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THE STRUCTURE OF A SPECIAL CLASS OF WEIGHTED TRANSLATION SEMIGROUPS

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A special class of weighted translation semigroups $\{S_t\}$ on $L^2(\mathcal{R}_+)$ is studied. The weakly closed algebra \mathcal{A} generated by the semigroup is maximal abelian and the spectra of elements of \mathcal{A} are studied. It is shown that each densely defined linear transformation commuting with \mathcal{A} is closable and that every transitive algebra containing \mathcal{A} is weakly dense in the full algebra of operators on $L^2(\mathcal{R}_+)$.

1. Introduction. A weighted translation semigroup $\{S_t\}$ with symbol ϕ is defined on $L^2(\mathcal{R}_+)$ by

$$(Sf)(x) = \begin{cases} \frac{\phi(x)}{\phi(x-t)} f(x-t) & \text{for } 0 \leq t \leq x \\ 0 & \text{for } t > x \end{cases}$$

where ϕ is a continuous, complex-valued function on \mathcal{R}_+ such that $\phi(x) \neq 0$ for x in \mathcal{R}_+ . These semigroups were studied in [2] and [3]. In [3] strongly continuous subnormal weighted translation semigroups are characterized as those for which ϕ^2 is a Laplace-Stieltjes Transform of a probability measure. In [4] a more general type of weighted translation semigroup is studied.

To insure the strong continuity of $\{S_t\}$ we assume that $\sup_{x \in \mathcal{R}_+} |\phi(x+t)/\phi(x)| \leq Me^{wt}$ for all t and some constants M and w [2, Lemma 2.1]. Two weighted translation semigroups with symbols ϕ and ρ are unitarily equivalent if and only if $|\phi/\rho|$ is constant [2, Theorem 2.5]. Thus without loss of generality we assume that ϕ is positive-valued and that $\phi(0) = 1$.

Throughout the paper unless otherwise indicated we shall assume further that $\int_0^x (\phi(x)/\phi(t)\phi(x-t))^2 dt$ is bounded and shall say that ϕ is of *bounded kernel type*. For such a ϕ and for each f in $L^2(\mathcal{R}_+)$ we define

$$(1) \quad A_f = \int_0^\infty \frac{f(t)}{\phi(t)} S_t dt.$$

In §2 we show that $\{A_f: f \in L^2(\mathcal{R}_+)\}$ is a subalgebra of $B(L^2)$, the full algebra of operators on $L^2(\mathcal{R}_+)$. We denote $\{A_f: f \in L^2(\mathcal{R}_+)\}$ by \mathcal{A}_0 and its closure in the weak operator topology by \mathcal{A} . In Theorem 2.6 we show that \mathcal{A} is a maximal abelian algebra and that \mathcal{A}_0 is a proper ideal of \mathcal{A} .

In §3 we establish a basic relation between the multiplicative linear functionals on \mathcal{A} and the elements of $L^2(\mathcal{R}_+)$ of the form $e^{i\lambda t}/\phi(t)$. This relation enables us in Theorem 3.5 to determine completely the spectrum of each element of \mathcal{A}_0 . In §4 it is shown that any densely defined linear transformation commuting with \mathcal{A} is closable. This result enables us to apply Arveson's Density Theorem to show that if $e^{i\lambda t}/\phi(t) \in L^2(\mathcal{R}_+)$ for some λ , then any transitive subalgebra of $B(L^2)$ which contains \mathcal{A} is weakly dense in $B(L^2)$. Finally in §5 certain function theoretic considerations related to ϕ are investigated. It is shown in Corollary 5.5 that if the associated semigroup is hyponormal then ϕ is not of bounded kernel type.

Throughout the paper the following notation is used: $H = \{\lambda: e^{i\lambda t}/\phi(t) \in L^2(\mathcal{R}_+)\}$, $E = \{g: g(t) = e^{i\lambda t}/\phi(t), \lambda \in H\}$ and $\alpha(\phi) = \sup\{\operatorname{Re} \lambda: \lambda \in H\}$ where $\alpha(\phi) = -\infty$ if H is empty. G will denote the infinitesimal generator of the semigroup $\{S_t\}$.

2. Basic facts about \mathcal{A} . In this section we shall show that each A_f is a bounded linear operator on $L^2(\mathcal{R}_+)$, that the mapping $f \rightarrow A_f$ of $L^2(\mathcal{R}_+)$ onto \mathcal{A}_0 is a continuous linear mapping, that \mathcal{A}_0 is an algebra, and that \mathcal{A} is a maximal abelian algebra.

LEMMA 2.1. $\|A_f\| \leq \rho \|f\|$ for all f in $L^2(\mathcal{R}_+)$ where

$$\rho = \sup_x \int_0^x \left(\frac{\phi(x)}{\phi(t)\phi(x-t)} \right)^2 dt.$$

Proof. Let $g \in L^2(\mathcal{R}_+)$. To see that A_f is well-defined we note that

$$\begin{aligned} (A_f g)(x) &= \int_0^\infty \frac{f(t)}{\phi(t)} (S_t g)(x) dt \\ (2) \qquad &= \int_0^x \frac{\phi(x)}{\phi(t)\phi(x-t)} f(t) g(x-t) dt \end{aligned}$$

and the integral exists since f and g are square integrable and ϕ is continuous and nonzero. We note further that

$$|(A_f g)(x)|^2 \leq \int_0^x \left(\frac{\phi(x)}{\phi(t)\phi(x-t)} \right)^2 dt \int_0^x |f(t)g(x-t)|^2 dt.$$

Therefore

$$\|A_f g\|^2 \leq \rho^2 \int_0^\infty \int_0^x |f(t)g(x-t)|^2 dt dx = \rho^2 \|f\|^2 \|g\|^2$$

so that A_f is a bounded linear operator on $L^2(\mathcal{R}_+)$ and $\|A_f\| \leq \rho \|f\|$.

LEMMA 2.2. (i) $A_{\alpha f + \beta g} = \alpha A_f + \beta A_g$,

(ii) $A_f g = A_g f$, and

(iii) $A_f A_g = A_{A_f g}$

for all f and g in $L^2(\mathcal{R}_+)$ and all complex numbers α and β .

Proof. (i) and (ii) follow immediately from equation (2). To prove (iii) we let $f, g, h \in L^2(\mathcal{R}_+)$ and note that

$$\begin{aligned} (A_f A_g h)(x) &= \int_{t=0}^x \frac{\phi(x)}{\phi(t)\phi(x-t)} f(t) (A_g h)(x-t) dt \\ &= \int_{t=0}^x \frac{\phi(x)}{\phi(t)\phi(x-t)} f(t) \int_{s=0}^{x-t} \frac{g(s)\phi(x-t)}{\phi(s)\phi(x-t-s)} h(x-t-s) ds dt \\ &= \int_{t=0}^x \frac{\phi(x)}{\phi(t)\phi(x-t)} f(t) \int_{s=t}^x \frac{g(s-t)\phi(x-t)}{\phi(s-t)\phi(x-s)} h(x-s) ds dt \\ &= \int_{s=0}^x \left[\int_{t=0}^s \frac{\phi(x)}{\phi(t)\phi(s-t)\phi(x-s)} f(t) g(s-t) h(x-s) dt \right] ds \\ &= \int_{s=0}^x \frac{\phi(x)}{\phi(s)\phi(x-s)} \left[\int_{t=0}^s \frac{\phi(s)}{\phi(t)\phi(s-t)} f(t) g(s-t) dt \right] h(x-s) ds \\ &= \int_{s=0}^x \frac{\phi(x)}{\phi(s)\phi(x-s)} (A_f g)(s) h(x-s) ds \\ &= (A_{A_f g} h)(x). \end{aligned}$$

Thus (iii) holds for all f and g .

It now follows immediately from Lemma 2.2 that \mathcal{A}_0 is a commutative algebra and from Lemma 2.1 that the mapping $f \rightarrow A_f$ is continuous. An easy computation shows that this mapping is one-to-one. We state these results in the following theorem.

THEOREM 2.3. \mathcal{A}_0 is a commutative algebra of operators on $L^2(\mathcal{R}_+)$ and the mapping $f \rightarrow A_f$ is a continuous, one-to-one, linear mapping of $L^2(\mathcal{R}_+)$ onto \mathcal{A}_0 .

It follows from the Open Mapping Theorem and Theorem 2.3 that \mathcal{A}_0 is closed in the uniform topology if and only if the mapping $f \rightarrow A_f$ is bicontinuous. It is an open question whether or not \mathcal{A}_0 is closed in the uniform operator topology.

LEMMA 2.4. If T is an operator on $L^2(\mathcal{R}_+)$ which is in the commutant of \mathcal{A}_0 , then $TA_f = A_{Tf}$ for each f in $L^2(\mathcal{R}_+)$.

Proof. Let f and g be elements of $L^2(\mathcal{R}_+)$. Then $TA_f g = TA_g f = A_g T f = A_{Tf} g$. Consequently $TA_f = A_{Tf}$ as desired.

LEMMA 2.5. $\{S_i\} \subset \mathcal{A} - \mathcal{A}_0$.

Proof. Let $f_n = n\phi\psi[r, r + 1/n]$ where $\psi[a, b]$ is the characteristic function of $[a, b]$. Then $f_n \in L^2(\mathcal{R}_+)$ and $A_{f_n} = n \int_r^{r+1/n} S_t dt$ which, because of the strong continuity of S_r , converges strongly to S_r . Consequently $S_r \in \mathcal{A}$. To see that $S_r \notin \mathcal{A}_0$ we assume the contrary: $S_r = \int_0^\infty (f(t)/\phi(t)) S_t dt$ for some f in $L^2(\mathcal{R}_+)$. Consequently, $S_r^*(g/\phi) = \int_0^\infty (\overline{f(t)}/\phi(t)) S_t^*(g/\phi) dt$ for each g in $L^2(\mathcal{R}_+)$ of compact support. Thus $g(x+r) = \int_0^\infty (\overline{f(t)}/\phi(t)) g(x+t) dt$. If we define $K(y, s) = \overline{f(s-y+r)}/\phi(s-y+r)$ for $y \geq r$ and 0 otherwise, we arrive at the integral equation $g(y) = \int_0^\infty K(y, s) g(s) ds$. Since the identity is not an integral operator on $L^2(\mathcal{R}_+)$ [5, p. 87], we arrive at a contradiction and our proof is complete.

An immediate consequence of Lemma 2.5 is that the weakly closed algebra \mathcal{A}_1 generated by $\{S_i\}$ is a subalgebra of \mathcal{A} . Since each element of \mathcal{A}_0 is obviously in \mathcal{A}_1 , we see that $\mathcal{A}_1 = \mathcal{A}$; that is, the weakly closed algebra generated by the semigroup $\{S_i\}$ is the same as the weakly closed algebra generated by $\{A_f\}$.

THEOREM 2.6. \mathcal{A} is a maximal abelian algebra and \mathcal{A}_0 is a proper ideal of \mathcal{A} .

Proof. That \mathcal{A} is abelian follows from the fact that \mathcal{A}_0 is

abelian. Thus by Lemma 2.4 if $T \in \mathcal{A}$, then $TA_f = A_{Tf} \in \mathcal{A}_0$, proving that \mathcal{A}_0 is an ideal of \mathcal{A} . Lemma 2.5 assures us that \mathcal{A}_0 is proper and that $I \in \mathcal{A}$. Choose a net g_λ such that A_{g_λ} converges weakly to the identity operator I . Then since $TA_{g_\lambda} = A_{Tg_\lambda}$, we have $T = \lim A_{Tg_\lambda}$, and hence $T \in \mathcal{A}$, proving that each element of the commutant of \mathcal{A} is an element of \mathcal{A} . The proof that \mathcal{A} is maximal abelian is complete.

We observe that no A_f is invertible since \mathcal{A}_0 is a proper ideal of the maximal abelian algebra \mathcal{A} . We shall study in more detail the spectral properties of elements of \mathcal{A}_0 in the next section.

3. Spectral properties of \mathcal{A} . In this section we first characterize certain multiplicative linear functionals on \mathcal{A} and then use this information to study the spectra of elements of \mathcal{A} . In particular we are able to show in Corollary 3.4 that whenever $g_\lambda \in L^2(\mathcal{R}_+)$ where $g_\lambda(x) = e^{\lambda x}/\phi(x)$, then g is an eigenvector for each element of \mathcal{A}^* . For an element A_f of \mathcal{A}_0 we then show in Theorem 3.5 that the eigenvalues corresponding to the g_λ together with the real number 0 make up the entire spectrum of A_f^* .

THEOREM 3.1. *If m is a multiplicative linear functional on \mathcal{A} , then there exists a unique g in $L^2(\mathcal{R}_+)$ such that*

- (i) $m(A_f) = \langle f, g \rangle$ and
- (ii) $A_f^*g = \langle g, f \rangle g$ for all f in $L^2(\mathcal{R}_+)$.

Conversely, if m and g satisfy (i) and (ii) and $g \neq 0$, then

- (iii) $A^*g = (\langle g, Ag \rangle / \|g\|^2)g$ for all A in \mathcal{A}

and m can be extended to a multiplicative linear functional K on \mathcal{A} such that

- (iv) $K(A) = \langle Ag, g \rangle / \|g\|^2$ for all A in \mathcal{A} .

Proof. Assume that m is a multiplicative linear functional on \mathcal{A} and define $L(f) = m(A_f)$ for each f in $L^2(\mathcal{R}_+)$. It follows from Theorem 2.3 that L is a continuous linear functional on $L^2(\mathcal{R}_+)$. By the Riesz Representation Theorem there exists a unique element g of $L^2(\mathcal{R}_+)$ such that $L(f) = \langle f, g \rangle$ for all f . Consequently $m(A_f) = \langle f, g \rangle$ for all f in $L^2(\mathcal{R}_+)$.

Assuming now that m is multiplicative, we have for all f and h in $L^2(\mathcal{R}_+)$ $\langle h, \langle g, f \rangle g \rangle = \langle f, g \rangle \langle h, g \rangle = m(A_f)m(A_h) = m(A_f A_h) = m(A_{A_f h}) = \langle A_f h, g \rangle = \langle h, A_f^* g \rangle$. Consequently $A_f^*g = \langle g, f \rangle g$ as desired.

Assume now that m and g satisfy (i) and (ii) and that $g \neq 0$. Reversing the computation in the preceding paragraph, we conclude that m is a multiplicative linear functional on \mathcal{A}_0 . We shall construct a multiplicative linear extension of m on \mathcal{A} . Let $A \in \mathcal{A}$

and $A_{f_\lambda} \rightarrow A$ weakly. Then $\langle g, Ag \rangle = \lim \langle g, A_{f_\lambda} g \rangle = \lim \langle A_{f_\lambda}^* g, g \rangle = \lim \langle g, f_\lambda \rangle \|g\|^2$ by (ii). Consequently $\lim \langle g, f_\lambda \rangle = \langle g, Ag \rangle / \|g\|^2$ and for each h in $L^2(\mathcal{R}_+)$, $\langle g, Ah \rangle = \lim \langle g, A_{f_\lambda} h \rangle = \lim \langle \langle g, f_\lambda \rangle g, h \rangle = \langle g, h \rangle \langle g, Ag \rangle / \|g\|^2$. Thus $A^*g = (\langle g, Ag \rangle / \|g\|^2)g$, proving the final assertion. We now define $K(A) = (\langle g, Ag \rangle / \|g\|^2)$ for each A in \mathcal{A} . A straightforward computation shows that K is a multiplicative linear functional on \mathcal{A} and that K is an extension of m .

We have shown that to each multiplicative linear functional on \mathcal{A} there corresponds a unique element g of $L^2(\mathcal{R}_+)$ which is a common eigenvector for the elements of \mathcal{A}^* , provided $g \neq 0$. In Theorem 3.3 we shall show that each such function g is necessarily of the form $e^{\lambda t} / \phi(t)$ for some complex number λ .

LEMMA 3.2. *If G is the generator of $\{S_t\}$ and λ is sufficiently large, then $A_f = (\lambda - G)^{-1}$ where $f(t) = e^{-\lambda t} \phi(t)$.*

Proof. Since $\{S_t\}$ is strongly continuous, there exist constants M and w so that $\sup_x |\phi(x+t)/\phi(x)| = \|S_t\| \leq Me^{wt}$ [2, Lemma 2.1]. Thus $\phi(t) \leq Me^{wt}$ and for λ sufficiently large $f(t) = e^{-\lambda t} \phi(t) \in L^2(\mathcal{R}_+)$. Then $A_f = \int_0^\infty (f(t)/\phi(t)) S_t dt = \int_0^\infty e^{-\lambda t} S_t dt = (\lambda - G)^{-1}$. [6, p. 344].

THEOREM 3.3. *If m is a multiplicative linear functional on \mathcal{A} and g satisfies*

- (i) $m(A_f) = \langle f, g \rangle$ and
- (ii) $A_f^* g = \langle g, f \rangle g$ for all f in $L^2(\mathcal{R}_+)$,

then either $g = 0$ or there exists a complex number λ such that $g(x) = e^{\lambda x} / \phi(x)$. Conversely, if $g(x) = e^{\lambda x} / \phi(x)$ and $g \in L^2(\mathcal{R}_+)$, then g satisfies (ii).

Proof. Let g satisfy (i) and (ii). By Lemma 3.2 $(\lambda^* - G^*)^{-1} g = \langle g, f \rangle g$ where $f(t) = e^{-\lambda t} \phi(t) \in L^2(\mathcal{R}_+)$. If $\langle g, f \rangle = 0$, then $g = 0$. Assume that $\langle g, f \rangle \neq 0$. Since $A_f^* g = \langle g, f \rangle g$, we have

$$(3) \quad \langle g, f \rangle g(x) = \frac{e^{\lambda x}}{\phi(x)} \int_x^\infty \frac{\phi(t)}{e^{\lambda t}} g(t) dt. \quad \text{a.e.}$$

Let $h(x) = \phi(x)g(x)/e^{\lambda x}$ and note that $h \in L^1(\mathcal{R}_+)$ since $\phi(x)/e^{\lambda x} \in L^2(\mathcal{R}_+)$ and $g \in L^2(\mathcal{R}_+)$. We now have $\langle g, f \rangle h(x) = \int_x^\infty h(t) dt$ a.e. Since h is integrable and $\langle g, f \rangle \neq 0$, we can conclude first that h is continuous and secondly that h is differentiable. Thus $\langle g, f \rangle h'(x) =$

$-h(x)$ and $h(x) = Ae^{\beta x}$ or equivalently $g(x) = Ae^{(\lambda+\beta)x}/\phi(x)$. It follows from (3) that $\langle g, f \rangle g(0) = (1/\phi(0)) \int_0^\infty (\phi(t)g(t)/e^{\lambda t}) dt = (1/\phi(0)) \langle g, f \rangle$. Thus $g(0) = 1/\phi(0)$, so that $A = 1$ and $g(x) = e^{(\lambda+\beta)x}/\phi(x)$, as desired. A straightforward computation shows that if g is of this form, then g satisfies (ii).

As an immediate consequence of Theorems 3.3 and 3.1 we have:

COROLLARY 3.4. *If $g_\lambda(t) = e^{\lambda t}/\phi(t) \in L^2(\mathcal{R}_+)$, then $A^*g_\lambda = (\langle g_\lambda, Ag_\lambda \rangle / \|g_\lambda\|^2)g_\lambda$ for all A in \mathcal{A} .*

In the remainder of the paper we let $H = \{\lambda : e^{\lambda t}/\phi(t) \in L^2(\mathcal{R}_+)\}$ and $E = \{g : g(t) = e^{\lambda t}/\phi(t), \lambda \in H\}$. We shall show that both sets are either empty or large: more precisely, either H is empty or H is a closed half-plane and at the same time either E is empty or its linear span is dense in $L^2(\mathcal{R}_+)$.

THEOREM 3.5. $\sigma(A_f) = \{\langle f, g \rangle : g \in E\} \cup \{0\}$.

Proof. In our comments following Theorem 2.6 we observed that $0 \in \sigma_{\mathcal{A}}(A_f)$ whenever $f \in L^2(\mathcal{R}_+)$. By Theorem 2.6 \mathcal{A} is a maximal abelian algebra and hence for each A in \mathcal{A} , $\sigma(A) = \sigma_{\mathcal{A}}(A) = \{m(A) : m \text{ a multiplicative linear functional on } \mathcal{A}\}$. By Theorems 3.1 and 3.3, $m(A_f) = \langle f, g \rangle$ for some g in E provided m is not identically zero on \mathcal{A}_0 , completing the proof.

We observed in the proof of Theorem 3.5 that $\sigma(A) = \{m(A) : m \text{ a multiplicative linear functional on } \mathcal{A}\}$ which implies that $\sigma(A) \supset \{\langle Ag, g \rangle / \|g\|^2 : g \in E\} \cup \{m_0(A)\}$ where m_0 is identically zero on \mathcal{A}_0 . It is not known whether this set is the entire spectrum of A ; equivalently, it is not known if m_0 is unique.

COROLLARY 3.6. *Among the conditions*

- (i) \mathcal{A} contains a nonzero quasinilpotent element;
- (ii) $\sigma((\beta - G)^{-1}) = \{0\}$ for some β such that $(\beta - G)^{-1}$ is bounded;
- (iii) $\sigma(A_f) = \{0\}$ for all f in $L^2(\mathcal{R}_+)$;
- (iv) $E = \emptyset$;
- (v) the linear span of E is not dense in $L^2(\mathcal{R}_+)$, the following implications hold: (i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v).

Proof. (i) \Rightarrow (v). If $\sigma(A) = \{0\}$, then by Theorem 3.1(iii) $A^*g = 0$ for each g in E . Thus if A is nonzero, the linear span of E is not dense

in $L^2(\mathcal{R}_+)$ and (v) holds. (v) \Rightarrow (ii). If $\sigma((\beta - G)^{-1}) \neq \{0\}$ for sufficiently large β , then by Theorem 3.5 there exist $g(t) = e^{\lambda t}/\phi(t) \in E$. If the linear span of E is not dense in $L^2(\mathcal{R}_+)$, then there exists a nonzero f such that

$$\begin{aligned} 0 &= \int \frac{e^{zt}}{\phi(t)} f(t) dt \text{ whenever } \operatorname{Re} z \leq \operatorname{Re} \lambda \\ &= \int e^{(z-\lambda)t} \frac{e^{\lambda t} f(t)}{\phi(t)} dt \\ &= \int e^{ixt} \frac{e^{\lambda t} f(t)}{\phi(t)} dt \text{ for } z = \lambda + ix, x \text{ real.} \end{aligned}$$

Thus the Fourier coefficients of the $L^1(\mathcal{R}_+)$ function $e^{\lambda t} f(t)/\phi(t)$ are zero, implying that $f = 0$ a.e. This contradiction completes the proof that (v) \Rightarrow (ii). (ii) \Rightarrow (iii) \Rightarrow (iv) by Theorem 3.5; (iv) \Rightarrow (v) trivially.

The following two examples demonstrate the two different types of symbols ϕ : in the first example $\alpha(\phi) > -\infty$ and H is a half plane and in the second example $\alpha(\phi) = -\infty$ and H is empty. Thus in the second example each A_f is quasinilpotent.

EXAMPLE 1. Let $\phi(x) = x + 1$. We shall show that ϕ is of bounded kernel type and $\alpha(\phi) = 0$.

$$\int_0^x \frac{\phi(x)^2}{\phi(t)^2 \phi(x-t)^2} dt = 2(x+1)^2 \left[\frac{\log(x+1)}{(x+3)^3} + \frac{x}{(x+2)^2(x+1)} \right]$$

which is bounded on \mathcal{R}_+ . To see that $\alpha(\phi) = 0$ we note that $\int_0^\infty |e^{\lambda x}/(x+1)|^2 dx$ converges for $\operatorname{Re} \lambda \leq 0$ and diverges for $\operatorname{Re} \lambda > 0$. Thus by Corollary 3.6 no element of \mathcal{A} is quasinilpotent and by Corollary 3.4 g is a common eigenvector for A^* whenever $g(x) = e^{\lambda x}/\phi(x)$, $\operatorname{Re} \lambda \leq 0$.

EXAMPLE 2. Let $\phi(x) = e^{-x^2/2}$. We shall show that ϕ is of bounded kernel type and $\alpha(\phi) = -\infty$. Obviously for each complex number λ $e^{\lambda t}/\phi(t) \notin L^2$ so that $\alpha(\phi) = -\infty$. To see that ϕ is of bounded kernel type we compute as follows

$$\int_0^x \left(\frac{\phi(x)}{\phi(t)\phi(x-t)} \right)^2 dt = \int_0^x \frac{e^{-x^2}}{e^{-t^2} e^{-(x-t)^2}} dt$$

$$\begin{aligned}
 &= \int_0^x e^{-2t(x-t)} dt \\
 &= e^{-x^2/2} \int_0^x e^{(x-2t)^2/2} dt \\
 &= e^{-x^2/2} \int_{-x}^x \frac{1}{2} e^{\frac{1}{2}s^2} ds \\
 &= e^{-x^2/2} \int_0^x e^{\frac{1}{2}s^2} ds.
 \end{aligned}$$

Thus by l'Hopital's Rule

$$\lim_{x \rightarrow \infty} \int_0^x \left(\frac{\phi(x)}{\phi(t)\phi(x-t)} \right)^2 dx = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \int_0^x e^{\frac{1}{2}s^2} dx}{\frac{d}{dx} e^{x^2/2}} = 0.$$

Consequently since $\int_0^x (\phi(x)/\phi(t)\phi(x-t))^2 dx$ is continuous and vanishes at ∞ , it is bounded.

We note also for this example that since $\|S_t^n\|^{1/n} = e^{-nt/2}$, each $S_t (t \neq 0)$ is quasinilpotent. The fact that each A_t is quasinilpotent follows from Corollary 3.6.

Although it appears difficult in general to determine which symbols ϕ are of bounded kernel type, in certain cases one can use information about the set H to show that ϕ is not of bounded kernel type. More precisely, if ϕ is of bounded kernel type, then H is a closed half plane. To see this we argue as follows. Assume $\lambda \in H$. Then $\int_0^\infty (e^{2\text{Re } \lambda x} / \phi(x)^2) dx = \int_0^\infty |e^{\lambda x} / \phi(x)|^2 dx < \infty$. Consequently if $\text{Re } z \leq \text{Re } \lambda$, then $z \in H$, proving that H is a half plane. Now choose β so that $f(t) = e^{-\beta t} \phi(t) \in L^2(\mathbb{R}_+)$ and $(\beta - G)^{-1}$ is bounded. By Lemma 3.2 and Theorem 3.5 $\sigma((\beta - G)^{-1}) = \{1/(\beta - \lambda) : \lambda \in H\} \cup \{0\}$. Thus $\{1/(\beta - \lambda) : \lambda \in H\} \cup \{0\}$ is compact and it easily follows that H is closed.

In [3] it was shown that $\phi(x) = (x + 1)^{-1/2}$ is the symbol for a subnormal weighted translation semigroup. Since $\int_0^\infty |e^{\lambda t} / \phi(t)|^2 dt$ converges for $\text{Re } \lambda < 0$ and diverges otherwise we see that H is not closed and hence ϕ is not of bounded kernel type. At the end of §5 we shall see that no subnormal weighted translation semigroup has symbol of bounded kernel type. Indeed a stronger conclusion is obtained: if $\{S_t\}$ is hyponormal ($S_t^* S_t \geq S_t S_t^*$ for each t), then the symbol ϕ of $\{S_t\}$ is not of bounded kernel type.

4. Transitivity. For clarity in this section we shall let \mathcal{A}_ϕ denote the weakly closed algebra generated by $\{S_i\}$, where ϕ is the symbol of $\{S_i\}$ and ϕ is of bounded kernel type.

Let T be a linear transformation with domain $D(T) \subseteq X$. We say that T commutes with A in $B(X)$ if $AD(T) \subseteq D(T)$ and $ATx = TAx$ for each x in $D(T)$. Also, T commutes with a set of operators S if it commutes with each operator in S . T is said to be *closable* if T has a closed extension.

THEOREM 4.1. *If T is a densely defined linear transformation commuting with \mathcal{A}_ϕ , then T is closable and TA_h is bounded for every h in $D(T)$.*

Proof. To prove that T is closable we must show that if $\{h_n\}$ is a sequence in $D(T)$ converging to 0 and $\{Th_n\}$ converges to some vector f , then $f = 0$. Note that if u is in $D(T)$ and v is in $L^2(\mathcal{R}_+)$, then $A_u v = A_v u$ is in $D(T)$. Let g be in $D(T)$. Then

$$TA_{h_n}g = A_{h_n}Tg \rightarrow 0.$$

But $TA_{h_n}g = TA_g h_n = A_g Th_n \rightarrow A_g f$. Thus for every g in $D(T)$, $A_g f = 0$. But since $D(T)$ is dense in $L^2(\mathcal{R}_+)$, $\{A_g : g \text{ in } D(T)\}$ is weakly dense in \mathcal{A}_ϕ . Since I is in \mathcal{A}_ϕ , we have $f = 0$. Hence T is closable.

Now suppose h is in $D(T)$. Since TA_h commutes with \mathcal{A}_ϕ , TA_h is closable. But TA_h is everywhere defined, so TA_h is in $B(L^2)$. In fact, since $TA_h f = TA_f h = A_f Th = A_{Th} f$, we have $TA_h = A_{Th}$.

Note that with T as in the above theorem and h in $D(T)$, $(TA_h)^*$ is, of course, bounded. Explicitly, $(TA_h)^* = T^* A_h^*$. To see this let f also be in $D(T)$ and let g be in $L^2(\mathcal{R}_+)$. Then

$$\begin{aligned} \langle Tf, A_h^* g \rangle &= \langle A_h Tf, g \rangle \\ &= \langle TA_h f, g \rangle \\ &= \langle f, (TA_h)^* g \rangle \end{aligned}$$

so that $A_h^* g$ is in $D(T^*)$ and $(TA_h)^* g = T^* A_h^* g$.

The properties of transformations commuting with the algebra \mathcal{A}_ϕ just developed are nicely applicable to the theory of transitive algebras. An algebra \mathcal{T} of operators on X is *transitive* if the only closed subspaces of X invariant under all the operators in \mathcal{T} are $\{0\}$ and X . For general discussions of transitive algebras see [1] and [7, Chapter 8]. The following result is an immediate corollary to Arveson's density theorem.

PROPOSITION. (Arveson). *If \mathcal{T} is a transitive algebra with the*

property that every linear transformation commuting with \mathcal{T} is a multiple of the identity, then \mathcal{T} is weakly dense in $B(X)$.

Now if T is a closed densely defined linear transformation commuting with the transitive algebra \mathcal{T} and either T or T^* has an eigenvector (other than 0), then T is a multiple of the identity. Since T^* commutes with $\mathcal{T}^* = \{A^* : A \text{ in } \mathcal{T}\}$ and \mathcal{T}^* is transitive if and only if \mathcal{T} is, it suffices to justify the above remark in the case $Tx = \lambda x$, $x \neq 0$. But then for every A in \mathcal{T} , $TAx = ATx = \lambda Ax$ so $T - \lambda I = 0$ on $\{Ax : A \text{ in } \mathcal{T}\}$ which is dense in X . But one sees easily that a closed transformation agreeing with a bounded operator on a dense set is in fact that bounded operator, and so $T = \lambda I$.

We now apply these remarks to certain algebras of the form \mathcal{A}_ϕ . Recall that $\alpha(\phi) = \sup \left\{ \lambda \text{ in } \mathcal{R} : \int_0^\infty (e^{2\lambda x} / \phi^2(x)) dx < \infty \right\}$.

THEOREM 4.2. *If ϕ is of bounded kernel type and $\alpha(\phi) > -\infty$, then every transitive algebra containing \mathcal{A}_ϕ is weakly dense in $B(L^2)$.*

Proof. We have seen that every densely defined linear transformation commuting with \mathcal{A}_ϕ is closable. It is easy to show that the minimal closed extension of a closable transformation L commutes with all the operators commuting with L . Let T be a closed linear transformation commuting with \mathcal{A}_ϕ . Then for each h in $D(T)$, we have seen that $T^*A_h^*$ is in \mathcal{A}_ϕ^* . Let $g(x) = e^{\alpha(\phi)x} / \phi(x)$. Then g is in $L^2(\mathcal{R}_+)$ and $A_h^*g = (g, f)g$ for each f in $L^2(\mathcal{R}_+)$. Now $T^*A_h^* = (TA_h)^* = A_{Th}^*$, so

$$\langle g, Th \rangle g = T^*A_h^*g = \langle g, h \rangle T^*g.$$

Thus g is an eigenvector for T^* ($\langle g, h \rangle$ cannot be 0 for all h in the dense set $D(T)$). It follows from the proposition preceding this theorem that every transitive algebra containing \mathcal{A}_ϕ is dense.

Question. What about transitivity considerations in the case $\alpha(\phi) = -\infty$?

5. Functional properties of ϕ . We now concentrate on some properties of the function $\phi \rightarrow \alpha(\phi)$. Throughout the following discussion we assume that ϕ is in $C^1([a, \infty))$ for some $a \geq 0$ and that $\phi(x) \neq 0$ for all $x \geq 0$. Note that Theorem 5.2 is not dependent upon ϕ being of bounded kernel type. Define

$$i(\phi) = \liminf_{t \rightarrow \infty} \frac{\phi'(t)}{\phi(t)}$$

$$s(\phi) = \limsup_{t \rightarrow \infty} \frac{\phi'(t)}{\phi(t)}.$$

LEMMA 5.1. *If ϕ is of bounded kernel type, then $\alpha(\phi) < \infty$.*

Proof. We have seen that for $\operatorname{Re} \lambda \leq \alpha(\phi)$ $g_\lambda(x) = e^{\lambda x}/\phi(x)$ is in $L^2(\mathcal{R}_+)$ and for every f in $L^2(\mathcal{R}_+)$ $(f, g_\lambda) = \int_0^\infty (e^{\lambda x}/\phi(x))f(x)dx$ which is the Laplace transform of f/ϕ , Lf , evaluated at $-\lambda$. Thus if $\alpha(\phi) = \infty$, Lf is entire for each f in $L^2(\mathcal{R}_+)$. But (f, g_λ) is in the spectrum of A_f so Lf is bounded and entire. By Liouville's Theorem Lf is constant for every f in L^2 . But then $f/\phi = 0$ and $f = 0$. Thus $\alpha(\phi) < \infty$.

THEOREM 5.2. *If ϕ is in $C^1[a, \infty)$ for some $a \geq 0$ and $\phi(x) \neq 0$ for all x then $i(\phi) \leq \alpha(\phi) \leq s(\phi)$.*

Proof. We prove only the inequality $i(\phi) \leq \alpha(\phi)$, the other inequality's validity being quite similarly (and symmetrically) ascertained. If $i(\phi) = -\infty$ the inequality holds. Assume first that $i(\phi)$ is finite. Let $\epsilon > 0$ and let $\lambda = i(\phi) - \epsilon$. Then since ϕ'/ϕ is continuous for $t \geq a$ we have $\phi'(t)/\phi(t) > \lambda + (\epsilon/2)$ for all $t \geq a$. Let $f(x) = e^{2\lambda x}/\phi^2(x)$. Then $f'(x)/f(x) = 2(\lambda - (\phi'(x)/\phi(x))) < -\epsilon$ for all $x \geq a$ hence $f(x) \leq f(a)e^{-\epsilon(x-a)}$ for all $x \geq a$. Since f is continuous on $[0, a]$, f is in $L^1(\mathcal{R}_+)$. Thus $i(\phi) - \epsilon \leq \alpha(\phi)$ for all $\epsilon > 0$ and so $i(\phi) \leq \alpha(\phi)$.

Now, if $i(\phi) = +\infty$ then we have $\lim_{t \rightarrow \infty} (\phi'(t)/\phi(t)) = +\infty$. But then we easily see that for any λ in \mathcal{R} the function f defined above is in L' and so $\alpha(\phi) = +\infty$.

COROLLARY 5.3. *If $\lim_{t \rightarrow \infty} (\phi'(t)/\phi(t))$ exists, then $\alpha(\phi) = \lim_{t \rightarrow \infty} (\phi'(t)/\phi(t))$.*

In order to see that strict inequalities in the above Theorem 5.2 are possible, even for ϕ of bounded kernel type, note that if h and $1/h$ are bounded continuous functions on \mathcal{R}_+ and ϕ is of bounded kernel type, then $h\phi$ also is of bounded kernel type. Moreover, one easily verifies that $\alpha(\phi) = \alpha(h\phi)$. However (assuming h is in $C^1([a, \infty))$) for $\rho = h\phi$, $\rho'/\rho = (h'/h) + (\phi'/\phi)$. If, for example, we let $\phi(x) = x + 1$ and $h(x) = 2 + \sin x$ then all requirements above are satisfied, $\lim_{t \rightarrow \infty} (\phi'(t)/\phi(t)) = 0$, $\liminf_{t \rightarrow \infty} (h'(t)/h(t)) = -\sqrt{3}/3$, $\limsup_{t \rightarrow \infty} (h'(t)/h(t)) = \sqrt{3}/3$, and so $i(\rho) = -\sqrt{3}/3$, $\alpha(\rho) = 0$, and $s(\rho) = \sqrt{3}/3$.

We conclude the paper by showing that the class of weighted

translation semigroups with symbol of bounded kernel type is disjoint from a rather large class of weighted translation semigroups, including the hyponormal (and of course subnormal) ones.

LEMMA 5.4. For ϕ of bounded kernel type in $C^1([a, \infty))$ for some $a > 0$, $\alpha(\phi) < s = \sup_{t \geq a} (\phi'(t)/\phi(t))$.

Proof. We have already seen that $\alpha(\phi) < \infty$ so the case $s = \infty$ is obvious. Suppose then that $s < \infty$. Then for $t \geq a$, $\phi'(t)/\phi(t) \leq s$ so $\phi(t)/\phi(a) \leq e^{s(t-a)}$ and hence $1/\phi^2(t) \geq (1/\phi^2(a))e^{2s(a-t)}$. We then have

$$\infty > \int_0^\infty \frac{e^{2\alpha(\phi)t}}{\phi^2(t)} dt \geq \int_0^a \frac{e^{2\alpha(\phi)t}}{\phi^2(t)} dt + \left(\int_a^\infty e^{2[\alpha(\phi)-s]t} dt \right) e^{2sa}.$$

Thus $\alpha(\phi) < s$, for otherwise the last integral diverges.

COROLLARY 5.5. If $\{S_t\}$ is a hyponormal weighted translation semigroup with symbol ϕ in $C^1([a, \infty))$ for some $a \geq 0$, then ϕ is not of bounded kernel type.

Proof. In [2] we showed that $\{S_t\}$ is hyponormal if and only if $\log \phi$ is convex. Thus ϕ'/ϕ is an increasing function and so $\lim_{t \rightarrow \infty} (\phi'(t)/\phi(t)) = \sup_{t \geq 0} (\phi'(t)/\phi(t))$. By Corollary 5.3 and Lemma 5.4 ϕ cannot be of bounded kernel type.

Note that if $\{S_t\}$ is subnormal, the condition of ϕ being in $C^1([a, \infty))$ holds automatically since ϕ has the form $\phi^2(x) = e^{\alpha x} \int_0^\infty e^{-tx} d\rho(t)$ where ρ is a probability measure.

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