COHOMOLOGY OF DEGREE 1 AND 2 OF THE SUZUKI GROUPS

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Let $V$ be the standard 4-dimensional module for $Sz(q)$, the Suzuki group based on the field of $q = 2^{2n+1}$ elements. In this paper we determine $H^2(Sz(q), V)$. This is usually $(q^32)$ of dimension one (otherwise zero) and is generated by a cocycle which is the restriction of a generator of $H^2(Sp_4(q), V)$. In addition, the well known groups $H^2(Sz(q), GF(q))$ and $H^1(Sz(q), V)$ are calculated. The proof involves the use of the Hochschild–Serre spectral sequence to determine the cohomology of the normalizer of a Sylow 2-subgroup acting on the various one-dimensional modules involved.

Let $K = GF(q)$, $q = 2^{2n+1}$, let $Sz(q)$ ($^2B_2(q)$) be the Suzuki group based on the field $K$ and let $B$ be a normalizer of a Sylow 2-subgroup of $Sz(q)$. In this paper we use the Hochschild–Serre spectral sequence to determine $H^i(B, V)$ $i = 1, 2$, where $V$ is a one dimensional $KB$-module, in terms of the solutions to certain equations in $\text{End}(K^*)$. These equations are solved when $V$ is trivial or involved in $K^4$, the standard four dimensional module for $KSz(q)$. Using this information we determine $H^2(Sz(q), K^4)$ as well as the previously known groups $H^2(Sz(q), K)$ and $H^1(Sz(q), K^4)$. These may be viewed as results concerning conjugacy classes in semi-direct products and concerning exact sequences of groups using the well known group-theoretic interpretation of cohomology of degree 1 and 2 [6].

We will assume all cocycles are normalized, i.e. vanish when any one of their arguments is the identity. When $[f] \in H^2(G, V)$, where $G$ is a group and $V$ is a left $G$-module, let $E(f)$ denote the extension of $V$ by $G$ using $f$, that is, $E(f) = \{(v, g) | v \in V, g \in G\}$ with multiplication $(v_1, g_1)(v_2, g_2) = (v_1 + g_1(v_2) + f(g_1, g_2), g_1g_2)$.

We use the explicit description of $Sz(q)$ given in [9]. Let $K_0$ be the prime subfield of $K$, $\Gamma = \text{Gal}(K/K_0)$ and $\theta \in \Gamma$ defined by $\theta: x \rightarrow x^{2^r}$. For $\alpha, u \in K$ and $t \in K^*$ put

$$J = \begin{bmatrix} 1 \\ t^{\theta} \\ t^{1-\theta} \\ t^{-\theta} \end{bmatrix}, \quad J = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & u^\theta & h & g \\ 1 & u & \alpha \\ 1 & u^\theta \\ 1 \end{bmatrix} = \begin{bmatrix} t^\theta \\ t^{1-\theta} \\ t^{-\theta} \\ 1 \end{bmatrix}$$
where \( h = h(\alpha, u) = u^{\theta+1} + \alpha \) and \( g = g(\alpha, u) = u^{2\theta+1} + u^\theta \alpha + \alpha^{2\theta} \). Set \( U = \{ (\alpha, u) \mid \alpha, u \in K \} \), \( T = \{ T(t) \mid t \in K^* \} \), \( B = UT \) so \( Sz(q) = \langle B, J \rangle \subset SL_4(q) \) (in \([9]\), \( U' \) is used in place of \( U \)). Then \( K^4 \) (columns) is the standard module on which \( Sz(q) \) acts as multiplication on the left. In fact \( Sz(q) \) is contained in the Symplectic group defined by \( J \).

Since \( U \) is a Sylow 2-subgroup of \( Sz(q) \) which is a T. I. set with normalizer \( B \), the Cartan–Eilenberg stability theorem tells us that if \( V \) is a \( KSz(q) \)-module then the restriction maps \( H^i(Sz(q), V) \to H^i(B, V) \to H^i(U, V)^T \) are isomorphisms for \( i > 0 \). Thus (after the case \( q = 2 \)) we shall replace \( Sz(q) \) by \( B \). Furthermore these isomorphisms show that when giving explicit cocycles it is sufficient to give their restrictions to \( U \) and show they are \( T \)-stable.

Assume first \( q = 2 \). Then \( Sz(q) \) is a group of order 20. Its Sylow 5-subgroup is cyclic, normal and a generator acts fixed-point-freely on \( K^4 \). This implies \( H^i(Sz(2), K^4) = 0 \) for \( i > 0 \) \([7]\). Henceforth we assume \( q \geq 8 \).

Throughout we assume \( \alpha, \beta, u, v \in K \) and \( t \in K^* \). We identify \( T \) with \( K^* \) via \( T(t) \leftrightarrow t \). It is seen that \( (\alpha, u)(\beta, v) = (\alpha + \beta + uv^\theta, u + v) \) and \( (\alpha, u)^{T(t)} = T(t)(\alpha, u)t^{-1}(\alpha, u)^{-1}(t\alpha, t\theta' u) \) where \( \theta' = 2 - 2\theta \). Also \( Z = \{ (\alpha, 0) \} \) is the center and derived subgroup of \( U \). Set \( A = U/Z \) and \( X = B/Z \) so \( X \) is the semidirect product \( AT \).

When \( V \) is a \( KT \)-module and \( \nu \in \text{End}(K^*) \) we say \( T \) acts with weight \( \nu \) on \( V \) provided \( T(t)v = t^\nu v \) for all \( t \in K^* \), \( v \in V \). The above formulas show \( Z \) and \( A \) are \( KT \)-modules of weight 1 and \( \theta' \) respectively. Observe \( \text{End}(K^*) = \mathbb{Z}/(q - 1)\mathbb{Z} \) and so is a commutative ring.

When \( V \) and \( W \) are (finite dimensional) \( K \)-modules \( \text{Hom}(W, V) = \bigoplus_{\sigma \in \mathbb{G}} H_\sigma(W, V) \) where \( H_\sigma(W, V) \) are the \( \sigma \)-semilinear maps from \( W \) to \( V \). If additionally \( V \) and \( W \) are \( KT \)-modules of weight \( \nu \) and \( \omega \) then \( H_\sigma(W, V) \) is a \( KT \)-module of weight \( \nu - \omega \sigma \).

Now fix \( V \), a one dimensional \( KB \)-module on which \( U \) acts trivially and \( T \) acts with weight \( \nu \). We shall often identify \( V \) with \( K \). From the (nonsplit) exact sequence of groups \( 1 \to Z \to B \to X \to 1 \) the Hochschild–Serre spectral sequence gives us the exact sequences of \( K \)-modules

\[
0 \to H^2(B, V)_0 \to H^2(B, V) \to H^2(Z, V)^X
\]

\( (\ast) \)

\[
0 \to H^1(X, V) \to H^1(B, V) \to H^1(Z, V)^X \to H^2(X, V)
\]

\[
\to H^2(B, V)_0 \to H^2(X, V) \to H^2(Z, V)^X \to H^3(X, V).
\]

Our aim is to determine \( H^2(B, V) \). In Lemmas 1, 2 and 3 we determine most of the other terms in (\ast) and study the maps \( \text{Res} \) and \( \Phi \).
**Lemma 1.** Let $W$ and $V$ (each identified with $K$) be one dimensional $KT$-modules of weight $\omega$ and $\nu$ respectively and regard $V$ as a trivial $W$-module. For $\sigma, \tau \in \Gamma$ define $h_\sigma: W \to V$ by $h_\sigma(w) = w^\sigma$ and $f_{(\sigma, \tau)}: W \times W \to V$ by $f_{(\sigma, \tau)}(w_1, w_2) = w_1^\sigma w_2^\tau$.

(a) $\{[h_\sigma]| \nu = \omega \sigma \} \sigma \in \Gamma$ is a $K$-base for $H^1(W, V)^T$.

(b) $\{[f_{(\sigma, \tau)}]| \nu = \omega (\sigma + \tau)\{\sigma, \tau\} \subseteq \Gamma$ is a $K$-base for $H^2(W, V)^T$.

**Proof.** (a) This statement is immediate since $H^1(W, V)^T = \text{Hom}(W, V)^T = \bigoplus H_\sigma(W, V)^T$ and $T$ acts on $H_\sigma(W, V) = Kh_\sigma$ with weight $\nu - \omega \sigma$.

(b) Since $W$ is abelian and trivial on $V$ we have an exact sequence of $KT$-modules $0 \to H^2_{ab}(W, V) \to H^2(W, V) \to \text{Alt}^2(W, V) \to 0$ where $\text{Alt}^2(W, V)$ is the group of alternate 2-forms: $W \times W \to V$ and $\Psi[f]: (w_1, w_2) \to f(w_1, w_2) - f(w_2, w_1)$. Furthermore $H^2_{ab}(W, V) \cong \text{Hom}(W, V)$. See [7] for the proofs of these statements. Taking $T$-cohomology of the above sequence gives the exact sequence of $K$-modules $0 \to H^2_T(W, V)^T \to H^2(W, V)^T \to \text{Alt}^2(W, V)^T \to 0 = H^1(T, \text{Hom}(W, V))$. We have seen $\dim_K \text{Hom}(W, V)^T = \# \{\sigma \in \Gamma| \nu = \omega \sigma\}$ and it can be seen that when $\nu = \omega \sigma$ then $f_{(\sigma/2, \sigma/2)}$ is a corresponding cocycle in $H^2_{ab}(W, V)^T = \text{Hom}(W, V)^T$.

In [5] it is shown that $\text{Alt}^2(W, V) = \bigoplus KF_{(\sigma, \tau)}$ where we sum over all sets $\{\sigma, \tau\} \subseteq \Gamma$, $\sigma \neq \tau$ and $F_{(\sigma, \tau)}: (w_1, w_2) \to w_1^\sigma w_2^\tau - w_1^\sigma w_2^\tau$. Since $T$ acts with weight $\nu - \omega (\sigma + \tau)$ on $KF_{(\sigma, \tau)}$, we have $\text{Alt}^2(W, V)^T = \bigoplus KF_{(\sigma, \tau)}$ summed over those $\{\sigma, \tau\}$ such that $\nu = \omega (\sigma + \tau)$. For such $\{\sigma, \tau\}$ it can be seen that $[f_{(\sigma, \tau)}] \in H^2(W, V)^T$ with $\Psi[f_{(\sigma, \tau)}] = F_{(\sigma, \tau)}$. Note $[f_{(\sigma, \tau)}] + [f_{(\sigma, \tau)}] = 0$ since $f_{(\sigma, \tau)} + f_{(\sigma, \tau)} = \delta g$ where $g(w) = w^{\sigma + \tau}$. This completes the proof.

Using Lemma 1 and the Cartan–Eilenberg stability theorem we can determine the terms of (*). We have $H^1(X, V) \cong H^1(A, V)^T = \text{Hom}(A, V)^T = \text{Hom}(U, V)^T = H^1(U, V)^T = H^1(B, V)$ has $K$-dimension $\# \{\sigma \in \Gamma| \nu = \theta' \sigma\}$. Also $H^1(X, H^1(Z, V)) = \bigoplus H_\sigma(A, H_\tau(Z, V))^T$ (summed over $\{\sigma, \tau\} \subseteq \Gamma \times \Gamma$) has $K$-dimension $\# \{(\sigma, \tau) \in \Gamma \times \Gamma| \nu = \theta' (\sigma + \tau)\}$ and $H^2(X, V) \cong H^2(A, V)^T$ has $K$-dimension $\# \{\{\sigma, \tau\} \subseteq \Gamma| \nu = \theta' (\sigma + \tau)\}$. Since $A$ acts trivially on $Z$ and $V$ we have $H^i(Z, V)^X \cong H^i(Z, V)^T$ has $K$-dimension $\# \{\sigma \in \Gamma| \nu = \sigma\}$ when $i = 1$, and $\# \{\{\sigma, \tau\} \subseteq \Gamma| \nu = \sigma + \tau\}$ when $i = 2$.

**Lemma 2.** If $\nu = \sigma + \tau$ for some $\sigma, \tau \in \Gamma$ assume $\nu$ is invertible in $\text{End}(K^*)$. Then $\text{Res} = 0$ in (*).

**Proof.** First we claim $\dim_K H^2(Z, V)^X \leq 1$. By the previous remarks this is evident if we show $\sigma + \tau = \varphi + \rho$ in $\text{End}(K^*)$, where $\sigma, \tau, \varphi, \rho \in \Gamma$, implies $\{\sigma, \tau\} = \{\varphi, \rho\}$. For this apply both sides to $(x + 1)$,
expand, cancel and see the same equality holds in $\text{End}(K^*)$. The claim follows from Dedekind's lemma.

Thus if $H^2(Z, V)^* \neq 0$ it is generated by some $\tilde{f}$ of the form $\tilde{f}((\alpha, 0), (\beta, 0)) = \alpha^\nu \beta^\tau$ with $\nu = \sigma + \tau$. If $\text{Res} \neq 0$ we can find $f \in Z^2(B, V)$ with $\text{Res} f = \tilde{f}$, that is, $f(\alpha, 0, \beta, 0) = \alpha^\nu \beta^\tau$ (we use $f(\alpha, u, \beta, v)$ for $f((\alpha, u), (\beta, v))$). Let $E = E(f)$, the extension using $f$, and let $\tilde{U}$ be its Sylow 2-subgroup. We show $\tilde{U}$ is a Suzuki 2-group of exponent 8 contradicting a theorem of G. Higman [3]. A Suzuki 2-group is a non-abelian 2-group with more than one involution and an automorphism $\varphi$ with $\langle \varphi \rangle$ transitive on the involutions.

Writing $(a, \alpha, u)$ for $(a, (\alpha, u)) \in \tilde{U}$ we see $(0, 0, 0) = (a, \alpha, u)^2 = (f(a, u, \alpha, u), u^{*+1}, 0)$ implies $u = 0$. Now $f(a, u, \alpha, u) = \alpha^{*+\tau} = 0$ implies $\alpha = 0$. Thus $V^* = \{(a, 0, 0) | a \in K^*\}$ is the set of involutions. There are $q - 1 > 1$ of them. It is easily seen that $(a, \alpha, u)$ is of exponent 8 when $u \neq 0$.

Choose $t$ with $\langle t \rangle = K^*$. Since $\nu$ is invertible in $\text{End}(K^*)$, we have $(1, 0, 0)^{\tau(t)} = \{(t^*, 0, 0) | t \in K^*\} = V^*$. Thus $T(t) \in \text{Aut}(\tilde{U})$ will serve as the required automorphism showing $\tilde{U}$ is a Suzuki 2-group. This completes the proof.

**Lemma 3.** In $(\ast)$ the map $\Phi$ is a surjection $\Leftrightarrow H^1(X, H^1(Z, V)) = 0$.

**Proof.** First we give the description of $\Phi$ as found in [7]. Choose a set splitting $S: X \to B$ with $\pi S = 1_X$, $S(1) = 1$. For $f \in Z^2(B, V)_0 = \{f \in Z^2(B, V) | f|Z \times Z = 0\}$ define $\tilde{f} \in C^1(X, Z^1(Z, V))$ by $\tilde{f}(x)(\alpha) = f(S(x), \alpha^{*+1}) = f(\alpha, S(x))$. Now $\tilde{f}$ induces a well defined map $\Phi$ on the classes (this uses only the fact that $Z$ is abelian).

Now assume $\text{Im} \Phi = H^1(X, H^1(Z, V)) \neq 0$ and choose a nonzero $[d] \in H^1(X, H^1(Z, V)) = \bigoplus H^1(A, H^1(Z, V))^\sigma$ of the form $d(u)(\alpha) = u^\sigma \alpha^\tau$ where $u \in A$, $\alpha \in Z$, $\sigma, \tau \in \Gamma$. Find $[f] \in H^2(B, V)_0$ with $\Phi[f] = [d]$. We no longer need the action of $T$ so replace $f$ by $f|U \times U$. We use $S$ defined by $S(u) = (0, u)$. Since $B^1(A, H^1(Z, V)) = 0$ we may assume $\tilde{f} = d$, that is

$$f(0, u, \alpha, 0) + f(\alpha, 0, 0, u) = u^\sigma \alpha^\tau. \ (1)$$

Let $E = E(f) = \{(a, \alpha, u) | a, \alpha, u \in K\}$, the extension of $V$ by $U$ using $f$, and let $\tilde{Z} = \{(a, 0, 0)\}$. Then $\tilde{Z} \triangleleft E$ and $\tilde{Z}$ is abelian since $f|Z \times Z = 0$. We have an exact sequence of groups $1 \to \tilde{Z} \to E \to A \to 1$. Define $\rho: A \to E$ by $\rho(u) = (0, 0, u)$ and let $g \in Z^2(A, \tilde{Z})$ be the corresponding cocycle, that is, $g(u, v) = \rho(u)\rho(v)\rho(u + v)^{-1}$. All multiplication in $E$ can be performed in terms of $f$ and it can be computed that $g = (g_1, g_2, 0)$ where $g_1(u, v) = f(uv^\sigma, u + v, (u + v)^{\sigma+1}, u + v)$ and $g_2(u, v) = uv^\sigma$. 


Similarly it can be computed that \((b, \alpha, 0)\psi(u) = (b + f(0, u, \alpha, 0) + f(0, u, u\theta^{-1}, u) + f(\alpha, u, u\theta^{-1}, u), \alpha, 0)\). Since \(f \in \mathcal{Z}(U, V)\) we have \(0 = \delta f((\alpha, 0), (0, u), (u\theta^{-1}, u)) = f(\alpha, 0, 0, u) + f(\alpha, u, u\theta^{-1}, u) + f(0, u, u\theta^{-1}, u) + f(\alpha, 0, 0, 0)\). Now use \(f(\alpha, 0, 0, 0) = 0\), equation (1) and the above expression for \((b, \alpha, 0)\psi(u)\) to obtain \((b, \alpha, 0)\psi(u) = (b + u\alpha r^\tau, \alpha, 0)\).

Using this expression for the action of \(A\) on \(\tilde{Z}\) the first slot of the equation \(0 = \delta g(u, v, w)\) implies

\[
0 = g_1(u, v) + g_1(u + v, w) + g_1(v, w) + u\sigma g_2(v, w)^* + g_1(u + v, w).
\]

Take \(u = v = w = 1\) and use the fact that \(g_1\) vanishes when either of its arguments is 0 to obtain \(0 = g_2(1, 1)^* = 1\), a contradiction. This completes the proof.

Let \(\{e_i\}, i = 1, 2, 3, 4\) be the standard base for \(K^4\) (columns) and put \(V_i = \langle e_1, \ldots, e_i\rangle / \langle e_1, \ldots, e_{i-1}\rangle\) as \(KB\)-module. Then \(V_i\) is a \(K\)-module on which \(U\) acts trivially and \(T\) acts with weight \(v_i\) where \(v_i = \theta, v_2 = 1 - \theta, v_3 = \theta - 1, v_4 = -\theta\). For convenience we set \(v_0 = 0\). In the following lemma we determine the terms occuring in (*) when \(v = v_i, i = 0, 1, 2, 3, 4\) by solving the equations following Lemma 1.

**Lemma 4.** The solutions are as indicated when \(q > 2\) and \(i \in \{0, 1, 2, 3, 4\}\).

(a) \(v_i = \theta' \sigma: (i, q, \sigma) = (2, q, 1/2); (4, 8, 1)\).

(b) \(v_i = \sigma: (i, q, \sigma) = (1, q, \theta); (3, 8, 1)\).

(c) \(v_i = \sigma \theta' + \tau: (i, q, \sigma, \tau) = (0, 8, \alpha, 2\sigma)\) (any \(\alpha \in \Gamma\));

\(\begin{align*}
& (1, q, \theta/2, 1/2); (2, 8, 1, 1); (3, 8, 4, 2); (4, 8, 2, 2); (4, 32, 2, 8); \\
& (4, 32, 1, 2).
\end{align*}\)

(d) \(v_i = \theta' (\sigma + \tau): (i, q, \{\sigma, \tau\}) = (1, q, \{1/2, \theta\}); (2, q, \{1/4\}); (3, 8, \{1, 2\}); (4, 8, \{1/2\}).

(e) \(v_i = \sigma + \tau: (i, q, \{\sigma, \tau\}) = (1, q, \{\theta/2\}); (2, 8, \{2, 3\}); (3, 8, \{4\}); (3, 32, \{2, 1\}); (4, 8, \{1, 4\}).

The following will be useful for solving these equations.

**Lemma 5.** Let \(\varphi_i \in \Gamma \hookrightarrow \text{End}(K^*)\) \(i = 1, 2, \cdots, m\). The following is arithmetic in \(\text{End}(K^*)\).

(a) If \(\varphi_1 + \varphi_2 = \varphi_3 + \varphi_4\) then \(\{\varphi_1, \varphi_2\} = \{\varphi_3, \varphi_4\}\).

(b) If the \(\varphi_i\)'s are distinct then \(\sum_{i=1}^n \varphi_i \notin \Gamma\).

(c) If \(\sum_{i=1}^n \varphi_i = 0\) then \(m \geq |\Gamma|\), and \(m = |\Gamma| \iff \{\varphi_i\}_{i=1}^m = \Gamma\).

**Proof of Lemma 5.** (a) A proof is included in the proof of Lemma 2.

For (b) and (c) write \(\varphi_i: x \to x^{2^n}\) for \(0 \leq n < |\Gamma|\).

(b) Here we assume the \(n_i\)'s are distinct. Then \(\sum \varphi_i \in \Gamma\) implies
for all \( x \in K \) we have \((x^{s_p} + 1) = (x + 1)^{s_p} = \Pi(x^{s_p} + 1) = \Sigma x^{s_p = r_i} \) where we sum over all \( J \subseteq \{1, 2, \cdots, m\} \). Cancelling the terms on the left with the corresponding terms on the right there remains a polynomial of degree less than \( 2^{|\Gamma|} \) with \( 2^{|\Gamma|} = |K| \) solutions.

(c) Assume \( m \) is minimal with \( \Sigma \phi_i = 0 \). Then the \( \phi_i \)'s are distinct since \( \phi_i + \phi_i = 2\phi_i \in \Gamma \). Then \( \Sigma \phi_i = 0 \) implies \((q - 1)|\Sigma 2^n \). Thus \( q - 1 = 2^{|\Omega|} - 2 = \Sigma \phi_i 2^i \leq \Sigma \phi_i 2^n \) implying \( m = |\Gamma| \) and \( \{\phi_i\} = \Gamma \).

We now indicate a proof of Lemma 4. Observe first that from their definitions we will have \( \theta \neq 1, 2\theta^2 = 1, \theta'(\theta + 1) = 1 \). Thus \( \theta', \theta + 1, 1 - \theta = \theta'/2 \) are invertible in \( \text{End}(K^*) \). Using these facts the equations can be manipulated to take advantage of Lemma 5 and reduce the problem to a few case by case investigations. We illustrate with the solution of \( \nu_i = \sigma\theta' + \tau \).

\[
i = 0: \quad 0 = \sigma\theta' + \tau \Rightarrow \theta = -\theta' = 2\theta - 2 \Rightarrow 2\theta = 2 + \tau\sigma^{-1}. \quad \text{Now Lemma 5 (b) says} \ 2 = \tau\sigma^{-1} \quad \text{so} \ \theta = 2, \ q = 8, \ \tau = 2\sigma.
\]

\[
i = 1: \quad \theta = \sigma\theta' + \tau = 2\sigma - 2\sigma\theta + \tau \Rightarrow \theta + 2\sigma\theta = 2\sigma + \tau \quad \text{and Lemma 5 (a) implies} \ \{\theta, 2\sigma\theta\} = \{2\sigma, \tau\}. \quad \text{Now} \ \theta \neq 1 \Rightarrow (\sigma, \tau) = (\theta/2, \theta^2) = (\theta/2, 1/2).
\]

\[
i = 2: \quad \text{Multiplying by} \ 1 + \theta \text{we obtain} \ 1/2 = \sigma + \tau\theta + \tau \quad \text{and Lemma 5 (b) says} \ \sigma, \ \tau\theta, \ \tau \text{are not distinct}. \ \theta \neq 1 \Rightarrow \tau\theta \neq \tau. \quad \sigma = \tau, \theta \Rightarrow 1/2 = 2\sigma + \tau \Rightarrow 2\theta = 1 \Rightarrow 1 = 2\theta^2 = \theta, \quad \text{a contradiction.} \quad \sigma = \tau \Rightarrow 1/2 = 2\sigma + 2\theta \Rightarrow 2 = \theta, \ q = 8 \quad \text{and it may be seen} \ \sigma = \tau = 1.
\]

\[
i = 3: \quad \text{Since} \ \nu_3 = -\nu_2 \text{we obtain} \ 0 = 1/2 + \sigma + \tau\theta + \tau \quad \text{and Lemma 5 (c) implies} \ |\Gamma| \leq 4. \quad \text{Thus} \ q = 8, \ \sigma = \tau = 1.
\]

\[
i = 4: \quad \text{Since} \ \nu_4 = -\nu_2 \text{we obtain} \ 2\sigma\theta = 2\sigma + \theta + \tau \quad \text{implying} \ 2\sigma, \ \theta, \ \tau \quad \text{are not distinct}. \quad 2\sigma = \theta \Rightarrow \theta' = 2\theta + \tau \Rightarrow 2\theta = \tau \Rightarrow \theta^2 = 4\theta, \ \theta = 4, \ q = 32, \ (\sigma, \tau) = (2, 8). \quad 2\sigma = \tau \Rightarrow 2\sigma\theta = 4\sigma + \theta \Rightarrow 4\sigma = \theta, \ \theta = 4, \ q = 32, \ (\sigma, \tau) = (1, 2). \quad \tau = \theta \Rightarrow 2\sigma\theta = 2\sigma + 2\theta \Rightarrow \sigma = \theta, \ \theta = 2, \ q = 8, \ (\sigma, \tau) = (2, 2).
\]

**Lemma 6.** When \( i \in \{1, 2, 3, 4\} \) we have

\[
\dim_k H^1(B, V_i) = \begin{cases} 1 & (i, q) = (2, q); (4, 8) \\
0 & \text{otherwise}, \end{cases}
\]

\[
\dim_k H^2(B, V_i) = \begin{cases} 1 & (i, q) = (2, q); (4, 8); (4, 32) \\
0 & \text{otherwise}. \end{cases}
\]
Proof. The first statement is immediate from Lemma 4 and the remarks following Lemma 1. For the second observe $\nu_i \in \{ \pm \theta, \pm \theta'/2 \}$ and so $\nu_i$ is invertible in $\text{End}(K^*)$. Now Lemmas 1, 2, 3 and 4 may be used to determine the relevant terms of sequences (*) when $\nu = \nu_i$. These considerations prove the claim except to show $H^2(B, V_i) \neq 0$ when $q = 32$. In this case it may be seen that $(\alpha, u), (\beta, v) \rightarrow u^2\beta^8 + u\beta^2 + u^3v^4 + u^2v^9$ gives a nonzero class in $H^2(U, V_i)^T = H^2(B, V_i)$.

We are now ready to proceed to the main results of this paper.

**Theorem 1.** Let $K$ be the trivial module for $Sz(q)$, $q \geq 8$. Then $\dim_K H^2(Sz(q), K)$ is 0 if $q > 8$, and is 2 if $q = 8$ with generators (on a Sylow 2-subgroup) any two of $f_\sigma: (\alpha, u), (\beta, v) \rightarrow (\alpha^2v + u^2\beta^4)^\sigma$, $\sigma \in \Gamma$.

Proof. We use $B$ in place of $Sz(q)$ and sequences (*) with $\nu = 0$ and $V = K$. According to Lemma 4 we have $H^2(Z, V)^x = H^2(X, V) = 0$ and $\dim_K H^1(X, H^1(Z, V)) = 0$ if $q > 8$, and $|\Gamma| = 3$ if $q = 8$. Now sequences (*) with Lemma 3 give the upperbound. For the lowerbound it is easily checked that $f_\sigma$ as given is a $T$-stable cocycle and when $\sigma \neq \tau$, $\Phi[f_\sigma]$ and $\Phi[f_\tau]$ are independent in $H^1(A, H^1(Z, V))^T = H^1(X, H^1(Z, V))$.

**Theorem 2.** Assume $q \geq 8$ and $K^4$ is the standard module for $Sz(q)$. Then $H^1(Sz(q), K^4)$ is of dimension one and is generated by the restriction of a generator of $H^1(Sp_4(q), K^4)$.

Proof. Define $[d] \in H^1(U, K^4)^T = H^1(B, K^4)$ by $d(\alpha, u) = (\alpha^6, u^{1/2}, 0, 0)^*$ ($^*$ denotes transpose). It can be checked explicitly that $d$ is a nontrivial $T$-stable cocycle defined on $U$ giving the claimed lowerbound. Furthermore it can be seen that if $v \in K^4$, $x \in U$, then $v^*x^*Jd(x) = (v^*J_0v + v^*x^*J_0xv)^{1/2}$ where $J_0$ is the $4 \times 4$ matrix with all entries 0 except $(J_0)_{41} = (J_0)_{23} = 1$. This means $d$ is the restriction of Dickson's derivation which generates $H^1(Sp_4(q), K^4)$ [8].

For the upperbound we use Lemma 6 to conclude $\dim_K H^1(B, K^4) \leq \sum_{i=1}^4 \dim_K H^1(B, V_i) = 1$ if $q > 8$, and 2 if $q = 8$. We are done at $q > 8$ and continue at $q = 8$.

Define $V_{12} = \langle e_1, e_2 \rangle$, $V_{34} = K^4/V_{12}$. We obtain the exact sequence of $K$-modules

$$0 \rightarrow H^1(B, V_{12}) \rightarrow H^1(B, K^4) \rightarrow H^1(B, V_{34}) \rightarrow H^2(B, V_{12}) \rightarrow H^2(B, K^4) \rightarrow H^2(B, V_{34}) \rightarrow .$$

The given cocycle shows $\dim_K H^1(B, V_{12}) = 1$ so it suffices to see $(\pi_1)_*$ =
Lemma 6 implies \( \dim_K H^1(B, V_{34}) \leq 1 \). It can be seen that \((\alpha, u) \mapsto (-, \alpha + u^3, u)^*\) is a nontrivial \( T \)-fixed cocycle in \( Z^1(U, V_{34})^T \) so its class generates \( H^1(U, V_{34})^T = H^1(B, V_{34}) \). If \((\sigma_i)_* \neq 0\) we can find \( f \in Z^1(U, K^4) \) of the form \( f(\alpha, u) = (f_1(\alpha, u), f_2(\alpha, u), \alpha + u^3, u)^* \). The \( e_2 \) coordinate of the equation \( \delta f((\alpha, u), (\beta, v)) = 0 \) gives the equation \( f_2(\alpha + \beta + uv^\theta, u + v) = f_2(\alpha, u) + f_2(\beta, v) + u(\beta + v^3) + \alpha v \). Set \( u = v = 0 \) to obtain \( f_2(\alpha, 0) = f_2(\alpha + \beta, 0) = f_2(\alpha, 0) + f_2(\beta, 0) \); and set \((\alpha, u) = (\beta, v)\) to obtain \( f_2(u^3, 0) = u^6\), that is, \( f_2(u, 0) = u^6 \). This is a contradiction as \( u \to u^6 \) is not an additive function.

**Theorem 3.** Let \( K^4 \) be the standard module for \( Sz(q) \). Then \( H^2(Sz(q), K^4) \) is zero if \( q = 8 \), and is of dimension one if \( q > 8 \) generated by a cocycle which is the restriction of a generator of \( H^2(Sp(q), K^4) \).

**Proof.** Landázuri (see [7]) has explicitly constructed (on a Sylow 2-subgroup) a nontrivial cocycle in \( Z^2(Sp_4(2^m), GF(2^m)^+) \) and further (see [5]) has shown \( H^2(Sp_4(2^m), GF(2^m)^+) \) is of dimension one when \( m > 1 \). Restricting his cocycle gives

\[
(f: (\alpha, u), (\beta, v) \mapsto \left( (\alpha^\theta u^\theta v^{1/2} + \alpha^\theta \beta^\theta + u^\theta \beta + u^\theta v^{\theta+1} + u^\theta \beta^\theta v^{1/2}), (uv)^{1/4}, 0, 0 \right)^*.
\]

We will see \( f \) is a coboundary only at \( q = 8 \). McLaughlin [7] has given a somewhat different argument to see \( \text{Res}(Sz(q), Sp_4(q)) \) is nonzero when \( q > 8 \) using the sufficient condition of Griess [2].

Consider now sequence (2). We have seen \((\sigma_i)_* = 0\) and \( \dim_K H^1(B, V_{34}) = 0 \) if \( q > 8 \), and 1 if \( q = 8 \). Next we show \( \dim_K H^2(B, V_{12}) = 1 \). The upper bound follows from Lemma 6 and the lower bound follows from the displayed cocycle \( f \). Also from Lemma (6), \( H^2(B, V_{34}) = 0 \) when \( q > 32 \). Using sequence (2) the proof is now complete when \( q > 32 \). Furthermore, the cases \( q = 8, 32 \) follow if we show there is no \( f \in Z^2(B, K^4) \) which has a nontrivial projection onto \( V_4 \).

Assuming we have such an \( f \), a contradiction is obtained by using the following: Let \( L = K^4/V_1 \) as \( KB \)-module.

\begin{itemize}
    \item[(a)] \( H^2(Z, L)^X = K \) generated by \((\alpha, \beta) \mapsto (-, \alpha^2 \beta^4, 0, 0)^*\) when \( q = 8 \) and by \((\alpha, \beta) \mapsto (-, 0, \alpha \beta^2, 0)^*\) when \( q = 32 \).
    \item[(b)] \( H^2(X, L^Z) = K \) generated by \((u, v) \mapsto (-, (uv)^{1/4}, 0, 0)^*\).
    \item[(c)] \( H^1(X, H^1(Z, L)) = 0 \).
\end{itemize}

We now assume (a), (b), (c). From the exact sequence of groups \( 1 \to Z \to B \to X \to 1 \) the Hochschild–Serre spectral sequence gives the exact sequences

\[
0 \to H^2(B, L)_0 \to H^2(B, L) \xrightarrow{\text{Res}} H^2(Z, L)^X \to H^2(X, L^Z) \to H^2(B, L)_0 \to H^2(X, H^1(Z, L)) \to .
\]
In general when we have a function whose range is $K^4$ let the subscript $i$ denote its projection onto $V_i$. Thus $f = (f_1, f_2, f_3, f_4)^*$. We are assuming $0 \neq [f_i] \in H^2(B, V_4)$. Let $\tilde{f}$ denote the projection of $f$ onto $L$. We write this as $\tilde{f} = (-f_2, f_3, f_4)$. Thus $\tilde{f} \in Z^2(B, L)$.

Assume first $\text{Res}[\tilde{f}] = 0$. Then using (c) and the above sequences $\tilde{f}$ is cohomologous to the image under the inflation map of a generator of $H^2(L, V^4)$, i.e. there is a $g \in C^1(B, L)$ with $(\tilde{f} - \delta f)((\alpha, u), (\beta, v)) = (-, (uv)^{1/4}, 0, 0)^*$. Using the fact that $(\alpha, u)$ is an upper triangular matrix it is easily seen that this equation implies $f_4 = \delta g_4 \in B^2(B, V_4)$, contradicting present assumptions.

Now we assume $\text{Res}[\tilde{f}] \neq 0$. Let $\tilde{f} = \text{Res}(f)$ so $[\tilde{f}] \in H^2(Z, K^4)^X$. Assume first $q = 8$. Now (a) tells us we may assume $\tilde{f}(\alpha, \beta) = (\tilde{f}_1(\alpha, \beta), \alpha^2 \beta^4, 0, 0)^*$. Let $u = (0, 1) \in U$. Then $(u-1) \cdot \tilde{f} = \delta g$ for some $g \in C^1(Z, K^4)$. Apply both sides to $(\alpha, \beta)$ and obtain

$$\begin{bmatrix}
\alpha^2 \beta^4 \\
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
g_1(\alpha + \beta) \\
g_2(\alpha + \beta) \\
g_3(\alpha + \beta) \\
g_4(\alpha + \beta)
\end{bmatrix} + \begin{bmatrix}
1 & 0 & \alpha & \alpha^4 \\
1 & 0 & \alpha & \alpha^4 \\
1 & 0 & \alpha & \alpha^4 \\
1 & 0 & \alpha & \alpha^4
\end{bmatrix} \begin{bmatrix}
g_1(\beta) \\
g_2(\beta) \\
g_3(\beta) \\
g_4(\beta)
\end{bmatrix} + \begin{bmatrix}
g_1(\alpha) \\
g_2(\alpha) \\
g_3(\alpha) \\
g_4(\alpha)
\end{bmatrix}.$$  

The third and fourth rows tell us $g_3$ and $g_4$ are additive; $\alpha = \beta$ in the second row tells us $0 = \alpha g_4(\alpha)$ implying $g_4 = 0$; $\alpha = \beta$ in the first row tells us $\alpha^5 = g_3(\alpha)$, contradicting the additivity of $g_3$.

Assume now $q = 32$. Here (a) tells us we may assume $\tilde{f}(\alpha, \beta) = (\tilde{f}_1(\alpha, \beta), 0, \alpha \beta^2, 0)^*$. Now, with $u = (0, 1) \in U$, the equation $(u-1) \cdot \tilde{f} = \delta g$ implies $(\alpha \beta^2, \alpha \beta^2, 0, 0)^* = \delta g(\alpha, \beta)$. As before $g_3$ and $g_4$ are additive. Set $\alpha = \beta$. The second coordinate implies $g_4(\alpha) = \alpha^2$; the first implies $\alpha^2 = g_3(\alpha) + \alpha^{2q+1}$; these imply $\alpha \rightarrow \alpha^{2q+1} = \alpha^9$ is additive, a contradiction.

We now prove (a), (b), (c). Note that if $x$ is an involution in some group and $d$ and $f$ are 1 and 2-cocycles from that group to some module then $\delta d(x, x) = 0$ and $\delta f(x, x, x) = 0$ imply $d(x) = -xd(x)$ and $f(x, x) = xf(x, x)$. Regard $L = K^3$ (columns) = $\langle e_2, e_3, e_4 \rangle$ on which $(\alpha, u)$ acts as multiplication by

$$\begin{bmatrix}
1 & u & \alpha \\
1 & u^q \\
1
\end{bmatrix}.$$  

(a) Take $[f] \in H^2(Z, L)^X$ and using our convention we have $f = (f_2, f_3, f_4)^*$. Since $[f_i] \in H^2(Z, V_4)^T$, by Lemma 4 (e) we may assume
\( f_4(\alpha, \beta) = \alpha \beta^* k_4 \) and \( k_4 = 0 \) when \( q = 32 \). The relation \( f(\alpha, \alpha) = af(\alpha, \alpha) \) implies \( k_4 = 0 \). Now \( [f_3] \in H^2(Z, V_3)^T \) and we may assume \( f_3(\alpha, \beta) = \alpha^\sigma \beta^* k_3 \) where \( \{\sigma, \tau\} = \{4\} \) if \( q = 8 \) and \( \{\sigma, \tau\} = \{2, 1\} \) if \( q = 32 \). Set \( u = (0, 1) \in U \). Then \( (u - 1). f = \delta g \) for \( g \in C^1(Z, L) \). In the usual way this equation implies \( g_3 \) and \( g_4 \) are additive. Setting \( \alpha = \beta \) we obtain \( \alpha^{\sigma + \tau} k_3 = ag_4(\alpha) \) implying \( k_3 = 0 \) or \( \alpha \to \alpha^{\sigma + \tau - 1} \) is additive. At \( q = 8 \) the latter is false implying \( k_3 = 0 \).

Since \( k_4 = 0 \) it follows that \( [f_2] \in H^2(Z, V_2)^T \) and by Lemma 4 (e) we may assume \( f_2(\alpha, \beta) = \alpha^2 \beta^* k_2 \) with \( k_2 = 0 \) when \( q = 32 \). This proves (a).

(b) We see \( L^Z = (e_2, e_3) = K^2 \) (columns) on which \( (0, u) \) acts as multiplication by \( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \). Take \( [f] \in H^2(X, K^2) \). By Lemma 4 (d) we may assume \( f_3(u, v) = uv^{2} k_3 \) with \( k_3 = 0 \) when \( q = 32 \). Now the relation \( uf(\bar{u}, \bar{u}) = f(\bar{u}, \bar{u}) \) implies \( f_3 = 0 \). Thus \( f_2 \in Z^2(X, V_2) \) and (b) follows from Lemma 4 (d).

(c) Take \( f \in Z^1(Z, L) \). Then \( f(\alpha) = af(\alpha) \) implies the image of \( f \) lies in \( L^{\sigma} = L^Z = (e_2, e_3) \). Thus \( f_4 = 0 \). Taking \( Z \)-cohomology of the exact sequence \( 0 \to L^Z \to L \to V_4 \to 0 \) gives the exact sequence of \( KK \)-modules \( 0 \to V_4 \to H^1(Z, L^Z) \to H^1(Z, L) \to H^1(Z, V_4) \to 0 \). We have just seen \( \pi_* = 0 \). Set \( V_{23} = L^Z \). It is easily seen that \( \operatorname{Im} \delta_* = \operatorname{Hom}_K(Z, V_2) \subset \operatorname{Hom}_K(Z, V_{23}) \subset \operatorname{Hom}(Z, L^Z) = H^1(Z, L^Z) \) showing \( H^1(Z, L) = \bigoplus_{\tau \neq 1} H_\tau(Z, V_{23}) \oplus H \) where

\[
H = \operatorname{Hom}_K(Z, V_{23})/\operatorname{Hom}_K(Z, V_2) = \operatorname{Hom}_K(Z, V_3).
\]

Now \( H^1(X, H) = \bigoplus H_\sigma(A, \operatorname{Hom}_K(Z, V_3))^T = 0 \) since by Lemma 4 (c) there is no \( \sigma \in \Gamma \) with \( \nu_3 = \sigma \theta^* + 1 \). Finally, we show \( H^1(X, H_\tau(Z, V_{23})) = 0 \) when \( \tau \neq 1 \). Take \( [f] \in H^1(A, H_\tau(Z, V_{23}))^T \). Taking \( u = v \) in the cocycle condition on \( f \) we see \( 0 = uf_3(u)(\alpha) \) showing \( f_3 = 0 \). Thus

\[
H^1(X, H_\tau(Z, V_{23})) = H^1(X, H_\tau(Z, V_2)) = \bigoplus H_\sigma(A, H_\sigma(Z, V_2))^T = 0
\]

since by Lemma 4 (c) there is no \( \sigma \in \Gamma \) with \( \nu_2 = \theta^* \sigma + \tau \) when \( \tau \neq 1 \).

REFERENCES


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