if any, such that \( y \in A \) whenever \( y = x(m n) \). If there exists no such \( n \) then let \( t_{A}(x) = +\infty \). Suppose \( C \subseteq \bigcup_{i} O_{k} \) where \( C \) is a clopen set (both closed and open in \( X \)) and each \( O_{k} \) is open. Define inductively

\[
C_{1} = \{c \in C/t_{O_{k}}(c) \leq t_{O_{k}}(c), k \neq 1\},
\]
\[
C_{n} = \{c \in C/(\bigcup_{i} C_{i})/t_{O_{k}}(c) \leq t_{O_{k}}(c), k \neq n\}.
\]

Then \( C = \bigcup_{i} C_{k}, C_{k} \subseteq O_{k} \) for all \( k \) and each \( C_{k} \) is clopen. Thus \( \mathcal{F} \) countably coallocates \( \mathcal{C} \), the lattice of clopen sets. Dubins is interested.
in measures defined on $A(\mathcal{E})$. These measures are called strategic measures. Strategic measures are always $\mathcal{E}$-inner regular.

**Theorem 4.11.** Let $X$ be a countable product of discrete topological spaces. For every countably additive strategic measure $u$, $u^{**}$ is a countably additive extension of $u$ to $\sigma(\mathcal{F})$.

Let $R$ be a ring of subsets in $X$. Define $\mathcal{L}(R)$ to be those subsets $b$ such that $b \cap r \in R$ for every $r \in R$. $\mathcal{L}(R)$ is an algebra containing $R$. $\mathcal{L}(R)$ certainly coallocates $R$ and if $R$ is a $\sigma$-ring then $\mathcal{L}(R)$ is an $\sigma$-algebra that countably coallocates $R$. For a measure (not necessarily finite valued on $R$) define $u_*(b) = \sup\{u(r)/r \subseteq b, r \in R\}$, and $u^{**}$ with respect to $\mathcal{L}(R)$. It is easy to see that $u^{**} = u_*$ on $\mathcal{L}(R)$. By Theorem 3.3 $u_*$ is an extension of $u$ to $\mathcal{L}(R)$. By Theorems 4.8 and 3.3 if $R$ is a $\sigma$-ring and $u$ is countably additive then $u_*$ is countably additive on $\mathcal{L}(R)$. $\mathcal{L}(R)$ is called the class of sets locally measurable with respect to $R$. The result for countably additive measures on a $\sigma$-ring is found in a paper by Berberian [2].

If $K \subseteq L$ is an $L$-ideal, then $A(L) \subseteq \mathcal{L}(R(K))$. Clearly $l \cap r$ belongs to $R(K)$ for all $l \in L$ and $r \in R(K)$. Suppose $b \cap r$ and $c \cap r$ belong to $R(K)$ for all $r \in R(K)$. Then $(b \cup c) \cap r$ belongs to $R(K)$ for all $r \in R(K)$. If $b \cap r \in R(K)$ then $b' \cup r'$ is in $A(K)$. Therefore $r \cap b' = r \cap (b' \cup r')$ belongs to $R(K)$. Thus $A(L)$ is contained in $\mathcal{L}(R(K))$. Also $\sigma(L)$ is contained in $\mathcal{L}(R_\sigma(K))$. Thus in a Hausdorff space $\sigma(\mathcal{F})$ is contained in the locally measurable sets of $R_\sigma(\mathcal{H})$ where $\mathcal{H}$ is the lattice of compact sets [Berberian and Jakobsen 3]. In a completely regular Hausdorff space $\sigma(\mathcal{F})$ is contained in the locally measurable sets of $R_\sigma(\mathcal{H})$. We also have, for any lattice $K$, $A(K) \subseteq \mathcal{L}(R(K))$ and $\sigma(K) \subseteq \mathcal{L}(R_\sigma(K))$. In the following theorems the measures need not be finite on any particular set.

**Theorem 4.12.** Any measure on $R(K)$ extends to a $R(K)$ inner regular measure on $A(K)$. Any countably additive measure on $R_\sigma(K)$ extends to a $R_\sigma(K)$ inner regular, countably additive measure on $\sigma(K)$.

**Theorem 4.13.** In a Hausdorff space any countably additive measure on $R_\sigma(\mathcal{H})$ has a countably additive, $R_\sigma(\mathcal{H})$ inner regular extension to $\sigma(\mathcal{F})$. In a completely regular Hausdorff space any countably additive measure on $R_\sigma(\mathcal{H}_5)$ can be extended to a countably additive, $R_\sigma(\mathcal{H}_5)$ inner regular measure on $\sigma(\mathcal{F})$.

**Theorem 4.14.** Let $K \subseteq L$ be a $L$-ideal. Then for every $R(K)$ inner regular measure on $A(K)$ has a $R(K)$ inner regular extension to
Every countably additive, $R_\sigma(K)$ inner regular measure on $\sigma(K)$ has a countably additive, $R_\sigma(K)$ inner regular extension to $\sigma(L)$.

In view of Theorem 4.14 the next example shows that coallocation is not necessary for every $K$ inner regular measure $u$ on $A(K)$ to have $u_*$ modular on $L'$. Also countable coallocation is not implied if $u_*$ is $\sigma$-smooth on $L'$ for every countably additive $K$ inner regular measure on $R_\sigma(K)$.

Topologize the set of real numbers as follows. For $x \neq 0$ or 2 a neighborhood basis for $x$ is the collection of open intervals containing $x$. A neighborhood basis for 0 is the collection of open intervals containing 0 and 1. Likewise a neighborhood basis for 2 is the collection of open intervals containing 1 and 2. The interval $[0, 2]$ is a compact closed set and the intervals $I_1 = (-1, 3/2)$ and $I_2 = (1/2, 3)$ are open sets. There does not exist a sequence $\{C_n\}$ of closed, compact sets such that $\bigcup_{n=1}^\infty C_n = [0, 2]$ and each $C_n$ is contained in either $I_1$ or $I_2$. Therefore the closed sets $\mathcal{F}$ do not coallocate or countably coallocate the lattice of compact closed sets though this lattice is an $\mathcal{F}$-ideal.

5. The extension of $\tau$-smooth measures. A measure on a lattice $L$ is $\tau$-smooth if for any net $\{l_\alpha\}$ decreasing to $\emptyset$, $\lim_\alpha u(l_\alpha) = 0$. We will study the measures on $A(L)$ which are $L$ inner regular, finite valued on $L$ and $\tau$-smooth on $L$. Denote these measures by $\mathcal{M}_\tau(L)$. $\mathcal{M}_\tau(L)$ are those measures in $\mathcal{M}_\tau(L)$ which are bounded.

For a lattice $L$, $\tau(L)$ is the smallest lattice containing $L$ that is closed under arbitrary intersections. We now show that every $u \in \mathcal{M}_\tau(L)$ extends to $u^{**}$, defined with respect to $\tau(L)$ on $A(L)$, and $\tau$-smooth on $\tau(L)$.

Lemma 5.1. Let $u$ be a measure on $A(L)$, $\tau$-smooth on $L$. For any $t$ in $\tau(L)$,

$$u_*(t') = \lim_\alpha u(l'_\alpha)$$

where $t' = \bigcup_\alpha l'_\alpha$ and $\{l'_\alpha\}$ is an increasing net of sets from $L'$.

Proof. Choose $l \subseteq t'$. Since $t \in \tau(L)$ there exists a net $\{l'_\alpha\}$ from $L'$ which is increasing and $\bigcup_\alpha l'_\alpha = t'$. Since $u$ is $\tau$-smooth, $\lim_\alpha u(l'_\alpha) = u(l) + \lim_\alpha u(l'_\alpha \cap l')$. Therefore $u_*(t') = \lim_\alpha u(l'_\alpha)$.

Theorem 5.2. Suppose $u$ is a measure on $A(L)$, $\tau$-smooth on $L$. Then $u_*$ is modular on $\tau(L)'$. 

Proof. Let \( s, t \in \tau(L) \). Then \( s' = \bigcup a h_a, t' = \bigcup \beta l_\beta \) where \( \{h_a\}, \{l_\beta\} \) are increasing nets from \( L' \).

Form the net \( \{k'_\gamma\} \) of unions \( k'_\gamma = h'_a \cup l'_\beta \). For the same \( \gamma, \alpha, \) and \( \beta \) define \( p'_\gamma = h'_a \cap l'_\beta \). \( \{k'_\gamma\} \) is a net increasing to \( t' \cup s' \) and \( \{p'_\gamma\} \) is a net increasing to \( t' \cap s' \). Thus

\[
U_*(t' \cup s') + U_*(t' \cap s') = \lim_\gamma (u(k'_\gamma) + u(p'_\gamma)) = \lim_\gamma (u(h'_a) + u(l'_\beta)) \leq u_*(t') + u_*(s').
\]

**Theorem 5.3.** Let \( u \in M^r_\mu(L) \). If \( u** \) is finite on \( L \) then it extends \( u \) to \( A(\tau(L)) \) and belongs to \( M^r(\tau(L)) \).

**Proof.** \( u** \) extends \( u \) according to Theorems 5.2 and 3.4. \( u** \) is \( \tau \)-smooth and finite on \( \tau(L) \) since each \( t \in \tau(L) \) is the intersection of sets from \( L \). Consider \( t, s \) from \( \tau(L) \). Choose \( v \) from \( \tau(L) \) such that \( s \subseteq v' \) and \( u**(v') - u**(s) < \epsilon \). Then \( u**(t \cap s') - u**(t \cap v) < \epsilon \). Every set in \( A(\tau(L)) \) is of the form \( \bigcup \alpha t_k \cap s'_k \) where \( s_k \) belongs to \( \tau(L) \) and either \( t_k \in \tau(L) \) or \( t_k = X \). Therefore \( u** \) is \( \tau(L) \) inner regular.

**Corollary 5.4.** Let \( u \) be a \( L \) inner regular, countably additive measure on \( R_\sigma(L), \tau \)-smooth and finite on \( L \). If \( u** \) is finite on \( L \) then \( u** \) is a countably additive extension of \( u \) to \( \sigma(\tau(L)) \) and \( u** \) is \( \tau \)-smooth and finite on \( \tau(L) \).

**Corollary 5.5.** Suppose \( X \) is a completely regular space. Suppose \( u \) is a \( L \) inner regular, countably additive measure on \( \sigma(X) \) that is \( \tau \)-smooth and finite on \( X \). Then \( u** \) is a countably additive extension of \( u \) to \( \sigma(X) \) and \( u** \) is \( \tau \)-smooth and finite on \( X \).

A collection of sets has the finite (countable) intersection property if every finite (countable) subcollection has a nonempty intersection. A lattice \( L \) is compact if every collection with the finite intersection property has a nonempty intersection. \( L \) is Lindelof if every collection with the countable intersection property has a nonempty intersection. A measure on a compact lattice is always \( \tau \)-smooth and any \( \sigma \)-smooth measure on a Lindelof lattice is \( \tau \)-smooth. \( M_r(L) \) are the \( L \) inner regular measures on \( A(L) \) that are finite on \( L \) and \( M^r_\mu(L) \) are those that are also \( \sigma \)-smooth on \( L \).
COROLLARY 5.6. If $L$ is compact then every $u \in \mathcal{M}(L)$ for which $u^\ast$ is finite on $L$ extends to $u^\ast \in \mathcal{M}(\tau(L))$. If $L$ is Lindelof then for every $u \in \mathcal{M}(L)$ such that $u^\ast$ is finite on $L$, $u^\ast \in \mathcal{M}(\tau(L))$ and extends $u$.

The result concerning compact lattices has been proved by using Zorn's lemma to show that $u^\ast$ on $A(\tau(L))$ is, in an appropriate sense, a maximal extension of $u$ [P. A. Meyer 10].

Suppose $u$ is a $L'$ outer regular measure on $A(L)$. Then for any decreasing net $\{l_\alpha\}$ from $L$ such that $\bigcap_\alpha l_\alpha \in A(L)$, $\lim_\alpha u(l_\alpha) = u(\bigcap_\alpha l_\alpha)$. If $L$ is a regular lattice then this property is a sufficient condition for a measure $u$ to be $L'$ outer regular.

DEFINITION 5.7. $L$ is $K$ regular if for any $l \in L$ there exists $\{h_\alpha\}$ from $L$ such that $l = \bigcap_\alpha h_\alpha$ and for each $\alpha$ there exists $k_\alpha$ from $K$ such that $l_\alpha \subseteq k_\alpha \subseteq h_\alpha$. If $L = K$ then $L$ is a regular lattice.

THEOREM 5.8. Assume $L$ is $K$ regular and that $K \subseteq A(L)$. If for any net $\{l_\alpha\}$ decreasing to $\bigcap_\alpha l_\alpha \in A(L)$, $\lim_\alpha u(l_\alpha) = u(\bigcap_\alpha l_\alpha)$ then $u$ is $K'$ outer regular on $L$. If $K = L$ then $u$ is $L'$ outer regular on $A(L)$. In addition, if $u$ is finite on $L$ and $L$ is regular, then $u$ is $L$ inner regular on $A(L)$.

Proof. The collection $\{l_\alpha\} \subseteq L$ such that $l_\alpha \supseteq k_\alpha \supseteq l$, is a net decreasing to $l$. Therefore

$$u(l) \leq \inf \{u(k_\alpha)/l \subseteq k_\alpha \subseteq l_\alpha\}$$

$$\leq \inf \{u(l_\alpha)/l \subseteq k_\alpha \subseteq l_\alpha\}$$

$$= u(l).$$

To give a similar result for measures on $\sigma(L)$ we need the following theorem. $\delta(L)$ is the smallest lattice containing $L$ closed under countable intersections.

THEOREM 5.9. Let $u$ be a countably additive, $\sigma$-finite measure on a ring $R$ containing $L$. If $u$ is $L$ inner regular then the countably additive extension of $u$ to $R_\sigma(R)$ is $\delta(L)$ inner regular.

Proof. Let $S$ be the collection of sets $s$ in $R_\sigma(R)$ such that $u(s) = \sup\{u(k)/l \subseteq s, \ l \in \delta(L)\}$. Then $R \subseteq S$. Let $\{s_\alpha\}$ be any sequence from $S$ such that $u(s_\alpha)$ is finite for all $k$. Then since $u$ is countably additive, $\bigcup_\alpha s_\alpha$ and $\bigcap_\alpha s_\alpha$ belong to $S$. 
Take any set \( b \) in \( R_\sigma(R) \) such that \( u(b) \) is finite. There exists a sequence \( \{r_k\} \) from \( R \) such that \( r = \bigcup_{i=1}^\infty r_k \) contains \( b \) and \( u(r) - u(b) < \epsilon \). There exists \( \{t_k\} \) from \( R \) such that \( t = \bigcup_{i=1}^\infty t_k \) contains \( r \setminus b \) and \( u(t) < \epsilon \). Then \( r \setminus t \subseteq b \) and \( u(b) - u(r \setminus t) < \epsilon \). For each \( k \), \( r \setminus t_k = \bigcup_{j=1}^\infty r_j \setminus t_k \) belongs to \( S \). Since \( r \setminus t = \bigcap_{i=1}^\infty r_i \setminus t_k \), \( r \setminus t \) belongs to \( S \). This implies that \( b \) belongs to \( S \).

Every \( b \in R_\sigma(R) \) is the countable union of sets \( b_k \) such that \( u(b_k) \) is finite. Therefore \( R_\sigma(R) = S \). A similar proof shows the extension of \( u \) is \( \delta(L)' \) outer regular when \( u \) is \( L' \) outer regular.

**Theorem 5.10.** Suppose \( L \) is a regular lattice. Let \( u \) be a countably additive, \( \sigma \)-finite measure on \( \sigma(L) \), finite on \( L \). If for any net \( \{l_\alpha\} \) decreasing to \( \bigcap_\alpha l_\alpha \in A(L) \), \( \lim_\alpha u(l_\alpha) = u(\bigcap_\alpha l_\alpha) \), then \( u \) is \( \delta(L) \) regular on \( \sigma(L) \).

**Corollary 5.11.** Let \( X \) be a topological space and \( u \) a countably additive, finite measure defined on \( \sigma(\mathcal{F}) \) such that for any decreasing net of closed sets \( \{f_\alpha\} \)

\[
\lim_\alpha u(f_\alpha) = u\left(\bigcap_\alpha f_\alpha\right).
\]

(i) If \( X \) is a regular space then \( u \) is \( \mathcal{F} \) regular.
(ii) If \( X \) is completely regular then \( u \) is \( \mathcal{F} \cap \mathcal{F}' \)-regular and for every closed set \( f \)

\[
u(f) = \inf\{u(z')/f \subseteq z', z \in \mathcal{F}\}.
\]

(iii) If \( X \) is 0-dimensional then \( u \) is \( \mathcal{F} \cap \mathcal{C}l \) regular where \( \mathcal{C}l \) is the lattice of clopen sets and for every closed set \( f \)

\[
u(f) = \inf\{u(c)/f \subseteq c, c \text{ clopen}\}.
\]

(iv) If \( X \) is a locally compact Hausdorff space then \( u \) is \( \mathcal{K}_8 \cap \mathcal{F} \) regular and for every closed set \( f \)

\[
u(f) = \inf\{u(k')/f \subseteq k', k \in \mathcal{K}_8\}.
\]

**Corollary 5.12.** Suppose \( X \) is a locally compact Hausdorff space and \( u \) a countably additive, finite measure on \( \sigma(\mathcal{F}) \) such that for any decreasing net \( \{z_\alpha\} \) of zero sets, where \( \bigcap_\alpha z_\alpha \in A(\mathcal{F}) \),
\[ \lim_{a} u(z_a) = u \left( \bigcap_{a} z_a \right). \]

Then \( u \) is \( \mathcal{H}_b \) regular.

Part (i) of 5.11 was proven by Gardner [6].

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